STABLE MANIFOLDS FOR MAPS*)

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Here we present a stable manifold theorem for non-invertible differentiable maps of finite dimensional manifolds. There is a long history of stable manifold theorems for hyperbolic fixed points and sets, see for instance [1]. More recently Pesin [3] has proven theorems of a general nature which rely on measure theoretic techniques. Pesin's results have been extended in [5]. The results described in the present paper were arrived at by the two authors along different paths. The first author starting from a treatment of differentiable maps in Hilbert space [6] specializes to the finite dimensional case while the second starting from seminar notes by Fahti, Herman and Yoccoz applies graph transform as in [1].

We say that a map is of class $C^{r,\theta}$ if its r-th derivative is Holder continuous of exponent θ (Lipschitz if $\theta = 1$). Similarly for manifolds. In what follows class $\mathcal{C}$ will mean class $C^{r,\theta}$ with integer $r \geq 1$ and $\theta \in (0,1]$, or class C^r with $r \geq 2$, or class C^∞, or class C^ω (real analytic), or (complex) holomorphic. [Class $\mathcal{C}^{-1}$ will be respectively $C^{r-1,\theta}$, C^{r-1}, C^∞, C^ω, or holomorphic].

Throughout what follows, M will be a locally compact $\mathcal{C}$-manifold and $f: M \to M$ a $\mathcal{C}$-map such that fM is relatively compact in M. (In particular, if $fM = M$, then M is a compact manifold). We introduce the inverse limit.

$\tilde{M} = \{(x_n)_{n\geq 0} : x_n \in M$ and $fx_{n+1} = x_n\}$ and define $\tilde{\pi}(x_n) = x_0$, $\tilde{f}(x_n) = (y_n)$ where $y_n = x_{n+1}$ for $n \geq 0$. Notice that $\tilde{M}$ is compact, $\tilde{\pi}$ is continuous $\tilde{M} \to M$ with image $\bigcap_{n\geq 0} f^n M$, and $\tilde{f}$ is a homeomorphism of $\tilde{M}$. Furthermore $f\,\tilde{\pi} = \tilde{\pi}\,\tilde{f}^{-1}$.

We state in (1), (2), (3) below some (easy) consequences of the multiplicative ergodic theorems**). Our main results are the stable and unstable manifold

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**) See Oseledec [2], Raghunathan [4].

theorems in (4), (5). It is likely that these results extend to general local fields (the multiplicative ergodic theorem does, see [4]). We have however not checked the ultrametric case.

(1) There is a Borel set $\Gamma \subset M$ such that $f\Gamma \subset \Gamma$, and $\rho(\Gamma) = 1$ for every f-invariant probability measure ρ. If $x \in \Gamma$, there are an integer $s \in [0,m]$, reals $\mu^{(1)} > \ldots > \mu^{(s)}$, and spaces $T_x M = V_x^{(1)} \supset \ldots \supset V_x^{(s)} \supset V_x^{(s+1)} \supset \{0\}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Tf^n(x)u\| = \mu^{(r)} \quad \text{if} \quad u \in V_x^{(r)} \setminus V_x^{(r+1)}$$

for $r = 1,\ldots,s$, and

$$\lim_{n\to\infty} \frac{1}{n} \log \|Tf^n(x)u\| = -\infty \quad \text{if} \quad u \in V_x^{(s+1)} .$$

The functions $x \to s$, $\mu^{(1)},\ldots,\mu^{(s)}$, $V_x^{(1)},\ldots,V_x^{(s)}$ are Borel and $x \to s$, $\mu^{(1)},\ldots,\mu^{(s)}$, $\dim V_x^{(1)},\ldots,\dim V_x^{(s)}$ are f-invariant.

(2) Similarly there is a Borel set $\tilde\Gamma \subset \tilde M$ such that $\tilde f \tilde\Gamma \subset \tilde\Gamma$ and $\tilde\rho(\tilde\Gamma) = 1$ for every $\tilde f$-invariant probability measure $\tilde\rho$. If $\tilde x = (x_n) \in \tilde\Gamma$, there are $s \in [0,m]$, $\mu^{(1)} > \ldots > \mu^{(s)}$ and $\{0\} = \tilde V_{\tilde x}^{(0)} \subset \tilde V_{\tilde x}^{(1)} \subset \ldots \subset \tilde V_{\tilde x}^{(s)} \subset T_{x_0} M$

such that if $(u_n)_{n\geq 0}$ satisfies $u_n \in T_{x_n} M$ and $Tf(x_{n+1})u_{n+1} = u_n$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \|u_n\| < +\infty$$

then $u_0 \in \tilde V_{\tilde x}^{(s)}$. Conversely, for every $u_0 \in \tilde V_{\tilde x}^{(s)}$ there is such a sequence (u_n), it is unique and

$$\lim_{n\to\infty} \frac{1}{n} \log \|u_n\| = -\mu^{(r)} \quad \text{if} \quad u_0 \in \tilde V_{\tilde x}^{(r)} \setminus \tilde V_{\tilde x}^{(r-1)}$$

for $r = 1,\ldots,s$.

(3) The map $\tilde\pi$ sends the $\tilde f$-invariant probability measures on $\tilde M$ <u>onto</u> the f-invariant probability measures on M. Almost everywhere with respect to every $\tilde f$-invariant probability measure $\tilde\rho$, the quantities $s \circ \tilde\pi$, $\mu^{(r)} \circ \pi$, $\dim V_{\tilde\pi(\cdot)}^{(r+1)}$ occurring in (1) are equal to $s, \mu^{(r)}$, $m - \dim \tilde V_{(\cdot)}^{(r)}$ in (2). This justifies the confusion in notation for s and $\mu^{(r)}$.

