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The Ergodic Theory of Axiom A Flows

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1. Introduction

Let M be a compact (Riemann) manifold and (f^t): $M \to M$ a differentiable flow. A closed (f^t)-invariant set $\Lambda \subset M$ containing no fixed points is hyperbolic if the tangent bundle restricted to Λ can be written as the Whitney sum of three (Tf^t)-invariant continuous subbundles

$$T_\Lambda M = E + E^s + E^u$$

where E is the one-dimensional bundle tangent to the flow, and there are constants $c, \lambda > 0$ so that

(a) $\|Tf^t(v)\| \leqq c\, e^{-\lambda t} \|v\|$ for $v \in E^s$, $t \geqq 0$ and

(b) $\|Tf^{-t}(v)\| \leqq c\, e^{-\lambda t} \|v\|$ for $v \in E^u$, $t \geqq 0$.

We can choose $t_0 > 0$ and change λ so that the above conditions hold with $c = 1$ when $t \geqq t_0$. We can also assume that, for such t, Tf^t (resp. Tf^{-t}) expands E at a smaller rate than it expands any element of E^u (resp. E^s). It is then said that the metric is *adapted* (see [14]) to f^{t_0}. We will always assume that $t_0 \leqq 1$ – this can be achieved by a rescaling of t $(t \to t' = t/t_0)$ which does not affect our main results.

A closed invariant set Λ is a *basic hyperbolic* set if

(a) Λ contains no fixed points and is hyperbolic;

(b) the periodic orbits of $f^t|\Lambda$ are dense in Λ;

(c) $f^t|\Lambda$ is a topologically transitive flow; and

(d) there is an open set $U \supset \Lambda$ with $\Lambda = \bigcap_{t \in \mathbb{R}} f^t U$.

These sets are the building blocks of the Axiom A flows of Smale [27]. We will especially be interested in *attractors*, basic hyperbolic sets Λ for which the U in (d) can be found satisfying $f^t U \subset U$ for all $t \geqq T_0$ (T_0 fixed) and hence $\Lambda = \bigcap_{t \geqq 0} f^t U$. This paper will study the average asymptotic behavior of orbits of points in the neighborhood U of a C^2-attractor.

Precisely we will find an ergodic probability measure μ_φ on a C^2 attractor Λ so that for almost all $x \in U$ w.r.t. Lebesgue measure and all continuous g: $U \to \mathbb{R}$ one has

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(f^t x)\, dt = \int g\, d\mu_\varphi \tag{1}$$

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(see Theorem 5.1). The measure μ_φ will be described as the unique equilibrium state for a certain function $\varphi = \varphi^{(u)}$ (defined by (2), Section IV) on Λ, i.e. the unique f^t-invariant probability measure μ on Λ which maximizes the expression

$$h_\mu(f^1) + \int \varphi \, d\mu$$

where $h_\mu(f^1)$ is measure theoretic entropy. This variational principle (which is formally identical with one in statistical mechanics [21]) is useful because it gives a description of μ_φ which persists when one lifts μ_φ to a symbol space for closer study.

This paper carries over to flows results previously obtained for diffeomorphisms with regard to equilibrium states [6, 7, 24] and attractors [24]. For Anosov flows ($\Lambda = M$) the measure μ_φ has been studied in [9, 16, 17, 20, 25, 26] and the theory of Gibbs states (a slightly different formalism from equilibrium states which yields the same measures for basic hyperbolic sets) has been developed in [26]. Some results obtained here for flows are new even for diffeomorphisms; this is the case of Theorem 5.6. Results for diffeomorphisms can be obtained from those for flows via suspension (or directly by simplification of the proofs).

The determination of the asymptotic behavior of orbits is a significant problem in the study of differentiable dynamical systems. In particular the asymptotic behavior of solutions of a differential equation is of central interest in physical applications. Here we consider only the case of Axiom A flows. In that case it is known that $f^t x$ often depends in a very sensitive or "unstable" manner on the initial condition x, and (1)—which describes the time-average of an "observable" g—is probably the best way of expressing the asymptotic behavior of $f^t x$. It is a natural problem to extend (1) to non Axiom A situations.

We shall show that $\mu_{\varphi^{(u)}}$ depends continuously on the flow (f^t) (Proposition 5.4). In the same direction, Sinai [26] has proved the stability of μ_φ under small stochastic perturbations for Anosov flows[1]. (1) holds almost everywhere for x in the basin of an Axiom A attractor; one can prove that, for a C^2 Axiom A flow, these basins (and those of point attractors) cover M up to a set of Lebesgue measure zero. Equivalently: if a basic set is not an attractor, its stable manifold has measure zero (Theorem 5.6).

It can be seen that, unless Λ is a periodic orbit, the entropy of μ_φ does not vanish; this indicates "strong ergodic properties" of the system (μ_φ, f^t). In fact, if (f^t) restricted to Λ is C-dense, (μ_φ, f^t) is a Bernoulli flow (see Remark 3.5). The correlation functions

$$\rho_{gg'}(t) = \int (g \circ f^t) \cdot g' \, d\mu_\varphi - \int g \, d\mu_\varphi \cdot \int g' \, d\mu_\varphi$$

are interesting to consider in physical applications. In the C-dense case we have $\lim_{t\to\infty} \rho_{gg'}(t) = 0$ if $g, g' \in L^2(\mu_\varphi)$ (Remark 3.5). Assuming that g, g' are C^1, does $\rho_{gg'}(t)$ tend to zero exponentially when $t \to \infty$? The methods of the present paper do not seem capable of answering this question. A positive answer has been obtained for diffeomorphisms ([24, 26]).

[1] The corresponding problem for attractors for Axiom A diffeomorphisms has been treated by Kifer (Sinai, private communication).

Terminology

The manifold M and the Riemann metric on M are C^∞. The flow (f^t) is called C^r $(r \geqq 1)$ if it corresponds to a C^r vector field on M; a basic hyperbolic set Λ for (f^t) is then called a C^r basic hyperbolic set. The flow (f^t) restricted to Λ is topologically transitive if it has a dense orbit.

For easy reference, we collect here the definitions of stable manifolds

$$W_x^s = \{y \in M: \lim_{t \to \infty} d(f^t x, f^t y) = 0\}$$

$$W_x^{cs} = \bigcup_{t \in \mathbb{R}} W_{f^t x}^s.$$

A distance on M is defined by

$$\delta_T(x, y) = \sup_{0 \leqq t \leqq T} d(f^t x, f^t y)$$

when $0 \leqq T < \infty$; $B_x(\varepsilon, T)$ is the closed ε-neighbourhood of x for that distance; also

$$W_x^s(\varepsilon) = W_x^s \cap B_x(\varepsilon, \infty).$$

Replacing t by $-t$ and s by u we obtain the definition of unstable manifolds. We also write

$$W_\Lambda^s(\varepsilon) = \bigcup_{x \in \Lambda} W_x^s(\varepsilon), \qquad \text{etc.}$$

The basic hyperbolic set Λ is C-dense if $W_x^s \cap \Lambda$ is dense in Λ for some (hence for all) $x \in \Lambda$.

In general we write $f^* \mu$ the image of a measure μ by a continuous map f.

2. Symbolic Dynamics

Let us recall the symbolic dynamics of a basic hyperbolic set Λ [4]. For $A = [A_{ij}]$ an $n \times n$ matrix of 0's and 1's we define

$$\Sigma_A = \{\mathbf{x} = (x_i)_{i=-\infty}^{+\infty} \in \{1, \ldots, n\}^{\mathbb{Z}}: A_{x_i x_{i+1}} = 1 \, \forall i \in \mathbb{Z}\}$$

and $\sigma_A: \Sigma_A \to \Sigma_A$ by $\sigma_A(\mathbf{x}) = (x_i')_{i=-\infty}^{\infty}$ where $x_i' = x_{i+1}$. If we give $\{1, \ldots, n\}$ the discrete topology and $\{1, \ldots, n\}^{\mathbb{Z}}$ the product topology, then Σ_A becomes a compact metrizable space and σ_A a homeomorphism. σ_A (or Σ_A) is called a *subshift of finite type* if $\sigma_A: \Sigma_A \to \Sigma_A$ is topologically transitive (i.e. for U, V non-empty open sets there is an $n > 0$ with $f^n U \cap V \neq \emptyset$).

For $\psi: \Sigma_A \to \mathbb{R}$ a positive continuous function one can define a special (or suspension) flow as follows. Let

$$Y = \{(\mathbf{x}, s): s \in [0, \psi(\mathbf{x})], \mathbf{x} \in \Sigma_A\} \subset \Sigma_A \times \mathbb{R}.$$

Identify the points $(\mathbf{x}, \psi(\mathbf{x}))$ and $(\sigma_A(\mathbf{x}), 0)$ for all $\mathbf{x} \in \Sigma_A$ to get a new space $\Lambda(A, \psi)$. Then $\Lambda(A, \psi)$ is a compact metric space (see [8] for a metric) and one can define a flow g^t on $\Lambda(A, \psi)$ by

$$g^t(\mathbf{x}, s) = (\mathbf{x}, s + t) \qquad \text{for } s + t \in [0, \psi(\mathbf{x})]$$

and remembering identifications. More precisely, if $z = q(\mathbf{x}, s)$ where $q: Y \to \Lambda(A, \psi)$ is the quotient map, then $g^t(z) = q(\sigma_A^k \mathbf{x}, v)$ where k is chosen so that

$$v = t + s - \sum_{j=0}^{k-1} \psi(\sigma_A^j \mathbf{x}) \in [0, \psi(\sigma_A^k \mathbf{x})].$$

The flow g^t on $\Lambda(A, \psi)$ will be important to us with ψ satisfying an additional condition. For $\psi: \Sigma_A \to \mathbb{R}$ let

$$\operatorname{var}_n \psi = \sup \{ |\psi(\mathbf{x}) - \psi(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in \Sigma_A,\ x_i = y_i \forall |i| \leqq n \}.$$

Let

$$\mathscr{F}_A = \{ \psi \in C(\Sigma_A) : \exists b > 0,\ \alpha \in (0,1) \text{ so that } \operatorname{var}_n \psi \leqq b\, \alpha^n \text{ for all } n \geqq 0 \}.$$

2.1. **Lemma.** *Let Λ be a basic hyperbolic set. Then there is a topologically mixing*[2] *subshift of finite type $\sigma_A: \Sigma_A \to \Sigma_A$, a positive $\psi \in \mathscr{F}_A$ and a continuous surjection $\rho: \Lambda(A, \psi) \to \Lambda$ so that*

$$\begin{array}{ccc} \Lambda(A,\psi) & \xrightarrow{g^t} & \Lambda(A,\psi) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \Lambda & \xrightarrow{f^t} & \Lambda \end{array}$$

commutes.

