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INTEGRAL REPRESENTATION OF MEASURES ASSOCIATED WITH A FOLIATION

by DAVID RUELLE

To Jean Leray

Let M be a compact differentiable manifold, $\mathscr{F}$ a foliation of codimension k, and $\mathscr{S}$ the set of open submanifolds of dimension k transversal to $\mathscr{F}$. A *transverse measure* ρ for $\mathscr{F}$ is a collection of real measures ρ_Σ on the $\Sigma \in \mathscr{S}$, such that these measures correspond to each other by the *canonical isomorphisms* defined by $\mathscr{F}$. For a discussion of these notions, and applications, see Plante [8], Ruelle and Sullivan [10], Schwartzmann [11], Edwards, Millett and Sullivan [6], Sullivan [14], Garnett [7]. We note that we can, as in [10], assume that $\mathscr{F}$ is only a partial foliation of M, and that the orientation assumptions of [10] are unnecessary here.

We generalize the notion of transverse measure by introducing measures associated with a cocycle. We call *cocycle* a family (f_τ) indexed by the canonical isomorphisms, such that:

(a) If τ maps $\Sigma \in \mathscr{S}$ onto $\Sigma' \in \mathscr{S}$, then f_τ is a continuous function on Σ' with strictly positive real values.

(b) If τ' maps Σ' onto Σ'', then:

$$f_{\tau' \circ \tau} = f_{\tau'} \cdot (f_\tau \circ \tau'^{-1}).$$

We say that a collection $\rho = (\rho_\Sigma)$ of real measures on the $\Sigma \in \mathscr{S}$ is a *measure associated with the cocycle* (f_τ), or is a (f_τ)*-measure,* if the image of the measure ρ_Σ by $\tau : \Sigma \to \Sigma'$ is $f_\tau \cdot \rho_{\Sigma'}$. Otherwise stated:

$$f_\tau = \frac{d(\tau\rho_\Sigma)}{d\rho_{\Sigma'}} \quad \text{a.e.}$$

for each local isomorphism $\tau : \Sigma \to \Sigma'$. The transverse measures are those associated with the trivial cocycle (1_τ). The notion of (f_τ)-measure occurs naturally in the work of Connes [5]; see also Bowen [1].

The (f_τ)-measures form a real vector space $\mathscr{J}$. We call *vague topology* the topology defined on $\mathscr{J}$ by the semi-norms:

$$\rho \mapsto |\rho_\Sigma(\varphi)|$$

where φ is a real continuous function with compact support in $\Sigma \in \mathscr{S}$. We write $\rho \geq 0$ if $\rho_\Sigma \geq 0$ for all $\Sigma \in \mathscr{S}$. With these definitions $\mathscr{J}$ is an ordered topological vector space.

Choose $\Sigma \in \mathscr{S}$ and a compact set $K \subset \Sigma$. There is a map α_K of $\mathscr{J}$ in the space $\mathscr{C}(K)^*$ of measures on K, such that $\alpha_K \rho$ is the restriction of ρ_Σ to K. The map α_K is linear and order-preserving.

Lemma 1. — *Let Σ, K be such that each leaf of $\mathscr{F}$ intersects the interior of K in Σ. Then α_K is an isomorphism of the ordered vector space $\mathscr{J}$ onto a subspace of $\mathscr{C}(K)^*$ closed for the vague topology.*

Remember that the vague topology is the w^*-topology of $\mathscr{C}(K)^*$ as dual of the space $\mathscr{C}(K)$ of real continuous functions on K. *Note that α_K need not be continuous for the vague topologies.*

To prove the lemma we remark that if K′ is compact in $\Sigma' \in \mathscr{S}$, there are finitely many open L_i in Σ' covering K′, and canonical isomorphisms $\tau_i : L_i$ into Σ such that the closure of $\tau_i L_i$ lies in the interior of K. Therefore, using a partition of unity, and the fact that ρ is associated with the cocycle (f_τ), we obtain an order preserving map π from the continuous functions on Σ' with support in K′ to the continuous functions on Σ with support in K, such that $\rho_{\Sigma'}(\varphi) = \rho_\Sigma(\pi\varphi) = (\alpha_K \rho)(\pi\varphi)$. Thus α_K is injective, and $\rho \geq 0$ if and only if $\alpha_K \rho \geq 0$. Furthermore, if $\alpha_K \rho$ tends to a limit vaguely, $\alpha_{K'} \rho$ also converges vaguely, hence ρ converges vaguely, and the limit is obviously associated with the cocycle (f_τ).

