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Probability Estimates for Continuous Spin Systems

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Abstract. Probability estimates for classical systems of particles with superstable interactions [1] are extended to continuous spin systems.

1. Notation and Assumptions

On a lattice $\mathbb{Z}^\nu$ we consider continuous d-dimensional spins. A *spin configuration* in $\Lambda \subset \mathbb{Z}^\nu$ is thus a function $s_\Lambda : \Lambda \mapsto \mathbb{R}^d$; its value at $x \in \Lambda$ will be denoted by s_x.

If $x = (x^1, \ldots, x^\nu) \in \mathbb{Z}^\nu$, we write $|x| = \max_i |x^i|$. If $s = (s^1, \ldots, s^d) \in \mathbb{R}^d$, we write $|s| = \left(\sum_i (s^i)^2\right)^{1/2} = \sqrt{s^2}$.

A measure $\mu \geqq 0$ on $\mathbb{R}^d$ is given such that

$$\int \mu(ds) e^{-\alpha s^2} < +\infty$$

if $\alpha > 0$, and μ is not identically 0.

We shall call *interaction* a real function U on all configurations in all finite $\Lambda \subset \mathbb{Z}^\nu$ satisfying the following conditions.

(a) U is $\otimes^\Lambda \mu$-measurable on $(\mathbb{R}^d)^\Lambda$ and invariant under translations of $\mathbb{Z}^\nu$.

(b) *Superstability.* There exist $A > 0$, $C \in \mathbb{R}$ such that if $s_\Lambda \in (\mathbb{R}^d)^\Lambda$ is a configuration on any finite Λ, then

$$U(S_\Lambda) \geqq \sum_{x \in \Lambda} [A s_x^2 - C] .$$

(c) *Regularity.* There exists a decreasing positive function Ψ on the natural integers such that

$$\sum_{x \in \mathbb{Z}^\nu} \Psi(|x|) < +\infty .$$

Furthermore if Λ_1, Λ_2 are disjoint finite subsets of $\mathbb{Z}^\nu$ and s_{Λ_1}, s_{Λ_2} the restrictions to Λ_1, Λ_2 of a configuration $s_{\Lambda_1 \cup \Lambda_2}$ on $\Lambda_1 \cup \Lambda_2$, then

$$|W(s_{\Lambda_1 \cup \Lambda_2})| \leqq \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} \Psi(|y - x|) \tfrac{1}{2} (s_x^2 + s_y^2)$$

where we have written

$$U(s_{\Lambda_1\cup\Lambda_2})=U(s_{\Lambda_1})+U(s_{\Lambda_2})+W(s_{\Lambda_1},s_{\Lambda_2}).$$

Condition (c) implies the following

(d) There are $r>0$ and $\lambda>0$ such that for all finite $\Lambda\subset\mathbb{Z}^\nu$

$$\int_{\Sigma^\Lambda}\left(\prod_{x\in\Lambda}\mu(ds_x)\right)\exp[-U(s_\Lambda)]>\lambda^{-\operatorname{card}\Lambda}$$

where $\Sigma=\{s\in\mathbb{R}^d:|s|\leqq r\}$. This is because, using (c), we have

$$U(s_\Lambda)\leqq\sum_{x\in\Lambda}U(s_x)+\left(\sum_{x\in\Lambda}s_x^2\right)\sum_y\Psi(|y|)$$

and, for sufficiently large r, $\int_{|s|\leqq r}\mu(ds)>0$.

Notice also that if there are $\varepsilon>0$, $B\in\mathbb{R}$ such that

$$U(s_\Lambda)\geqq\sum_{x\in\Lambda}[(A+\varepsilon)s_x^2-B|s_x|]$$

then (b) holds with $C=B/4\varepsilon$.

2. Probability Estimates

Let $\Delta\subset\Lambda\subset\mathbb{Z}^\nu$, Λ finite. We denote by s_Δ the restriction to Δ of a configuration s_Λ on Λ, and write

$$\varrho_\Delta^{(\Lambda)}(s_\Delta)=Z_\Lambda^{-1}\int\left(\prod_{x\in\Lambda/\Delta}\mu(ds_x)\right)\exp[-U(s_\Lambda)] \tag{1}$$

where

$$Z_\Lambda=\int\left(\prod_{x\in\Lambda}\mu(ds_x)\right)\exp[-U(s_\Lambda)].$$

The probability estimates of this section are bounds on $\varrho_\Delta^{(\Lambda)}$, given in Theorem 2.2. below. To obtain these bound we imitate the arguments of [1]. That paper in effect treats a special case of the problem considered here, where $d=1$ and μ is carried by the natural integers. In [1], the probability estimates are obtained on the basis of technical results, which carry over immediately to the present case if the variable n is allowed to vary in $\mathbb{R}^d$ rather than take natural integer values. As an example we transcribe below (Proposition 2.1) the main technical estimate of [1].