This is from [4], Section 2, except for the mixing condition on σ_A. If $\sigma_A: \Sigma_A \to \Sigma_A$ is *not* mixing, then for some $m > 0$ $\Sigma_A = X_1 \cup \cdots \cup X_m$, a disjoint union of closed sets with $\sigma_A(X_i) = X_{i+1}$ and $\sigma_A^m | X_i: X_i \to X_i$ conjugate to a mixing subshift of finite type (see e.g. 2.7 of [2]).

Identifying $\sigma_A^m: X_1 \to X_1$ with some $\sigma_B: \Sigma_B \to \Sigma_B$ and defining $\psi': \Sigma_B \to \mathbb{R}$ by

$$\psi'(x) = \psi(x) + \psi(\sigma_A x) + \cdots + \psi(\sigma_A^{m-1} x)$$

one can see that $\Lambda(B, \psi')$ is homeomorphic to $\Lambda(A, \psi)$ in a natural way and $\psi' \in \mathscr{F}_B$.

There are other properties of the map ρ which we shall recall as we need them. Throughout the remainder of the paper ψ will always denote a positive function in $\mathscr{F}_A$ and σ_A a mixing subshift of finite type.

For any homeomorphism f the set of f-invariant Borel probability measures will be denoted $M(f)$. If $F = (f^t)_{t \in \mathbb{R}}$ is a continuous flow we will write $M(F) = \bigcap_{t \in \mathbb{R}} M(f^t)$.

3. Equilibrium States

Let us review the definition of topological pressure for a homeomorphism $f: X \to X$ of a compact metric space and a continuous function $\varphi: X \to \mathbb{R}$ [23, 28]. For given $\varepsilon > 0$ and $n > 0$, a subset $E \subset X$ is called *(ε, n)-separated* if

$$x, y \in E, \quad x \neq y \Rightarrow d(f^k x, f^k y) > \varepsilon \quad \text{for some } k \in [0, n].$$

[2] A homeomorphism $F: X \to X$ is topologically mixing if, for U, V open nonempty in X, $U \cap F^n V \neq \emptyset$ for all sufficiently large n.

One defines

$$Z_n(f,\varphi,\varepsilon)=\sup\left\{\sum_{x\in E}\exp\sum_{k=0}^{n-1}\varphi(T^k x)\colon E \text{ is } (\varepsilon,n)\text{-separated}\right\}$$

$$P(f,\varphi,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log Z_n(f,\varphi,\varepsilon)$$

and

$$P(f,\varphi)=\lim_{\varepsilon\to 0}P(f,\varphi,\varepsilon).$$

For $\varphi=0$ the number $P(f,\varphi)$ is just the topological entropy $h(f)$ of f; the theory of topological pressure generalizes that of topological entropy. The main general result is that

$$P(f,\varphi)=\sup_{\mu\in M(f)}\left(h_\mu(f)+\int\varphi\,d\mu\right).$$

This was proved by Walters [28]; for f expansive there is a $\mu\in M(f)$ with $h_\mu(f)+\int\varphi\,d\mu=P(f,\varphi)$.

An *equilibrium state* for $\varphi\colon X\to\mathbb{R}$ with respect to $f\colon X\to X$ is a $\mu\in M(f)$ with $h_\mu(f)+\int\varphi\,d\mu=P(f,\varphi)$, i.e. a $\mu\in M(f)$ maximizing the quantity $h_\mu(f)+\int\varphi\,d\mu$

Now we will consider the case of a flow $F=(f^t\colon X\to X)$ and $\varphi\colon X\to\mathbb{R}$. A set $E\subset X$ is (ε, T)-*separated* if

$$x,y\in E,\qquad x\neq y\ \Rightarrow\ d(f^t x, f^t y)>\varepsilon\qquad\text{for some } t\in[0,T].$$

Then we define

$$Z_T(F,\varphi,\varepsilon)=\sup\left\{\sum_{x\in E}\exp\int_0^T\varphi(f^t x)\,dt\colon\ E \text{ is } (\varepsilon,T)\text{-separated}\right\}$$

$$P(F,\varphi,\varepsilon)=\limsup_{T\to\infty}\frac{1}{T}\log Z_T(F,\varphi,\varepsilon)$$

and

$$P(F,\varphi)=\lim_{\varepsilon\to 0}P(F,\varphi,\varepsilon).$$

The definition of $P(F,\varphi)$ is independent of the choice of metric on M. It is a straightforward exercise to check that if one lets $\varphi^1(x)=\int_0^1\varphi(f^t x)\,dt$, then $P(F,\varphi)=P(f^1,\varphi^1)$; also, for $\mu\in M(F)$ one has $\int\varphi\,d\mu=\int\varphi^1\,d\mu$. As $M(F)\subset M(f^1)$ one has

$$h_\mu(f^1)+\int\varphi\,d\mu=h_\mu(f^1)+\int\varphi^1\,d\mu\leqq P(f^1,\varphi^1)=P(F,\varphi)$$

for $\mu\in M(F)$. There is an argument ([28], see also [10], p. 359–360) to show that for any $\mu'\in M(f^1)$ one can find a $\mu\in M(F)$ with

$$h_\mu(f^1)+\int\varphi^1\,d\mu\geqq h_{\mu'}(f^1)+\int\varphi^1\,d\mu'.$$

Hence it follows that

$$P(F,\varphi)=\sup_{\mu\in M(F)}\left(h_\mu(f^1)+\int\varphi\,d\mu\right).$$

By an *equilibrium state* for φ (with respect to F) we mean a $\mu\in M(F)$ with

$$h_\mu(f^1)+\int\varphi\,d\mu=P(F,\varphi).$$

For $G=\{g^t\}$ the special flow on $\Lambda(A,\psi)$ there is a well-known bijection between $M(G)$ and $M(\sigma_A)$. For $\nu\in M(\sigma_A)$ and m Lebesgue measure, $\nu\times m$ gives measure 0 to the identifications on $Y\to\Lambda(A,\psi)$ and so $\mu_\nu=(\nu\times m(Y))^{-1}\,\nu\times m|Y$ gives a probability measure on $\Lambda(A,\psi)$. One can check that $\nu\in M(\sigma_A)$ implies $\mu_\nu\in M(G)$ and that $\nu\to\mu_\nu$ defines a bijection $M(\sigma_A)\to M(G)$.

It is known that any function $\gamma\in\mathscr{F}_A$ has a unique equilibrium state ν w.r.t. σ_A [6, 11, 22, 24] and that ν depends continuously on γ (weak topology for ν, uniform topology for γ). We will now state the corresponding condition on $\varphi\colon\Lambda(A,\psi)\to\mathbb{R}$ which guarantees a unique equilibrium state.

3.1 **Proposition**[3]. *Let $\varphi\colon\Lambda(P,\psi)\to\mathbb{R}$ be continuous, $\Phi(\mathbf{x})=\int_0^{\psi(\mathbf{x})}\varphi(\mathbf{x},t)\,dt$ and $c=P(G,\varphi)$. Assume that $\Phi\in\mathscr{F}_A$. Then there is a measure $\mu_\varphi\in M(G)$ so that*

(a) *μ_φ is the unique equilibrium state for φ with respect to G.*

(b) *$\mu_\varphi=\mu_{\nu_0}$ where ν_0 is the unique equilibrium state for $\Phi-c\psi$ on Σ_A.*

(c) *μ_φ is ergodic and positive on non-empty open sets and*

(d) *for $\varepsilon>0$ there is a $C_\varepsilon>0$ so that*

$$\mu_\varphi(B_{x,G}(\varepsilon,T))\geqq C_\varepsilon\exp\left(-cT+\int_0^T\varphi(g^t x)\,dt\right)$$

for all $x\in\Lambda(A,\psi)$, $T\geqq 0$ where

$$B_{x,G}(\varepsilon,T)=\{y\in\Lambda(A,\psi)\colon d(g^t y,g^t x)\leqq\varepsilon \text{ for all } t\in[0,T]\}.$$

Let $\gamma=\Phi-c\psi$. As $\Phi,\psi\in\mathscr{F}_A$ we have $\gamma\in\mathscr{F}_A$. This guarantees that γ has a unique equilibrium state ν_0. By Fubini's theorem, for any $\nu\in M(\sigma_A)$, $(\nu\times m)(Y)=\int\psi\,d\nu$ and $\int\varphi\,d\mu_\nu=\dfrac{\int\Phi\,d\nu}{\int\psi\,d\nu}$. A theorem of Abramov [1] states that

$$h_{\mu_\nu}(g^1)=\frac{h_\nu(\sigma_A)}{\int\psi\,d\nu}.$$

Hence

$$\begin{aligned}c=P(G,\varphi)&=\sup_{\mu\in M(G)}\left(h_\mu(g^1)+\int\varphi\,d\mu\right)\\&=\sup_{\nu\in M(\sigma_A)}\frac{h_\nu(\sigma_A)+\int\Phi\,d\nu}{\int\psi\,d\nu}.\end{aligned}$$

Thus $P(\sigma_A,\gamma)=\sup_\nu\left(h_\nu(\sigma_A)+\int(\Phi-c\psi)\,d\nu\right)=0$ with ν attaining the supremum (i.e. $\nu=\nu_0$) precisely if μ_ν is the unique equilibrium state for φ. This shows that $\mu_\varphi=\mu_{\nu_0}$ satisfies (a) and (b); (c) is true because ν_0 has these same properties ([6] or [24], Appendix B).