Lemma 2. — *Let $\mathscr{G}$ be a linear subspace of the space $\mathscr{C}(K)^*$ of real measures on the compact set K. If $\rho \in \mathscr{G}$ implies $|\rho| \in \mathscr{G}$, then the cone $\mathscr{G}_+$ of positive measures in $\mathscr{G}$ is simplicial. If ρ, ρ' belong to distinct extremal generatrices of the cone $\mathscr{G}_+$, they are disjoint measures.*

Remember that a cone C in a real vector space is simplicial if the order that it defines on itself is a lattice (any two points have a min and a max). The easy proof of Lemma 2 is left to the reader.

Theorem. — *The cone C of positive elements of $\mathscr{J}$ is simplicial. If ρ, ρ' belong to distinct extremal generatrices of C, then the measures ρ_Σ, ρ'_Σ are disjoint for all $\Sigma \in \mathscr{S}$.*

In view of Lemma 1, the theorem immediately follows from Lemma 2 applied to $\mathscr{G} = \alpha_K \mathscr{J}$.

The cone C is closed and has a basis B which is convex, compact, and metrizable. For instance, if Σ, K are as in Lemma 1, let φ have compact support in Σ, $\varphi \geq 0$, and $\varphi(x) = 1$ if $x \in K$; one can take:

$$B = \{\rho \in \mathscr{J} : \rho \geq 0 \text{ and } \rho(\varphi) = 1\}.$$

According to Choquet's theory [4], the theorem implies that each $\rho \geq 0$ has a unique integral representation in terms of extremal elements of B:

$$\rho = \int_B \sigma m_\rho(d\sigma)$$

where m_ρ is carried by the set of extremal points of B. The arbitrariness in the choice of B corresponds to the fact that there is no natural normalization of positive (f_τ)-measures, but all choices of B give equivalent decompositions. If ρ is an extremal point of some B (*i.e.* if $\rho \neq 0$ and ρ belongs to an extremal generatrix of C) we say that ρ is a *pure* (f_τ)*-measure* (respectively a *pure transverse measure* in the case of the trivial cocycle). The theorem gives thus a unique decomposition of (f_τ)-measures into pure (f_τ)-measures, and states that two pure (f_τ)-measures are either proportional or disjoint [1].

Given a positive (f_τ)-measure ρ, we let $\mathscr{A}_\rho$ be the algebra of classes of bounded real functions on M which are constant on leaves of $\mathscr{F}$, and such that their restriction to each $\Sigma \in \mathscr{S}$ is ρ_Σ-measurable. Two functions are in the same class if their restrictions to each $\Sigma \in \mathscr{S}$ are equal ρ_Σ-almost everywhere.

Proposition. — *A positive* (f_τ)*-measure* ρ *is pure if and only if* $\mathscr{A}_\rho$ *is trivial* (consisting of the constant functions).