Given $\alpha>0$, we can choose an integer $P_0>0$ and for each $j\geqq P_0$ an integer $l_j>0$ such that

$$|l_{j+1}/l_j-(1+2\alpha)|<\alpha.$$

We use the notation

$$[j]=\{x\in\mathbb{Z}^\nu:|x|\leqq l_j\},\qquad V_j=(2l_j+1)^\nu$$

2.1. Proposition. *Let $\varepsilon>0$ and $C\geqq 0$ be given, and let Ψ be a decreasing positive function on the natural integers such that*

$$\sum_{x\in\mathbb{Z}^\nu} \Psi(|x|)<+\infty .$$

If α is sufficiently small one can choose an increasing sequence (ψ_j) such that $\psi_j\geqq 1$, $\psi_j\to\infty$, and fix $P>P_0$ so that the following is true.

Let $n(\cdot)$ be a function from $\mathbb{Z}$ to the reals $\geqq 0$. Suppose that there exists q such that $q\geqq P$ and q is the largest integer for which

$$\sum_{x\in[q]} n(x)^2\geqq \psi_q V_q .$$

Then

$$\sum_{x\in[q+1]} C+\sum_{x\in[q+1]}\sum_{y\notin[q+1]} \Psi(|y-x|)\tfrac{1}{2}(n(x)^2+n(y)^2)\leqq\varepsilon\sum_{x\in[q+1]} n(x)^2 .$$

This differs from Proposition 2.1 of [1] mostly by the fact that $n(\cdot)$ has real rather than integer values. Lemmas 2.2, 2.3, 2.4, and Proposition 2.5 of [1] similarly carry over to the present case.

To adapt Proposition 2.6 of [1] to $\varrho_\Lambda^{(A)}$ some care is needed because we do not have in general $\varrho(\{0\})>0$. Since however we have (d) and the regularity condition (c) (rather than only lower regularity in [1]), we can write $\varrho_\Lambda^{(A)}(s_\Lambda)=\varrho'+\varrho''$ where (3.30) and (3.31) of [1] are replaced (see Appendix) by

$$\varrho'\leqq C'\exp\left[\sum_{y\in\mathbb{Z}^\nu}\Psi(|y|)-A)s_x^2\right]\cdot\varrho_{\Lambda\backslash\{x\}}^{(A)}(s_{\Lambda\backslash\{x\}}) \tag{2}$$

$$\varrho''\leqq\sum_{q\geqq P} e^{-C''\psi_{q+1}V_{q+1}+D''V_{q+1}}\cdot\exp\sum_{x\in[q+1]\cap\Lambda}[-(A-3\varepsilon)s_x^2]\cdot\varrho_{\Lambda\backslash[q+1]}^{(A)}(s_{\Lambda\backslash[q+1]}) \tag{3}$$

with some constants C', C'', D''. Therefore, by induction on card Λ,

$$\varrho_\Lambda^{(A)}(s_\Lambda)\leqq\exp\sum_{x\in\Lambda}(Es_x^2+F) \tag{4}$$

with some constants E, F.

We show now, following Proposition 2.7 of [1], that for any $\varepsilon>0$ one can choose δ independent of (A), Λ, s_Λ such that

$$\varrho_\Lambda^{(A)}(s_\Lambda)\leqq\exp\sum_{x\in\Lambda}[-(A-3\varepsilon)s_x^2+\delta] . \tag{5}$$

We may assume $A>3\varepsilon$. Let $\delta=(E+A-3\varepsilon)\psi_P V_P+F$. If $|s_x|\leqq(\psi_P V_P)^{1/2}$ for each $x\in\Lambda$, then (5) follows from (4). If $|s_x|>(\psi_P V_P)^{1/2}$ for some x, we put x at the origin by a translation. Then $\varrho'=0$, and $\varrho_\Lambda^{(A)}(s_\Lambda)=\varrho''$ so that, using (3) and induction,