We now verify (d). Let $x\in\Lambda(A,\psi)$ be represented by $x=(\mathbf{x},t_1)$, $t_1\in[0,\psi(\mathbf{x}))$ and $g^T x=(\sigma_A^n\mathbf{x},t_2)$, where $n=n(x)$ is such that

$$t_2=T+t_1-\sum_{k=0}^{n-1}\psi(\sigma_A^k\mathbf{x})\in[0,\psi(\sigma_A^n\mathbf{x})).$$

[3] Another proof of (a), (b), (d) has now been obtained by E. Franco-Sanchez, Berkeley thesis, 1974.

Given $\varepsilon>0$ one can find $\delta_\varepsilon>0$ and $s_\varepsilon>0$ (not depending on x or T) so that[4]

$$B_{x,G}(\varepsilon, T)\supset\{(\mathbf{y}, t)\colon |t-t_1|\leqq s_\varepsilon \text{ and } d(\sigma_A^k \mathbf{y}, \sigma_A^k \mathbf{x})\leqq\delta_\varepsilon \text{ for all } k\in[0, n-1]\}.$$

We do not go through the details but do point out that, for any $\alpha>0$, $\psi\in\mathscr{F}_A$ implies that for δ small enough

$$d(\sigma_A^k \mathbf{x}, \sigma_A^k \mathbf{y})\leqq\delta\,\forall k\in[0, n-1] \;\Rightarrow\; \sum_{k=0}^{n-1}|\psi(\sigma_A^k \mathbf{x})-\psi(\sigma_A^n \mathbf{y})|<\alpha.$$

Then

$$\mu_\varphi B_{x,G}(\varepsilon, T)\geqq\frac{s_\varepsilon}{\int\psi\, d\nu_0}\nu_0\{\mathbf{y}\colon d(\sigma_A^k \mathbf{y}, \sigma_A^k \mathbf{x})\leqq\delta_\varepsilon\forall k\in[0, n-1]\}.$$

By [6] Lemma 5, this right side is at least

$$a_\varepsilon \exp\left(\sum_{k=0}^{n-1}\gamma(\sigma_A^k \mathbf{x})-nP(\sigma_A, \gamma)\right)=a_\varepsilon \exp\sum_{k=0}^{n-1}\gamma(\sigma_A^k \mathbf{x})$$

for some $a_\varepsilon>0$. Since

$$\sum_{k=0}^{n-1}\psi(\sigma_A^k \mathbf{x})+t_2=T+t_1$$

and

$$\int_0^{t_1}\varphi(\mathbf{x}, t)\, dt+\int_0^T\varphi(g^t x)\, dt-\int_0^{t_2}\varphi(\sigma_A^n \mathbf{x}, t)\, dt=\sum_{k=0}^{n-1}\Phi(\sigma_A^k \mathbf{x}),$$

one sees that $\sum_{k=0}^{n}\gamma(\sigma_A^k \mathbf{x})$ differs from $-cT+\int_0^T\varphi(g^t x)\, dt$ by at most $2\|\psi\|(|c|+\|\varphi\|)$.

This proves (d).

3.2. *Remark.* Special flows are simple enough that parts (a) and (b) above could have been derived without appealing so much to general (and harder) results on topological pressure.

3.3 **Theorem.** *Assume that Λ is a basic hyperbolic set for F and that $\varphi\colon \Lambda\to\mathbb{R}$ satisfies a Hölder condition of positive exponent. Then φ has a unique equilibrium state μ_φ. Furthermore, μ_φ is ergodic and positive on non-empty open sets of Λ, and for any $\varepsilon>0$ there is a $C_\varepsilon>0$ so that*

$$\mu_\varphi(B_{x,F|\Lambda}(\varepsilon, T))\geqq C_\varepsilon \exp\left(-P(F|\Lambda, \varphi)\, T+\int_0^T\varphi(f^t x)\, dt\right)$$

for all $x\in\Lambda$, $T\geqq 0$.

We apply the preceding proposition to the function $\varphi^*=\varphi\circ\rho$ on $\Lambda(A, \psi)$. There are $b_1>0$ and $\tau\in(0, 1)$ so that

$$d(\rho(\mathbf{x}, 0), \rho(\mathbf{y}, 0))\leqq b_1\,\tau^N$$

if $x_i=y_i$ for all $|i|\leqq N$ (see [4], Lemma 2.2.(i)).

[4] In this formula $(\mathbf{y}, t)$ has to be replaced by $(\sigma_A^{-1}\mathbf{y}, \psi(\sigma_A^{-1}\mathbf{y})+t)$ resp. by $(\sigma_A \mathbf{y}, t-\psi(\mathbf{y}))$ when $t<0$ resp. $t>\psi(\mathbf{y})$.

Since F is a differentiable flow, there is a constant b_2 so that

$$d(f^t x, f^t y) \leqq b_2\, d(x, y) \qquad \text{provided } t \in [0, \|\psi\|].$$

The Hölder condition on φ states that

$$|\varphi(x) - \varphi(y)| \leqq b_3\, d(x, y)^\alpha \quad \text{with} \quad \alpha > 0.$$

Combining these estimates, when $x_i = y_i \forall i \in [-N, N]$ we have

$$\begin{aligned} &\left| \int_0^{\psi(\mathbf{x})} \varphi^*(\mathbf{x}, t)\, dt - \int_0^{\psi(\mathbf{y})} \varphi^*(\mathbf{y}, t)\, dt \right| \\ &\qquad \leqq \|\varphi\|\, |\psi(\mathbf{x}) - \psi(\mathbf{y})| + \int_0^{\psi(\mathbf{x})} \left|\varphi\big(f^t \rho(\mathbf{x}, 0)\big) - \varphi\big(f^t \rho(\mathbf{y}, 0)\big)\right| dt \\ &\qquad \leqq \|\varphi\|\, |\psi(\mathbf{x}) - \psi(\mathbf{y})| + \|\psi\|\, b_3 (b_2 b_1)^\alpha (\tau^\alpha)^N. \end{aligned}$$

Since $\psi \in \mathscr{F}$, this gives $\Phi^* \in \mathscr{F}$ where $\Phi^*(\mathbf{x}) = \int_0^{\psi(\mathbf{x})} \varphi^*(\mathbf{x}, t)\, dt$. So φ^* has a unique equilibrium state μ_{φ^*} as in the preceding proposition.

We recall that there are closed subsets $A_s = \rho^{-1}(\Delta^s \mathscr{M})$ and $A_u = \rho^{-1}(\Delta^u \mathscr{M})$ of $\Lambda(A, \psi)$ so that (see [4])

(a) $A_s \neq \Lambda(P, \psi) \neq A_u$

(b) $g^t A_s \subset A_s$, $g^{-t} A_u \subset A_u \forall t \geqq 0$ and

(c) ρ is one-to-one off $\bigcup_{t \in \mathbb{R}} g^t(A_u \cup A_s) = \bigcup_{n \in \mathbb{Z}} g^n(A_u \cup A_s)$.

Because μ_{φ^*} is positive on non-empty open sets $\mu_{\varphi^*}(A_s) \neq 1 \neq \mu_{\varphi^*}(A_u)$; since μ_{φ^*} is ergodic and each of these sets is invariant under one direction of time, $\mu_{\varphi^*}(A_s) = 0 = \mu_{\varphi^*}(A_u)$. By (c) then ρ gives a conjugacy of the measurable flows (G, μ_{φ^*}) and $(F(\Lambda, \mu_\varphi))$ where $\mu_\varphi = \rho^* \mu_{\varphi^*}$. In particular $h_{\mu_\varphi}(f^1) = h_{\mu_{\varphi^*}}(g^1)$ and

$$h_{\mu_\varphi}(f^1) + \int \varphi\, d\mu_\varphi = h_{\mu_{\varphi^*}}(g^1) + \int \varphi^*\, d\mu_{\varphi^*} = P(G, \varphi^*).$$

As $F|\Lambda$ is the quotient of G and $\varphi^* = \varphi \circ \rho$, one has $P(F|\Lambda, \varphi) \leqq P(G, \varphi^*)$ (see Walters [28], Theorem 2.2); because $h_{\mu_\varphi}(f^1) + \int \varphi\, d\mu_\varphi = P(G, \varphi^*)$ one has $P(F|\Lambda, \varphi) = P(G, \varphi^*)$ and μ_φ is an equilibrium state for φ. If μ were another equilibrium state for φ, then $\mu = \rho^* \mu'$ for some $\mu' \in M(G)$ (by an easy application of the Hahn-Banach and Markov-Kakutani theorems) and

$$h_{\mu'}(g^1) + \int \varphi^*\, d\mu' \geqq h_\mu(f^1) + \int \varphi\, d\mu = P(F|\Lambda, \varphi) = P(G, \varphi^*).$$

So μ' is an equilibrium state for φ^*, $\mu' = \mu_{\varphi^*}$ and $\mu = \mu_\varphi$. Thus μ_φ is the unique equilibrium state. The remaining properties for μ_φ follow from the corresponding ones for μ_{φ^*} in Proposition 3.1.

3.4 *Remark.* For $\varphi = 0$ the uniqueness of equilibrium state just says that $F|\Lambda$ has a unique invariant measure maximizing entropy. This was proved earlier in [5].