If ρ is not pure, let $\rho = \rho^1 + \rho^2$ with non proportional (f_τ)-measures $\rho^1, \rho^2 \geq 0$. Choose Σ, K as in Lemma 1, and let $\sigma^i = \rho^i_\Sigma - \inf(\rho^1_\Sigma, \rho^2_\Sigma)$. There are ρ_Σ measurable functions $\psi_1, \psi_2 \geq 0$ such that $\sigma^i = \psi^i \rho_\Sigma$. We have $\psi^1 + \psi^2 \neq 0$ (because

$$\sigma^1 + \sigma^2 = \sup(\rho^1_\Sigma, \rho^2_\Sigma) \neq 0)$$

and $\psi_1 . \psi_2 = 0$ a.e. (because σ^1, σ^2 are disjoint). Choosing some Riemann metric d on the leaves of $\mathscr{F}$, let:

$$\Psi^i(x) = \lim_{n\to\infty} \min\{\psi_i(y), y \in \mathrm{K}, d(x,y) \leq n\}.$$

Clearly Ψ^1, Ψ^2 belong to $\mathscr{A}_\rho$ and are not proportional, so that $\mathscr{A}_\rho$ is non trivial. Conversely, if $\mathscr{A}_\rho$ is non trivial, it is immediate that ρ is not pure.

Interpretation of the decomposition. — Let h be a diffeomorphism of a compact manifold B, and $\mathscr{F}$ be the foliation by the orbits of the suspension of h. We identify B with a submanifold of codimension 1 of M, transverse to $\mathscr{F}$. The transverse measures of $\mathscr{F}$ correspond then to the h-invariant measures on B. The pure transverse measures correspond to the h-ergodic measures, and the decomposition into pure transverse measures corresponds to the ergodic decomposition. The integral representation of positive (f_τ)-measures appears thus as an extension of ergodic theory. A different, deeper, relation is with the theory of Gibbs states in statistical mechanics, as discussed in the following example.

Example. — Let $\mathrm{A} \in \mathrm{SL}_n(\mathbf{Z})$ be hyperbolic, *i.e.* the spectrum of A is disjoint from $\{z : |z| = 1\}$. Let V^s (respectively V^u) be the subspace of $\mathbf{R}^n$ associated with the eigenvalues less than 1 (respectively larger than 1) in absolute value. We call $\hat{\mathrm{A}}$ the map

[1] For cases where there is only one pure (f_τ)-measure, see Bowen and Marcus [2], and also the Example below.

induced by A on $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$, and W^s, W^u the images of V^s, V^u in $\mathbf{T}^n$. It is readily seen that $G = W^s \cap W^u$ is a n-generator subgroup of $\mathbf{T}^n$, G is dense in $\mathbf{T}^n$ because W^s, W^u are dense.

Choose $a_1, \ldots, a_n \in \mathbf{R}^n$ such that their images in $\mathbf{T}^n$ are generators of G. Write $a_i = (a_{i1}, \ldots, a_{in})$, take $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, and define:

$$\Gamma_x = \{(x_1 + \sum_i t_i a_{i1}, \ldots, x_n + \sum_i t_i a_{in}, t_1, \ldots, t_n) \in \mathbf{R}^{2n} : t_1, \ldots, t_n \in \mathbf{R}\}.$$

The images F_x of the Γ_x in $M = \mathbf{T}^n \times \mathbf{T}^n$ constitute a codimension n foliation of M, with holonomy group G with respect to the Section $\mathbf{T}^n = \mathbf{T}^n \times \{0\}$. We shall define functions $f_\tau : \mathbf{T}^n \to \mathbf{R}$ when $\tau \in G$, i.e. for the canonical isomorphisms of the section $\mathbf{T}^n$. It is easy to extend this definition to that of a cocycle for $\mathscr{F}$.

Let φ be a real Hölder continuous function on T^n. We let:

$$f_\tau(x) = \exp \sum_{k=-\infty}^{\infty} (\varphi(\hat{A}^k \tau^{-1} x) - \varphi(\hat{A}^k x)).$$

There is one and only one measure ρ *associated with this cocycle.* In fact:

$$\rho_{\mathbf{T}^n} = \lim_{m \to +\infty} \frac{1}{N_m} (\exp \sum_{k=-m}^{m} \varphi(\hat{A}^k x))\, dx$$

where dx is Haar measure on $\mathbf{T}^n$, and N_m a normalizing factor. These statements have their origin in a relation between statistical mechanics and differentiable dynamical systems introduced by Sinai: $\rho_{\mathbf{T}^n}$ is a *Gibbs state* for the function φ (see Sinai [13], Capocaccia [3], Ruelle [9], Chapter 7). We notice that if $\varphi = 0$ then $\rho_{\mathbf{T}^n} = dx$, and uniqueness follows from the fact that G is a dense subgroup of $\mathbf{T}^n$. For the general case the reader is referred to the papers quoted above.