$$\begin{aligned}\varrho_\Lambda^{(A)}(s_\Lambda)&\leqq\exp\sum_{x\in\Lambda}[-(A-3\varepsilon)s_x^2]\\&\quad\sum_{q\geqq P} e^{-C''\psi_{q+1}V_{q+1}+DV_{q+1}}\,e^{\delta\,\mathrm{card}(\Lambda\backslash[q+1])}\\&\leqq\exp\sum_{x\in\Lambda}[-(A-3\varepsilon)s_x^2]\cdot e^{\delta\,\mathrm{card}(\Lambda\backslash[q+1])+F}\end{aligned}$$

and (4) follows. We have proved the following

2.2. Theorem. *Let $\varrho_\Delta^{(\Lambda)}(s_\Delta)$ be defined by* (1) *for an interaction U satisfying* (a), (b), (c). *Given $A^* < A$, there exists δ independent of Λ, Δ, s_Δ such that*

$$\varrho_\Delta^{(\Lambda)}(s_\Delta) \leqq \exp \sum_{x \in \Delta} [-A^* s_x^2 + \delta] .$$

2.3. Corollary. *Let $\gamma \geqq 2$, and suppose that the superstability condition is strengthened to*

$$U(s_\Lambda) \geqq \sum_{x \in \Lambda} [A|s_x|^\gamma - C] .$$

Then the conclusion of Theorem 2.2 can be strengthened to

$$\varrho_\Delta^{(\Lambda)}(s_\Delta) \leqq \exp \sum_{x \in \Delta} [-A^*|s_x|^\gamma + \delta]$$

Define $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ by

$$Fs = \begin{cases} s & \text{if} \quad |s| \leqq 1 \\ (|s|^{2/\gamma - 1})s & \text{if} \quad |s| \geqq 1 \end{cases}$$

and write $F(s_x)_{x \in \Lambda} = (Fs_x)_{x \in \Lambda}$.

Let $\tilde{\mu}$ be the image by F of the measure μ, and let $\tilde{U}(s_\Lambda) = U(Fs_\Lambda)$. Then $\tilde{U}$ is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu}$. In particular

$$\begin{aligned} \tilde{U}(s_\Lambda) = U(Fs_\Lambda) &\geqq \sum_{x \in \Lambda} [A|Fs_x|^\gamma - C] \\ &\geqq \sum_{x \in \Lambda} [As_x^2 - A - C] \end{aligned}$$

and

$$\begin{aligned} |\tilde{W}(s_{\Lambda_1 \cup \Lambda_2})| &\leqq \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} \Psi(|y - x|) \tfrac{1}{2} (|Fs_x|^2 + |Fs_y|^2) \\ &\leqq \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} \Psi(|y - x|) \tfrac{1}{2} (s_x^2 + s_y^2) . \end{aligned}$$

Therefore

$$\begin{aligned} \varrho_\Delta^{(\Lambda)}(s_\Delta) = \tilde{\varrho}_\Delta^{(\Lambda)}(F^{-1}s_\Delta) &\leqq \exp \sum_{x \in \Delta} [-A^*|F^{-1}s_x|^2 + \delta] \\ &\leqq \exp \sum_{x \in \Delta} [-A^*|s_x|^\gamma + \delta] . \end{aligned}$$

2.4. Corollary. *Suppose that*

$$U(s_\Lambda) = \tilde{U}(s_\Lambda) + \sum_{x \in \Lambda} V(s_x)$$

and that $\tilde{U}$ is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu} = e^{-V}\mu$. Then Theorem 2.2 can be replaced by

$$\varrho_\Delta^{(\Lambda)}(s_\Delta) \leqq \exp \sum_{x \in \Delta} [-A^*|s_x|^\gamma + \delta - V(s_x)| .$$

This is because

$$\varrho_\Delta^{(\Lambda)}(s_\Delta) = \exp \left[-\sum_{x \in \Delta} V(s_x) \right] \tilde{\varrho}_\Delta^{(\Lambda)}(s_\Delta)$$

where $\tilde{\varrho}$ is defined by (1) with μ, U replaced by $\tilde{\mu}$, $\tilde{U}$.

Appendix

We sketch here the proofs of (2) and (3), using notation which is either that of [1], or has obvious meaning.