3.5. *Remark.* It was proved in [2] and [24] that (σ_A, ν_0) is isomorphic to a Bernoulli shift where ν_0 is the equilibrium state of $\gamma \in \mathscr{F}_A$. The corresponding result for $(F|\Lambda, \mu_\varphi)$ follows from results proved elsewhere. If $F|\Lambda$ is C-dense (i.e. $W^u(x) \cap \Lambda$ is dense in Λ for every $x \in \Lambda$ where $W^u(x) = \{y\colon d(f^{-t} x, f^{-t} y) \to 0$ as $t \to +\infty\}$), then G is also C-dense and Sinai [26] p. 48–9, applied a theorem of

Gurevič [12] to show that $(G, \mu_{\varphi*})$ is a K-flow. We mention that, although Sinai uses the formalism of Gibbs states instead of equilibrium states, the measure $\mu_{\varphi*}$ is the same as the one he constructs. M. Ratner [20] (also Bunimovič [9]) has proved in this C-dense case that $(G, \mu_{\varphi*})$ is actually Bernoulli (i.e. $(g^t, \mu_{\varphi*})$ is isomorphic to a Bernoulli shift for each $t \neq 0$). Since $(F|\Lambda, \mu_\varphi) \approx (G, \mu_{\varphi*})$, $(F|\Lambda, \mu_\varphi)$ is Bernoulli when $F|\Lambda$ is C-dense and φ is Hölder continuous. In that case we have

$$\lim_{t\to\infty} \int (g \circ f^t) \cdot g' \, d\mu_\varphi = \int g \, d\mu_\varphi \cdot \int g' \, d\mu_\varphi$$

for all $g, g' \in L^2(\mu_\varphi)$. [This follows from the fact that (μ_φ, f^t) is equivalent to a Bernoulli shift for each t, and the continuity of the flow (f^t)].

4. Attractors

Now assume Λ is a C^2 basic hyperbolic set. For $x \in \Lambda$ let $\lambda_t(x)$ be the Jacobian of the linear map $Df^t: E_x^u \to E_{f^t x}^u$ using inner products induced by the Riemannian metric.

Define

$$\varphi^{(u)}(x) = -\left.\frac{d \ln \lambda_t(x)}{dt}\right|_{t=0} = -\left.\frac{d\lambda_t(x)}{dt}\right|_{t=0} \tag{2}$$

which exists and depends differentiably on E_x^u (hence continuously on x) as f^t is a C^2 flow. Since $\lambda_{T+t}(x) = \lambda_t(f^T x)\, \lambda_T(x)$ one has

$$-\ln \lambda_t(f^T x) = -\ln \lambda_{T+t}(x) + \ln \lambda_T(x)$$

and so

$$\varphi^{(u)}(f^T x) = -\left.\frac{d \ln \lambda_s(x)}{ds}\right|_{s=T}.$$

This implies that

$$\int_0^T \varphi^{(u)}(f^t x)\, dt = -\ln \lambda_T(x).$$

This integral is the one appearing in Theorem 3.3 for $\varphi = \varphi^{(u)}$.

4.1. **Lemma.** *For Λ a C^2 basic hyperbolic set and $\varphi^{(u)}: \Lambda \to \mathbb{R}$ as above, $\varphi^{(u)}$ satisfies a Hölder condition of positive exponent.*

$x \to E_x^u$ is Hölder continuous (3.1 of [19]) and $E_x^u \to \varphi^{(u)}(x)$ is differentiable, so the composition $x \to \varphi^{(u)}(x)$ is Hölder.

4.2. **Lemma** (Volume lemma). *Let Λ be a C^2 basic hyperbolic set and define*

$$B_x(\varepsilon, T) = \{y \in M: d(f^t x, f^t y) \leqq \varepsilon \text{ for all } t \in [0, T]\}.$$

For small $\varepsilon > 0$ there is a constant $c_\varepsilon > 1$ so that

$$m(B_x(\varepsilon, T))\, \lambda_T(x) \in [c_\varepsilon^{-1}, c_\varepsilon]$$

for all $x \in \Lambda$ and $T \geqq 0$, where m is the measure on M derived from the Riemann metric.

4.3. **Lemma** (Second volume lemma). *For small* ε, $\delta>0$ *there is* $d=d(\varepsilon,\delta)>0$ (d independent of n) *so that*

$$m(B_y(\delta,n))\geqq d\cdot m(B_x(\varepsilon,n))$$

whenever $x\in\Lambda$ *and* $y\in B_x(\varepsilon,n)$.

These two lemmas are proved in the Appendix.

4.4. **Proposition.** (a) *Let* Λ *be a* C^2 *basic hyperbolic set and define*

$$B_\Lambda(\varepsilon,T)=\bigcup_{x\in\Lambda}B_x(\varepsilon,T).$$

Then (for sufficiently small ε*)*

$$P(F|\Lambda,\varphi^{(u)})=\limsup_{T\to\infty}\frac{1}{T}\log m(B_\Lambda(\varepsilon,T))\leqq 0. \tag{3}$$

(b) *Define*

$$W_x^s(\varepsilon)=\{y\in M:\ \lim_{t\to\infty}d(f^t y,f^t x)=0 \text{ and } d(f^t y,f^t x)\leqq\varepsilon\,\forall t\geqq 0\}$$

when $x\in\Lambda$, *and let* $W_\Lambda^s(\varepsilon)=\bigcup_{x\in\Lambda}W_x^s(\varepsilon)$. *If* $m(W_\Lambda^s(\varepsilon))>0$, *then* $P(F|\Lambda,\varphi^{(u)})=0$ *and*

$$h_{\mu_{\varphi^{(u)}}}(f^1)=-\int\varphi^{(u)}\,d\mu_{\varphi^{(u)}}. \tag{4}$$

This is true in particular if Λ *is an attractor.*

Let $0<\delta\leqq\varepsilon$. If E is a maximal (δ,T)-separated set for $F|\Lambda$, then

$$\bigcup_{x\in E}B_x(\delta/2,T)\subset B_\Lambda(\varepsilon,T)\subset\bigcup_{x\in E}B_x(\delta+\varepsilon,T)$$

where the $B_x(\delta/2,T)$ are disjoint, and the second inclusion follows from $\Lambda\subset\bigcup_{x\in E}B_x(\delta,T)$. Thus, assuming ε small enough and using the volume lemma,

$$c_{\delta/2}^{-1}\sum_{x\in E}\lambda_T(x)^{-1}\leqq m(B_\Lambda(\varepsilon,T))\leqq c_{\delta+\varepsilon}\sum_{x\in E}\lambda_T(x)^{-1}.$$

Therefore

$$c_{\delta/2}^{-1}Z_T(F|\Lambda,\varphi^{(u)},\delta)\leqq m(B_\Lambda(\varepsilon,T))\leqq c_{\delta+\varepsilon}Z_T(F|\Lambda,\varphi^{(u)},\delta)$$

and

$$P(F|\Lambda,\varphi^{(u)},\delta)=\limsup_{T\to\infty}\frac{1}{T}\log m(B_\Lambda(\varepsilon,T)).$$

The limit $\delta\to 0$ yields (3), proving (a).

By Theorem 3.3 and Lemma 4.1, $\varphi=\varphi^{(u)}$ has a unique equilibrium state $\mu_{\varphi^{(u)}}$. By the definition of equilibrium states, (4) is equivalent to $P(F|\Lambda,\varphi^{(u)})=0$. The latter statement follows from (3) with

$$m(B_\Lambda(\varepsilon,T))\geqq m(W_\Lambda^s(\varepsilon))>0.$$

There is a neighbourhood V of Λ so that

$$W_\Lambda^s(\varepsilon)\supset\{y\in M:\ f^t y\in V \text{ for all } t\geqq 0\}.$$

Indeed 5.1 of [13] gives this for diffeomorphisms and [13] indicates how to do the proof for flows. If Λ is an attractor, one can find a small neighbourhood U' of Λ so that $f^t y\in V$ for all $t\geqq 0$ whenever $y\in U'$. Then $U'\subset W_\Lambda^s(\varepsilon)$ and therefore $m(W_\Lambda^s(\varepsilon))>0$.

4.5. *Remark.* A rescaling of t: $t \to t' = t/t_0$ does not change the invariant measures, it replaces $h_\mu(f^1)$ by $h_\mu(f^{t_0}) = t_0\, h_\mu(f^1)$ (see [1]) and $\varphi^{(u)}$ by $\varphi'^{(u)} = t_0\, \varphi^{(u)}$. Therefore – as indicated in the Introduction – rescaling of t does not change $\mu_{\varphi^{(u)}}$ or the main results below. A change of Riemann metric on M changes $\varphi^{(u)}$ but not $\mu_{\varphi^{(u)}}$, or $P(F|\Lambda, \varphi^{(u)})$ as one readily sees.

4.6. **Corollary.** *Let Λ be a C^2 basic hyperbolic set. For sufficiently small $\varepsilon > 0$ there is a constant c'_ε such that*

$$m(B_x(2\varepsilon, T)) \leqq c'_\varepsilon\, \mu_{\varphi^{(u)}}(B_x(\varepsilon, T))$$

for all $x \in \Lambda$, $T \geqq 0$.

This follows from Theorem 3.3, Lemma 4.2, and Proposition 4.4(a), with $c'_\varepsilon = c_{2\varepsilon}/C_\varepsilon$.

5. Main Results

5.1. **Theorem.** *Let Λ be a C^2 hyperbolic attractor, W^s_Λ its basin. Then for m-almost all points $x \in W^s_\Lambda$ one has*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(f^t x)\, dt = \int g\, d\mu_{\varphi^{(u)}}$$

for all continuous g: $M \to \mathbb{R}$ (i.e. x is a generic point for $\mu_{\varphi^{(u)}}$).

We can replace W^s_Λ by a neighbourhood U of Λ such that $f^t U \subset U$ for all $t \geqq t_0$ and $\bigcap_{t\geqq 0} f^t U = \Lambda$.