In view of the frequent non-uniqueness of Gibbs states we conjecture that, for the foliation discussed here, there exist cocycles with several non proportional associated measures.

Invariance under a diffeomorphism. — Let g be a diffeomorphism of M preserving $\mathscr{F}$ (i.e. permuting the leaves). Suppose that (f_τ) is a cocycle *compatible* with g, *i.e.* such that:

$$f_{g \circ \tau \circ g^{-1}} = f_\tau \circ g^{-1}.$$

This condition is for instance always satisfied by the trivial cocycle (1_τ).

If $\rho = (\rho_\Sigma)$ is a (f_τ)-measure, then $g\rho = (g\rho_{g^{-1}\Sigma})$ is again a (f_τ)-measure. This is because:

$$\begin{aligned}\tau(g\rho_{g^{-1}\Sigma}) &= g(g^{-1} \circ \tau \circ g)\, \rho_{g^{-1}\Sigma} = g(f_{g^{-1}\tau g}\, \rho_{g^{-1}\Sigma}) \\ &= g((f_\tau \circ g) \cdot \rho_{g^{-1}\Sigma}) = f_\tau \cdot (g\rho_{g^{-1}\Sigma}).\end{aligned}$$

Thus $g\mathscr{J} = \mathscr{J}$, and in fact $gC = C$, where C is the cone of positive measures in $\mathscr{J}$. Suppose $\mathscr{J} \neq 0$, and let B be a compact basis of C. We have $B = C \cap \{\rho : \lambda(\rho) = 1\}$

for some continuous linear functional ρ on $\mathscr{J}$. The map $\rho \mapsto g\rho/\lambda(g\rho)$ has a fixed point $\rho_0 \in B$. Therefore $g\rho_0 = \lambda_0 \rho_0$, where $\lambda_0 = \lambda(g\rho_0) > 0$, and λ_0 is in general different from 1.

Consider now the case of the trivial cocycle, *i.e.* of transverse measures. Under suitable conditions, discussed in [10], [14], λ_0 is an eigenvalue of the action of g on cohomology, and the corresponding class is associated with a geometric current determined by ρ_0. If the class is nonzero, λ_0 is thus an algebraic number (in fact, a unit in the ring of algebraic integers).

Question: under what conditions do the numbers λ_0 associated with the transverse measures of a foliation form a finite set of algebraic numbers? A. Connes has pointed out to me that this is not always the case.

Diffeomorphisms which expand leaves. — Let the foliation $\mathscr{F}$ contain a leaf with polynomial growth (*i.e.* the Riemann volume of a ball $B(x, r) \subset L$ increases polynomially with its radius r) then Plante [8] has shown that $\mathscr{F}$ has a transverse measure $\rho \neq 0$ with support in the closure of L.

If the diffeomorphism g preserves $\mathscr{F}$ and expands the leaves (*i.e.* multiplies sufficiently small distances on leaves, with respect to some Riemann metric, by a factor $> C > 1$), then the leaves have polynomial growth. This was proved by Sullivan and Williams [15]; see also Shub [12]. In particular $\mathscr{F}$ has a transverse measure $\rho \neq 0$, and by the preceding Section we may assume that $g\rho_0 = \lambda_0 \rho_0$. We recover thus a result stated in another context by Sullivan (see [14], III, 13): *if the diffeomorphisms g preserves $\mathscr{F}$ and expands the leaves, there is a transverse measure $\rho_0 \neq 0$ such that $g\rho_0 = \lambda_0 \rho_0$.*

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