Proof of (2).

$$\begin{aligned}
\varrho' &= Z_\Lambda^{-1} \int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta}) \exp[-U(s_x) - U(s_{\Lambda\setminus\{x\}}) - W(s_x, s_{\Lambda\setminus\{x\}})] \\
&\leqq e^{-U(s_x)} Z_\Lambda^{-1} \int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta}) \exp[-U(s_{\Lambda\setminus\{x\}}) - W(s'_x, s_{\Lambda\setminus\{x\}})] \\
&\quad \cdot \exp\left[\left(\tfrac{1}{2}\sum_y \Psi(|y|)\right)(s_x^2 + s_x'^2) + 2D'\right] \\
&\leqq \lambda e^{2D'} \exp\left[-As_x^2 + C + \left(\tfrac{1}{2}\sum_y \Psi(|y|)\right) s_x^2\right] \\
&\quad \cdot \sup_{s'_x \in \Sigma} \exp\left[\left(\tfrac{1}{2}\sum_y \Psi(|y|)\right) s_x'^2\right] \\
&\quad \cdot Z_\Lambda^{-1} \int_\Sigma \mu(ds'_x) \int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta}) \exp[-U(s_\Lambda^*)] \\
&\leqq C' \exp\left[\left(\sum_y \Psi(|y|) - A\right) s_x^2\right] \cdot \varrho_{\Delta\setminus\{x\}}^{(\Lambda)}(s_{\Lambda\setminus\{x\}}) .
\end{aligned}$$

Proof of (3).

$$\begin{aligned}
\varrho'' &= \sum_{q \geqq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda|\Delta}) \exp(-U(s_{[q+1]\cap\Lambda})) \\
&\quad \cdot \exp(-W(s_{[q+1]\cap\Lambda}, s_{\Lambda\cap[q+1]})) \exp(-U(s_{\Lambda\setminus[q+1]})) \\
&\leqq \sum_{q \geqq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta}) \exp \sum_{x \in [q+1]\cap\Lambda} [-As_x^2 + C] \\
&\quad \cdot \exp \sum_{x \in [q+1]\cap\Lambda} \sum_{y \in \Lambda[q+1]} \Psi(|y-x|) \tfrac{1}{2}(s_x^2 + s_y^2) \\
&\quad \cdot \exp \sum_{x \in [q+1]\cap\Lambda} \sum_{y \in \Lambda\setminus[q+1]} \Psi(|y-x|) \tfrac{1}{2}(s_x'^2 + s_y^2) \\
&\quad \cdot \exp[-W(s'_{[q+1]\cap\Lambda}, s_{\Lambda\setminus[q+1]}) - U(s_{\Lambda\setminus[q+1]})] \\
&\leqq \sum_{q \geqq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta}) \\
&\quad \cdot \exp\left[-(A - 3\varepsilon) \sum_{x \in [q+1]\cap\Lambda} s_x^2 - C''\Psi_{q+1} V_{q+1}\right] \\
&\quad \cdot \exp\left(\tfrac{1}{2}\sum_y \Psi(|y|) \sum_{x \in [q+1]\cap\Lambda} s_x'^2\right. \\
&\quad \cdot \exp[-W(s'_{[q+1]\cap\Lambda}, s_{\Lambda\setminus[q+1]}) - U(s_{\Lambda\setminus[q+1]})] \\
&\leqq \sum_{q \geqq P} \exp \sum_{x \in [q+1]\cap\Delta} [-(A - 3\varepsilon)s_x^2] \\
&\quad \cdot e^{-C''} \psi_{q+1} V_{q+1} [\textstyle\int \mu(ds) e^{-(A-3\varepsilon)s^2}]^{|[q+1]\cap\Lambda\setminus\Delta|} \\
&\quad \cdot \left(\sup_{s' \in \Sigma} \exp\left[\left(\tfrac{1}{2}\sum_y \Psi(|y|)\right) s'^2\right]\right)^{|[q+1]\cap\Lambda|} \lambda^{|[q+1]\cap|\Delta|} \\
&\quad \cdot Z_\Lambda^{-1} \int_{\Sigma^{[q+1]\cap\Lambda}} \mu^{[q+1]\cap\Lambda}(ds'_{[q+1]\cap\Lambda}) \int \mu^{\Lambda\setminus[q+1]}(ds_{\Lambda\setminus[q+1]\setminus\Delta}) \, e^{-U(s_\Lambda^*)} \\
&\leqq \sum_{q \geqq P} \exp \sum_{x \in [q+1]\cap\Lambda} [-(A - 3\varepsilon)s_x^2] \\
&\quad \cdot e^{-C''\psi_{q+1}V_{q+1} + D''V_{q+1}} \varrho_{\Delta\setminus[q+1]}^{(\Lambda)}(s_{\Lambda\setminus[q+1]}) .
\end{aligned}$$

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Reference

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