Let us write

$$\bar{g}(T, x) = \frac{1}{T} \int_0^T g(f^t x)\, dt \quad \text{and} \quad \bar{g} = \int g\, d\mu_\varphi.$$

Let

$$E(g, \delta) = \{x \in U\colon \limsup_{T\to\infty} |\bar{g}(T, x) - \bar{g}| \geqq \delta\}.$$

Choose $\varepsilon > 0$ so small that $|g(f^t x) - g(f^t x')| < \delta/4$ whenever $d(x, x') \leqq \varepsilon$ and $0 \leqq t \leqq 1$. If we set $C_n(g, \delta') = \{x \in U\colon |\bar{g}(n, x) - \bar{g}| > \delta'\}$, then

$$E(g, \delta) \subset \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} C_n\left(g, \frac{3\delta}{4}\right) \subset E\left(g, \frac{3\delta}{4}\right).$$

Now fix $N > 0$ and choose finite subsets $S_N, S_{N+1}, \ldots,$ of Λ successively as follows. Let $S_n (n \geqq N)$ be a maximal subset of $C_n(g, \delta/2) \cap \Lambda$ satisfying the conditions:

(a) $B_x(\varepsilon, n) \cap B_y(\varepsilon, k) = \emptyset$ for $x \in S_n$, $y \in S_k$, $N \leqq k < n$ and

(b) $B_x(\varepsilon, n) \cap B_{x'}(\varepsilon, n) = \emptyset$ for $x, x' \in S_n$, $x \neq x'$.

(Notice that each S_n is finite.) Choose $\alpha > 0$ so that

$$B_\Lambda(\alpha) \subset W^s_\Lambda(\varepsilon) = \bigcup_{z\in\Lambda} W^s_z(\varepsilon)$$

($B_\Lambda(\alpha)$ is the closed α-neighbourhood of Λ). If

$$y \in B_\Lambda(\alpha) \cap C_n\left(g, \frac{3\delta}{4}\right) \quad (n \geqq N)$$

and $y \in W_z^s(\varepsilon)$ with $z \in \Lambda$, then $z \in C_n(g, \delta/4)$ because of the way ε was chosen. By the maximality of S_n one has

$$B_z(\varepsilon, n) \cap B_x(\varepsilon, k) \neq \emptyset \quad \text{for some } x \in S_k,\ N \leqq k \leqq n$$

and then $B_x(2\varepsilon, k) \supset B_z(\varepsilon, n) \supset W_z^s(\varepsilon) \ni y$. Thus one has

$$B_\Lambda(\alpha) \cap \bigcup_{n=N}^{\infty} C_n\left(g, \frac{3\delta}{4}\right) \subset \bigcup_{k=N}^{\infty} \bigcup_{x \in S_k} B_x(2\varepsilon, k).$$

Using Lemma 4.2 one has

$$m\left(B_\Lambda(\alpha) \cap \bigcup_{n=N}^{\infty} C_n\left(g, \frac{3\delta}{4}\right)\right) \leqq A'_{2\varepsilon} \sum_{k=N}^{\infty} \sum_{x \in S_k} \exp \int_0^k \varphi(f^t x)\, dt. \tag{5}$$

The definition of S_n implies that $V_N = \bigcup_{k=N}^{\infty} \bigcup_{x \in S_k}^{\infty} B_x(\varepsilon, k)$ is a disjoint union. The choice of ε gives that $B_\varepsilon(x, k) \subset C_k(g, \delta/4)$ for $x \in S_k \subset C_k(g, \delta/2)$ and so $V_N \subset \bigcup_{k=N}^{\infty} C_k(g, \delta/4)$. Because $\mu_{\varphi^{(u)}}$ is ergodic,

$$0 = \mu_{\varphi^{(u)}}\left(E\left(g, \frac{\delta}{4}\right)\right) \geqq \mu_{\varphi^{(u)}}\left(\bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} C_k\left(g, \frac{\delta}{4}\right)\right) = \lim_{N \to \infty} \mu_{\varphi^{(u)}}\left(\bigcup_{k=N}^{\infty} C_k\left(g, \frac{\delta}{4}\right)\right)$$

and $\lim_{N \to \infty} \mu_{\varphi^{(u)}}(V_N) = 0$. By Theorem 3.3 (and $P(F|\Lambda, \varphi^{(u)}) = 0$, confer Proposition 4.4)

$$\mu_{\varphi^{(u)}}(V_N) \geqq C_\varepsilon \sum_{k=N}^{\infty} \sum_{x \in S_k} \exp \int_0^k \varphi(f^t x)\, dt.$$

Hence the sum on the right converges to 0 as $N \to \infty$ and using (5) above we get

$$\lim_{N \to \infty} m\left(B_\Lambda(\alpha) \cap \bigcup_{n=N}^{\infty} C_n\left(g, \frac{3\delta}{4}\right)\right) = 0.$$

This in turn gives $m(B_\Lambda(\alpha) \cap E(g, \delta)) = 0$.

Now $f^t E(g, \delta) \subset E(g, \delta)$ for all $t \geqq 0$ and $f^t(U) \subset B_\Lambda(\alpha)$ for some $t > 0$. As f^t is a diffeomorphism, $m(f^t E(g, \delta)) \leqq m(B_\Lambda(\alpha) \cap E(g, \delta)) = 0$ implies $m(E(g, \delta)) = 0$. Letting $\{g_k\}_{k=1}^{\infty}$ be a dense sequence of continuous functions $\bar{U} \to \mathbb{R}$, we get that for x outside the m-null set

$$\bigcup_{k, m \geqq 1} E\left(g_k, \frac{1}{m}\right)$$

one has $\lim_{T \to \infty} \bar{g}_k(T, x) = \bar{g}_k$; as the g_k are dense, it follows that $\lim_{T \to \infty} \bar{g}(T, x) = \bar{g}$ for all continuous $g: \bar{U} \to \mathbb{R}$.

5.2. *Remark.* For the special case of an Anosov flow ($\Lambda = M$) with an invariant (probability) measure μ' absolutely continuous w.r.t. m, this theorem implies the known fact that $\mu' = \mu_{\varphi^{(u)}}$.

5.3. **Theorem.** *Let Λ be a C^2 attractor, W_Λ^s its basin, and let ν be a probability measure absolutely continuous with respect to m and with support in W_Λ^s. If the flow F restricted to Λ is C-dense, then*

$$\lim_{t \to \infty} \int (g \circ f^t)\, d\nu = \int g\, d\mu_{\varphi^{(u)}}$$

for all continuous $g: M \to \mathbb{R}$.

We may choose U as in the proof of Theorem 5.1, and assume that $\operatorname{supp} \nu \subset U$. Define $f^{*t} \nu$ by

$$(f^{*t} \nu)(g) = \nu(g \circ f^t)$$

and write $\mu_{\varphi^{(u)}} = \mu$. We have to prove that weak $\lim_{t \to \infty} f^{*t} \nu = \mu$. In showing this we may assume that $\nu = r \cdot m$ where $r \geqq 0$ is bounded (by density of bounded functions in L^1).

Given $\varepsilon > 0$ we find as in the proof of Proposition 4.4 that if U' is a sufficiently small neighbourhood of Λ, then $U' \subset W^s_\Lambda(\varepsilon)$. We can choose $t(\varepsilon) > 0$ so that $f^{t(\varepsilon)} U \subset U'$ and therefore

$$\operatorname{supp} f^{*t(\varepsilon)} \nu \subset W^s_\Lambda(\varepsilon).$$

Let $E \subset \Lambda$ be a maximal (T, ε)-separated set for $F|\Lambda$, we have thus

$$\operatorname{supp} f^{*t(\varepsilon)} \nu \subset \bigcup_{x \in E} B_x(2\varepsilon, T).$$

Let $(\psi_x)_{x \in E}$ be a non-negative measurable partition of unity on $\operatorname{supp} f^{*t(\varepsilon)} \nu$ subordinate to the covering by the $B_x(2\varepsilon, T)$ and let χ_x be the characteristic function of $B_x(\varepsilon, T)$. We write

$$\nu_{\varepsilon, T} = \sum_{x \in E} \left(\frac{\int \psi_x \, d(f^{*t(\varepsilon)} \nu)}{\int \chi_x \, d\mu} \right) \cdot \chi_x \, \mu.$$

The measure $\nu_{\varepsilon, T}$ is a probability measure absolutely continuous with respect to μ, with density bounded independently of T. This is because

$$\frac{\int \psi_x \, d(f^{*t(\varepsilon)} \nu)}{\int \chi_x \, d\mu} \leqq \|r_{t(\varepsilon)}\|_\infty \frac{m(B_x(2\varepsilon, T))}{\mu(B_x(\varepsilon, T))} \leqq C'_\varepsilon \|r_{t(\varepsilon)}\|_\infty$$

by Corollary 4.6 ($r_{t(\varepsilon)}$ denotes the density of $f^{*t(\varepsilon)} \nu$ with respect to m).

Notice that $\nu_{\varepsilon, T}$ is obtained by redistributing the mass of $f^{*t(\varepsilon)} \nu$ in such a manner that all that goes to $B_x(\varepsilon, T)$ comes from $B_x(2\varepsilon, T)$. Therefore also $f^{*t} \nu_{\varepsilon, T}$ is obtained by redistributing the mass of $f^{*(t+t(\varepsilon))} \nu$ in such a manner that all that goes to $f^t B_x(\varepsilon, T)$ comes from $f^t B_x(2\varepsilon, T)$. The diameter of $f^t B_x(2\varepsilon, T)$ is at most 4ε when $0 \leqq t \leqq T$; therefore if $\mathcal{N}$ is a closed weak neighbourhood of the origin in the space of real measures on M we have

$$f^{*(t+t(\varepsilon))} \nu - f^{*t} \nu_{\varepsilon, T} \in \mathcal{N}$$

when $0 \leqq t \leqq T$, provided ε has been chosen sufficiently small.

Remember now that $\nu_{\varepsilon, T} = s_{\varepsilon, T} \cdot \mu$ where $s_{\varepsilon, T} \in L^\infty(\mu)$ and $\|s_{\varepsilon, T}\|_\infty$ is bounded independently of T. We can thus choose $T_n \to \infty$ such that $s_{\varepsilon, T_n} \to s_\varepsilon$ in $L^\infty(\mu)$ with its topology of weak dual of $L^1(\mu)$. We have thus

$$f^{*(t+t(\varepsilon))} \nu - f^{*t}(s_\varepsilon \cdot \mu) \in \mathcal{N} \tag{6}$$

for all $t \geqq 0$. We use now Remark 3.5: (μ, f^t) is a Bernoulli flow and

$$\lim_{t \to \infty} \int s_\varepsilon \cdot (g \circ f^t) \, d\mu = \mu(g)$$

for all continuous $g: \Lambda \to \mathbb{R}$. There is thus $t_{\mathcal{N}}$ such that

$$f^{*t}(s_\varepsilon \cdot \mu) - \mu \in \mathcal{N} \tag{7}$$

for $t \geqq t_{\mathcal{N}}$. From (6) and (7) we obtain

$$f^{*t}\, v - \mu \in 2\mathcal{N}$$

when $t \geqq t(\varepsilon) + t_{\mathcal{N}}$. Therefore $f^{*t}\, v$ tends weakly to μ when $t \to \infty$.

5.4. **Proposition.** *Let Λ be a C^2 basic hyperbolic set. The measure $\mu_{\varphi^{(u)}}$ depends continuously on the C^2 flow F for the weak topology on measures and the C^1 topology on flows. Also the pressure of $\varphi^{(u)}$ and the entropy of $\mu_{\varphi^{(u)}}$ with respect to F depend continuously on F for the C^1 topology on flows.*

Let $\mu = \mu_{\varphi^{(u)}}$, and μ' be the corresponding measure for a flow F'. We have to show that $\mu' \to \mu$ weakly when $F' \to F$ in the C^1 sense; μ' is a measure carried by the F'-basic set Λ' close to Λ.

Going back to Lemma 2.1 and using [4] we have a special flow G' on $\Lambda'(A, \psi')$ and a continuous surjection ρ': $\Lambda'(A, \psi') \to \Lambda'$. By [4] and the Ω-stability theorem [5], we can construct $\Lambda'(A, \psi')$ from the same subshift σ_A: $\Sigma_A \to \Sigma_A$ which was used for $\Lambda(P, \psi)$. When $F' \to F$, we have $\psi' \to \psi$ uniformly, and $\rho'(\mathbf{x}, t) \to \rho(\mathbf{x}, t)$ (uniformly in $(\mathbf{x}, t)$ for $0 \leqq t \leqq \min\{\psi(\mathbf{x}), \psi'(\mathbf{x})\}$). Furthermore $E'^u_{\rho'(\mathbf{x},t)} \to E^u_{\rho(\mathbf{x},t)}$ where E'^u is the unstable subbundle for F' (by Theorem (6.1) of [15]).

We know that

$$\begin{aligned} \Phi(\mathbf{x}) &= \int_0^{\psi(\mathbf{x})} \varphi^{(u)}(\rho(\mathbf{x}, t))\, dt = \int_0^{\psi(\mathbf{x})} \varphi^{(u)}(f^t \rho(\mathbf{x}, 0))\, dt \\ &= -\ln \lambda_{\psi(\mathbf{x})}(\rho(\mathbf{x}, 0)) \end{aligned}$$

and correspondingly

$$\Phi'(\mathbf{x}) = -\ln \lambda'_{\psi'(\mathbf{x})}(\rho'(\mathbf{x}, 0)).$$

Therefore when $F' \to F$, we have $\Phi' \to \Phi$ uniformly.

According to Section 3 we have

$$P(F|\Lambda, \varphi^{(u)}) = P(G, \varphi^{(u)} \circ \rho) = \sup_{v \in M(\sigma_A)} \frac{h_v(\sigma_A) + \int \Phi\, dv}{\int \psi\, dv}.$$

Therefore $P(F|\Lambda, \varphi^{(u)})$ depends continuously on F. Using the notation of Proposition 3.1, we let v_0 be the unique equilibrium state for $\Phi - P(F|\Lambda, \varphi^{(u)}) \cdot \psi$. By the continuity of the equilibrium state indicated just before Proposition 3.1, $v_0' \to v_0$ when $F' \to F$. Thus $\mu_{v_0'} \to \mu_{v_0}$ and $\mu' = \rho'^* \mu_{v_0'} \to \mu = \rho^* \mu_{v_0}$.

Finally, the entropy of $\mu_{\varphi^{(u)}}$ is

$$P(F|\Lambda, \varphi^{(u)}) - \int \varphi^{(u)}\, d\mu_{\varphi^{(u)}}$$

where

$$\int \varphi^{(u)}\, d\mu_{\varphi^{(u)}} = \int (\varphi^{(u)} \circ \rho)\, d\mu_{v_0} = \int \Phi\, dv_0 / \int \psi\, dv_0$$

depends continuously on F.

5.5. **Proposition.** *Let Λ be a C^1 basic hyperbolic set and let $\varepsilon > 0$.*

(a) *If $W_x^u(\varepsilon) \subset \Lambda$ for some $x \in \Lambda$, then Λ is an attractor.*

(b) *If Λ is not an attractor, there exists $\gamma > 0$ such that for all $x \in \Lambda$, there is $y \in W_x^u(\varepsilon)$ with $d(y, \Lambda) > \gamma$.*

If $W_x^u(\varepsilon) \subset \Lambda$, and $u > 0$, the set

$$U_x = \bigcup \{W_y^s(\varepsilon) \colon y \in \bigcup_{|t| \leqq u} f^t W_x^u(\varepsilon)\} \tag{8}$$

is a neighbourhood of x in M (see [13], Lemma 4.1). Choose a periodic point $p \in U_x \cap \Lambda$ and let t_0 be its period. For some $\beta \in (0, \varepsilon]$, we have $W_p^u(\beta) \subset U_x$, hence

$$W_p^u(\beta) \subset W_\Lambda^s(\varepsilon) \cap W_\Lambda^u(\varepsilon) = \Lambda$$

(by [18], Theorem 3.2). Now

$$W_p^u = \bigcup_{n=1}^{\infty} f^{-nt_0} W_p^u(\beta) \subset \Lambda$$

and

$$W_p^{cu} = \bigcup_{0 \leqq t \leqq t_0} W_{f^t p}^u$$

is dense in Λ (see for instance [3], p. 11–13).

For each $x \in W_p^{cu}$, the set U_x defined by (8) is a neighbourhood of x in M. As $W_x^s(\varepsilon)$, $W_x^u(\varepsilon)$ depend continuously on $x \in \Lambda$, one can find $\delta > 0$ independent of x such that $U_x \supset B_x(2\delta)$ for all $x \in W_p^{cu}$ (see [13], Lemma 4.1). In view of this, and the density of W_p^{cu} in Λ,

$$B_\Lambda(\delta) \subset \bigcup \{U_x \colon x \in W_p^{cu}\}.$$

Therefore, if $z \in B_\Lambda(\delta)$ there exist $x \in W_p^{cu}$ and $y \in \bigcup_{|t| \leqq u} f^t W_x^u(\varepsilon) \subset W_p^{cu} \subset \Lambda$ such that $z \in W_y^s(\varepsilon)$. When $t \to \infty$ then $d(f^t z, f^t y) \to 0$ uniformly in z. Therefore

$$\bigcap_{t \geqq 0} f^t B_\Lambda(\delta) = \Lambda,$$

which shows that Λ is an attractor and proves (a).

To prove (b), notice that the set

$$V_\gamma = \{x \in \Lambda \colon d(y, \Lambda) > \gamma \text{ for some } y \in W_x^u(\varepsilon)\}$$

is open in Λ since $W_x^u(\varepsilon)$ varies continuously with x. Also V_γ increases as γ decreases and, by part (a) of the present proposition, $\bigcup_{\gamma > 0} V_\gamma = \Lambda$. Therefore, by compactness, $V_\gamma = \Lambda$ for some $\gamma > 0$.

5.6. **Theorem**[5]. *Let Λ be a C^2 basic hyperbolic set. The following conditions are equivalent:*

(a) *Λ is an attractor;*

(b) *$m(W_\Lambda^s) > 0$;*

(c) *$P(F|\Lambda, \varphi^{(u)}) = 0$.*

[5] Some of the ideas in the proof of this theorem were earlier discovered by J. Franks and R. F. Williams.

Since $W_\Lambda^s = \bigcup_{n=0}^{\infty} f^{-n} W_\Lambda^s(\varepsilon)$, (b) can be replaced by $m(W_\Lambda^s(\varepsilon))>0$ for any small $\varepsilon>0$. In Proposition 4.4 (b) we have seen that (a) ⇒ (b) ⇒ (c). To complete the proof we assume that Λ is not an attractor, and show that $P(F|\Lambda, \varphi^{(u)})<0$.

Given a small $\varepsilon>0$, choose γ as in Proposition 5.5(b). There is $t>0$ such that if $x\in\Lambda$, $f^t W_x^u(\gamma/4)\supset W_{f^t x}^u(\varepsilon)$. Let $E\subset\Lambda$ be (γ, T)-separated. For any $x\in E$ and $T>0$, there is $y(x,T)\in B_x(\gamma/4, T)$ such that $d(f^{t+T}(y(x,T)), \Lambda)>\gamma$ [because $f^T B_x(\gamma/4, T)\supset W_x^u(\gamma/4)$, hence $f^{T+t} B_x(\gamma/4, T)\supset W_{f^t x}^u(\varepsilon)$]. Choose $\delta\in(0, \gamma/4]$ such that $d(f^t z, f^t y)<\gamma/2$ whenever $d(z,y)\leqq\delta$. Then

$$B_{y(x,T)}(\delta, T)\subset B_x(\gamma/2, T)$$

$$f^{T+t} B_{y(x,T)}(\delta, T)\cap B_\Lambda(\gamma/2)=\emptyset,$$

hence

$$B_{y(x,T)}(\delta, T)\cap B_\Lambda(\gamma/2, T+t)=\emptyset.$$

Using the second volume lemma we have thus

$$\begin{aligned} m(B_\Lambda(\gamma/2, T)) - m(B_\Lambda(\gamma/2, T+t)) &\geqq \sum_{x\in E} m(B_{y(x,T)}(\delta, T)) \\ &\geqq d(3\gamma/2, \delta)\sum_{x\in E} m(B_x(3\gamma/2, T)) \geqq d(3\gamma/2,\delta)\, m(B_\Lambda(\gamma/2, T)) \end{aligned}$$

and therefore

$$m(B_\Lambda(\gamma/2, T+t))\leqq (1-d(3\gamma/2,\delta))\, m(B_\Lambda(\gamma/2, T))$$

so that by Proposition 4.4(a)

$$P(F|\Lambda, \varphi^{(u)})\leqq \frac{1}{t}\log(1-d(3\gamma/2,\delta))<0.$$

5.7. **Corollary.** *Let F be a C^2 Axiom A flow on the compact manifold M.*

(a) *The closures of the basins of the attractors cover M.*

(b) *If Λ is a basic hyperbolic set and $m(\Lambda)>0$, then Λ is a connected component of M and $F|\Lambda$ is an Anosov flow.*

Since F satisfies Axiom A, $M=\bigcup\{W_\Lambda^s\colon \Lambda$ is a basic hyperbolic set$\}$.

The complement of the basins of the attractors is $\bigcup\{W_\Lambda^s\colon \Lambda$ is not an attractor$\}$, it has measure 0, and therefore contains no open set. This proves (a).

If Λ is a basic set and $m(\Lambda)>0$, then $m(W_\Lambda^s)>0$ and $m(W_\Lambda^u)>0$ so that Λ is an attractor for both F and the opposite flow F^{-1}. Since Λ is an attractor for F, then $W_\Lambda^u=\Lambda$. Since Λ is an attractor for F^{-1}, then W_Λ^u is open. Therefore Λ is open and closed. Also, Λ is connected since W_p^{cu} is dense in Λ for periodic p (see [3], p. 11–13). This proves (b).

Appendix

Throughout what follows Λ will be a hyperbolic set for the C^r flow (f^t) on the manifold M $(r\geqq 1)$.

We recall that, by assumption, the Riemann metric on M is adapted to f^1 (see Introduction). Notice also that there is $K>0$ such that $\|Tf^n|E\|\leqq K$ for all $n\geqq 0$.

Denote by indices 0, 1, 2 the components in E_x, E_x^s, E_x^u of a vector in $T_x M$. We shall use in this Appendix a new scalar product in $T_x M$ defined by

$$\langle u, v\rangle_x = \sum_0^2 (u_i, v_i). \tag{A.1}$$

We write $\|u\| = \|u\|_x = (\langle u, u\rangle_x)^{1/2}$. If E' is a subspace of $T_x M$, $E'(\varepsilon)$ will denote the closed ε-ball centered at the origin of E' for this metric.

A.1. **Convenient Charts.** *For sufficiently small $\varepsilon>0$ and each $x\in\Lambda$, let us define a C^r chart φ_x: $T_x M(\varepsilon)\to M$ such that*[6]

$$\varphi_x(E_x+E_x^s)(\varepsilon)\subset W_x^{cs}, \qquad \varphi_x(E_x+E_x^u)(\varepsilon)\subset W_x^{cu}$$

and the map $F=\varphi_{fx}^{-1}\circ f^1\circ\varphi_x$ is tangent to $T_x f^1$ at the origin of $T_x M$.

If $\exp_x^{-1} W_x^{cs}$ is (in a neighbourhood of the origin of $T_x M$) the graph of a function ψ': $E_x\times E_x^s\to E_x^u$ we set

$$\varphi'(u)=u_0+u_1+\big(u_2+\psi'(u_0,u_1)\big).$$

If $\varphi'^{-1}\exp_x^{-1} W_x^{cu}$ is the graph of ψ'': $E_x\times E_x^u\to E_x^s$ we set

$$\varphi''(u)=u_0+\big(u_1+\psi''(u_0,u_2)\big)+u_2.$$

Then $\varphi_x=\exp_x\circ\varphi'\circ\varphi''$ has the desired properties.

If $u\in T_x M(\varepsilon)$, ε sufficiently small, Taylor's formula yields

$$\|F_2(u)\| = \|F_2(u_0+u_1+u_2)-F_2(u_0+u_1)\| \geqq \gamma^{-1}\|u_2\| \tag{A.2}$$

for some $\gamma\in(0,1)$ independent of x. Similarly if $0<\omega\leqq 1$, ε can be chosen so small that

$$\|F_0(u)+F_1(u)-F_0(v)-F_1(v)\| \leqq \omega\,\|F_2(u)-F_2(v)\| \tag{A.3}$$

whenever

$$\|u_0+u_1-v_0-v_1\|\leqq\omega\,\|u_2-v_2\| \quad\text{and}\quad u,v\in T_x M(\varepsilon).$$

We define

$$D_x(\varepsilon,n)=\{u\in T_x M\colon \|F^k u\|_{f^k x}\leqq\varepsilon \text{ for } k=0,1,\dots,n\}. \tag{A.4}$$

Let $u\in D_x(\varepsilon,n)$, then (A.2) yields

$$\|(F^k)_2(u)\|\leqq\gamma^{n-k}\|(F^n)_2(u)\|\leqq\varepsilon\gamma^{n-k}$$

for $k=0,1,\dots,n$. Let $v=u_0+u_1$. We may assume that $v\in D_x\big((K+1)\varepsilon,n\big)$ and, for ε suitably small, apply (A.3) with $\omega=1$.

We obtain

$$\|(F^k)_0(u)+(F^k)_1(u)-(F^k)_0(v)-(F^k)_1(v)\|\leqq\|(F^k)_2(u)\|\leqq\varepsilon\gamma^{n-k}$$

and therefore

$$\|F^k(u)-F^k(v)\|\leqq 2\varepsilon\gamma^{n-k} \tag{A.5}$$

for $k=0,1,\dots,n$. (In particular $v\in D_x(3\varepsilon,n)$.)

[6] $W_x^{cs}=\bigcup\{W_y^s\colon y\in\text{orbit of } x\}$, $W_x^{cu}=\bigcup\{W_y^u\colon y\in\text{orbit of } x\}$.

A.2. **Lemma.** *Let $r=1$ and $\psi\colon M\to\mathbb{R}$ be C^1. Given $\theta>0$ there is $\delta>0$ such that: if $x\in\Lambda$, $y\in M$, $n>0$ and $d(f^k y, f^k x)\leqq\delta$ for $k=0,1,\dots,n$, then*

$$\left|\int_0^n \psi(f^t y)\,dt-\int_0^n \psi(f^t x)\,dt\right|<\theta.$$

Because of the C^1 assumptions there exists $C>0$ such that

$$\left|\int_0^1 \psi(f^t p)\,dt-\int_0^1 \psi(f^t q)\,dt\right|\leqq C\,d(p,q).$$

Given ε, one can choose δ so small that $d(x,y)<\delta$ implies $u=\varphi_x^{-1}y\in T_x M(\varepsilon/2)$ for $x\in\Lambda$ and $y\in M$. Define $v=u_0+u_1$ and $z=\varphi_x v\in W_x^{cs}$. We may then assume that $z\in W^s_{f^s x}(\varepsilon)$ with $|s|<C_0\,\varepsilon$ (C_0 independent of ε). Then

$$d(f^k z, f^{k+s} x)<C_1\,\varepsilon\,\gamma'^k$$

for some $\gamma'\in(0,1)$. If

$$d(f^k x, f^k y)\leqq\delta \qquad \text{for } k=0,1,\dots,n,$$

(A.5) yields

$$d(f^k y, f^k z)<C_2\,\varepsilon\,\gamma^{n-k}$$

with $C_2>0$. Thus

$$\begin{aligned}
\left|\int_0^n \psi(f^t y)-\int_0^n \psi(f^t x)\right| &\leqq \sum_{k=0}^{n-1}\left|\int_0^1 \psi(f^t f^k y)-\int_0^1 \psi(f^t f^k z)\right|\\
&\quad+\sum_{k=0}^{n-1}\left|\int_0^1 \psi(f^t f^k z)-\int_0^1 \psi(f^t f^{k+s} x)\right|\\
&\quad+\left|\int_0^n \psi(f^{t+s} x)-\int_0^n \psi(f^t x)\right|\\
&\leqq C\left[\frac{C_2\,\varepsilon}{1-\gamma}+\frac{C_1\,\varepsilon}{1-\gamma'}\right]+2C_0\,\varepsilon\,\|\psi\|.
\end{aligned}$$

A.3. **Lemma.** *Let $\pi\colon G^q M\to M$ be the Grassmannian bundle of q-dimensional subspaces in TM and $G^q f^t\colon G^q M\to G^q M$ be the diffeomorphism induced by Tf^t. We assume that $r=2$ so that $G^q f$ is a C^1 flow on $G^q M$. If $q=\dim E_x^u$ and $\Lambda^*=\{E_x^u\colon x\in\Lambda\}$, then Λ^* is a hyperbolic set for the flow $G^q f$.*

For $x\in\Lambda$, define the manifolds

$$V_x^{*s}=\{E\in\pi^{-1}(W_x^s(\varepsilon))\colon d(E-E_x^u)<\varepsilon\};\qquad V_x^{*u}=TW_x^u(\varepsilon)$$

in $G^q M$. The manifold $\pi^{-1}(x)$ contains E_x^u, and $G^q f^t[\pi^{-1}(x)]=\pi^{-1}(f^t x)$. It is known (and easily seen) that $G^q f^1$ contracts a neighbourhood of E_x^u in $\pi^{-1}(x)$ (see B.1 of [19]). Since f^1 contracts $W_x^s(\varepsilon)$, it follows that, when ε is sufficiently small, Gf^1 contracts V_x^{*s}. Thus TGf^1 is a contraction of $E^{*s}=TV_x^{*s}$. Clearly $G^q f^{-1}$ contracts V^{*u} and therefore $TG^q f^{-1}$ is a contraction of $E^{*u}=TV_x^{*u}$. We have

$$\begin{aligned}
\dim E^{*s}+\dim E^{*u}&=\dim W_x^s(\varepsilon)+\dim\pi^{-1}(x)+\dim W_x^u(\varepsilon)\\
&=\dim M-1+\dim\pi^{-1}(x)=\dim G^q M-1
\end{aligned}$$

which concludes the proof.

A.4. **Lemma.** *Let $r=2$. Given $\theta>0$, there is $\varepsilon>0$ so that the following holds.*

If $x\in\Lambda$ and $v\in D_x(\varepsilon,n)$ (defined by (A.4)), *let $E_v^*\in G^q M$ be the tangent at φ_x^v to the manifold $\varphi_x(v+E_x^u(\varepsilon))$. Then*

$$e^{-\theta}\leqq\frac{\operatorname{Jac}Tf^n|E_v^*}{\operatorname{Jac}Tf^n|E_x^u}\leqq e^\theta \tag{A.6}$$

and

$$e^{-\theta}\leqq\frac{\operatorname{Jac}D_v(F^n|v+E_x^u)}{\operatorname{Jac}D_0(F^n|E_x^u)}\leqq e^\theta. \tag{A.7}$$

(D_0, D_v are derivatives in charts.)

The Jacobian in (A.7) is computed with respect to the scalar product (A.1). Clearly the estimate (A.7) differs from (A.6) only by bounded factors and it suffices to prove (A.6). To do this we apply Lemma A.2 to the hyperbolic set Λ^* for the C^1 flow $G^q f$ (cf. Lemma A.3), with the replacement $x\to E_x^u$, $y\to E_v^*$. We have to check that (for sufficiently small ε),

$$d(Tf^k E_x^u, Tf^k E_v^*)\leqq\delta$$

for $k=0,1,\dots,n$. Using the charts φ_x, this results from (A.4) and (A.3). To conclude the proof it suffices to define

$$\psi(E)=\frac{d}{dt}\ln(\operatorname{Jac}Tf^t|E)\Big|_{t=0}$$

and remark that

$$\int_0^n\psi(Tf^t E_v)\,dt=\ln\operatorname{Jac}Tf^n|E_v.$$

A.5. **Proof of the Volume Lemma.** We shall show that for sufficiently small $\varepsilon>0$ there exist $b_\varepsilon, b'_\varepsilon>0$ so that

$$b_\varepsilon\leqq m_x(D_x(\varepsilon,n))\cdot\operatorname{Jac}D_0(F^n|E_x^u)\leqq b'_\varepsilon \tag{A.8}$$

for all $x\in\Lambda$, $n>0$. Here m_x denotes the measure on T_xM associated with the scalar product (A.1). This will prove the volume lemma because the use of the charts φ_x multiplies all distances, measures and Jacobians (see Lemma A.4) by positive factors bounded away from 0 and ∞.

If $v\in(E_x+E_x^s)(\varepsilon)$ define

$$N_v(\varepsilon,n)=\{u\in T_xM\colon u_0+u_1=v\ \text{and}\ (F^k)_2(u)\in E_{f^kx}^u(\varepsilon)\ \text{for}\ k=0,1,\dots,n\}.$$

If ε is sufficiently small, $\|F^k v\|\leqq(K+1)\varepsilon$ for all $k\geqq 0$.

Also, using (A.3) with $\omega\leqq 1$ and induction on n, we find that $F^n N_v(\varepsilon,n)$ is the graph of a C^1 function $g\colon E_{f^nx}^u(\varepsilon)\to(E_{f^nx}+E_{f^nx}^s)((K+2)\varepsilon)$ such that $\|Dg\|\leqq\omega$[7]. In particular we obtain the second inclusion of:

$$D_x(\varepsilon,n)\subset\bigcup_v N_v(\varepsilon,n)\subset D_x((K+3)\varepsilon,n).$$

[7] This is an easy adaptation of the first part of the proof of Theorem 2.3 of [14].

To prove (A.8) it suffices thus to show that

$$c_\varepsilon \leqq m_x\Big(\bigcup_v N_v(\varepsilon, n)\Big) \cdot \operatorname{Jac} D_0(F^n | E_x^u) \leqq c'_\varepsilon \tag{A.9}$$

for some $c_\varepsilon, c'_\varepsilon > 0$.

Since $\|Dg\| \leqq \omega$, the measure of $F^n N_v(\varepsilon, n)$ (induced on the manifold by the metric (A.1) on $T_{f^n x} M$) is contained between bounds $d_\varepsilon, d'_\varepsilon > 0$. In view of (A.7), the measure of $N_v(\varepsilon, n)$ multiplied by $\operatorname{Jac} D_0(F^n | E_x^u)$ is contained between $d_\varepsilon e^{-\theta}$ and $d'_\varepsilon e^{\theta}$.

Finally, using Fubini's theorem to integrate over $v \in (E_x + E_x^s)(\varepsilon)$ yields (A.9).

A.6. **Proof of the Second Volume Lemma.** Let $w \in D_x(\varepsilon, n)$, and define

$$D_{xw}(\delta, n) = \{u \in T_x M\colon \|F^k u - F^k w\|_{f^k x} \leqq \delta \text{ for } 0, 1, \ldots, n\}.$$

It will suffice to show that there is $b_\delta > 0$ so that

$$m_x\big(D_{xw}(\delta, n)\big) \cdot \operatorname{Jac} D_0(F^n | E_x^u) \geqq b_\delta \tag{A.10}$$

for all $x \in \Lambda$, $w \in D_x(\varepsilon, n)$, $n > 0$. Furthermore it suffices to prove (A.10) for $\delta < \varepsilon$.

Let

$$\Delta = \{u \in T_{f^n x} M\colon u_0 = (F^n w)_0,\ u_2 = (F^n w)_2,\ u_1 \in E_{f^n x}^s(3\varepsilon)\}.$$

For each $v \in \Delta$, let $\Gamma_v = \{f^t \varphi_{f^n x} v\colon |t| < \alpha\}$ so that $\varphi_{f^n x}^{-1} \Gamma_v$ is a "piece of trajectory" through v. Then, for small ε and suitable α,

$$W = \bigcup_{v \in \Delta} (\varphi_{f^n x}^{-1} \Gamma_v)$$

is a C^2 manifold in $T_{f^n x} M$, which is the graph of a function ψ defined on a subset of $E_{f^n x} + E_{f^n x}^s$ with values in $E_{f^n x}^u$ and such that $\|D\psi\| \leqq \omega$ with $\omega \in (0, 1)$.

We may assume that the domain of ψ contains $E_{f^n x}(2\varepsilon) + E_{f^n x}^s(2\varepsilon)$ and let W' be the graph of the restriction of ψ to $E_{f^n x}(2\varepsilon) + E_{f^n x}^s(2\varepsilon)$. We write

$$W' = \bigcup_{\tau\colon |\tau| < 2\varepsilon} W'_\tau$$

where $W'_\tau \subset \{u\colon u_0 = \tau\}$. Let

$$W''_\tau = \{u \in T_x M\colon F^n u \in W'_\tau \text{ and } (F^k)_1(u) \in E_{f^k x}^s(2\varepsilon) \text{ for } k = 0, 1, \ldots, n\}$$

$$W'' = \bigcup_{\tau\colon |\tau| < 2\varepsilon} W''_\tau.$$

Applying (A.3) and the argument in A.5 to $F^n W''_\tau$, F^{-1} instead of $N_v(\varepsilon, n)$, F we find that W''_τ is the graph of a function $g_\tau\colon E_x^s(2\varepsilon) \to (E_x + E_x^u)(2(K+1)\varepsilon)$ such that $\|Dg_\tau\| \leqq \omega$. On the other hand W'' is a union of "pieces of trajectories" which are graphs of maps $E_x \to E_x^s + E_x^u$ with derivative $\leqq \omega$ (for sufficiently small ε). The C^2 manifold W'' is thus the graph of a function ψ'' defined on a subset of $E_x + E_x^s$ with values in E^u and such that $\|D\psi''\| \leqq 1$ (for small ε, hence small ω). The manifold W'' imitates a piece of center-stable manifold through w.

Notice that F^k contracts or expands a "piece of trajectory" in W'' by a factor bounded away from 0 and ∞ (contained between $(K+1)^{-1}$ and $(K+1)$, say). From this and the above properties of W'' it follows that the domain of ψ'' contains a ball B of radius β around $w_0 + w_1$, in $E_x + E_x^s$ for sufficiently small β

and (taking $\beta < \delta/4(K+1)$) we have

$$d(F^k \psi'' v, F^k w) \leqq \frac{\delta}{2}$$

for $k=0, 1, \ldots, n$ whenever $v \in B$.

For each $v \in B$, define

$$N_v^*\left(\frac{\delta}{2}, n\right) = \left\{u \in T_x M : u_0 + u_1 = v \text{ and } \|(F^k)_2(u) - (F^k)_2(\psi'' v)\| \leqq \frac{\delta}{2} \text{ for } k=0, 1, \ldots, n\right\}.$$

We have

$$\bigcup_{v \in B} N_v^*\left(\frac{\delta}{2}, n\right) \subset D_{xw}(\delta, n)$$

and proceeding as in A.5 we find that

$$m_x\left(\bigcup_v N_v^*\left(\frac{\delta}{2}, m\right)\right) \cdot \operatorname{Jac} D_0(F^n | E_x^u) \geqq c_\delta$$

for some $c_\delta > 0$. This proves (A.10) and therefore the second volume lemma.

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