(4) Local stable manifolds

Let Θ, λ, r be f-invariant Borel functions on Γ with $\Theta > 0$, $\lambda < 0$, r integer $\in [0,s]$, and

$$\mu^{(r+1)} < \lambda < \mu^{(r)}$$

(where $\mu^{(0)} = +\infty$, $\mu^{(s+1)} = -\infty$). Replacing possibly Γ by a smaller set retaining the properties of (1) one may construct Borel functions $\beta > \alpha > 0$ on Γ with the following properties.

(a) If $x \in \Gamma$ the set $W_x^\lambda = \{y \in M\colon d(x,y) \le \alpha(x)$ and $d(f^n x, f^n y) \le \beta(x) e^{n\lambda(x)}$ for all $n > 0\}$ is contained in Γ and is a C-submanifold of the ball $\{y \in M\colon d(x,y) \le \alpha(x)\}$. For each $y \in W_x^\lambda$, we have $T_y W_x^\lambda = V_y^{(r+1)}$. More generally, for every $t \in [0,s]$, the function $y \to V_y^{(t+1)}$ is of class C^{1} on W_x^λ.

(b) If $y,z \in W_x^\lambda$, then

$$d(f^n y, f^n z) \le \gamma(x)\ d(y,z)\ e^{n\lambda(x)}.$$

(c) If $x \in \Gamma$, then $\alpha(f^n x)$, $\beta(f^n x)$ decrease less fast with n than the exponential $e^{-n\Theta}$.

The manifolds W_x^λ do not in general depend continuously on x, but the construction implies measurability properties on which we shall not elaborate here.

(5) Local unstable manifolds

Let Θ, μ, r be $\tilde{f}$-invariant Borel functions on $\tilde{\Gamma}$ with $\Theta > 0$, $\mu > 0$, r integer $\in [0,s]$, and

$$\mu^{(r+1)} < \mu < \mu^{(r)}$$

(where $\mu^{(0)} = +\infty$, $\mu^{(s+1)} = -\infty$). Replacing possibly $\tilde{\Gamma}$ by a smaller set retaining the properties of (2), one may construct Borel functions $\tilde{\beta} > \tilde{\alpha} > 0$ and $\tilde{\gamma} > 1$ on $\tilde{\Gamma}$ with the following properties.

(a) If $\tilde{x} = (x_n) \in \tilde{\Gamma}$ the set

$$\tilde{W}^\mu_{\tilde{x}} = \{\tilde{y} = (y_n) \in \tilde{M} : d(x_0,y_0) \le \tilde{\alpha}(\tilde{x}) \text{ and } d(x_n,y_n) \le \tilde{\beta}(\tilde{x}) e^{-n\mu(\tilde{x})}$$

for all $n > 0\}$ is contained in $\tilde{\Gamma}$; the map $\tilde{\pi}$ restricted to $\tilde{W}^\mu_{\tilde{x}}$ is injective and $\tilde{\pi}\tilde{W}^\mu_{\tilde{x}}$ is a C-submanifold of the ball $\{y \in M\colon d(x_0,y) \le \tilde{\alpha}(\tilde{x})\}$. For each

$\tilde{y} = (y_n) \in \widetilde{W}^{\mu}_{\tilde{x}}$, we have $T_{y_0} \tilde{\pi} \widetilde{W}^{\mu}_{\tilde{x}} = \tilde{V}^{(r)}_{\tilde{y}}$. More generally, for every $t \in [0,s]$, the function $y \to \tilde{V}^{(t)}_{\tilde{\pi}^{-1} y}$ is of class C^{-1} on $\tilde{\pi} \widetilde{W}^{\mu}_{\tilde{x}}$.

(b) If (y_n), $(z_n) \in \widetilde{W}^{\mu}_{\tilde{x}}$, then

$$d(y_n, z_n) \leq \tilde{\gamma}(\tilde{x}) \; d(x_0, y_0) e^{-n\mu(x)} .$$

(c) If $\tilde{x} \in \tilde{\Gamma}$, then $\tilde{\alpha}(\hat{f}^b \tilde{x})$, $\tilde{\beta}(\hat{f}^n \tilde{x})$ decrease less fast with n than the exponential $e^{-n\Theta}$.

(6) Global stable and unstable manifolds exist under obvious transversality conditions (for instance, if $T_x f$ is a linear isomorphism), Under these conditions they are immersed submanifolds.

(7) The results described above for maps apply immediately to flows, via a time T map.

REFERENCES

[1] M. Hirsch, C. Pugh and M. Shub, Invariant manifolds, Lecture Notes in Math. no. 583, Springer, Berlin, 1977.

[2] V.I. Oseledec, Multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems. Trudy Moskov., Mat. Obsc. 19, 179-210 (1968). English transl. Trans. Moscow Math. Soc., 19, 197-221 (1968).

[3] Ya. B. Pesin, Invariant manifold families which correspond to non-vanishing characteristic exponents. Izv. Akad. Nauk SSSR, Ser. Mat. 40 no. 6, 1332-1379. (1976), English transl. Math. USSSR izv. 10, no. 6, 1261-1305 (1976).

[4] M.S. Raghunathan, A proof of Oseledec multiplicative ergodic theorem, Israel J. Math. To appear.

[5] D. Ruelle, Ergodic theory of differentiable dynamical systems. I.H.E.S. Publications Mathematiques. To appear.

[6] D. Ruelle, Invariant manifolds for flows in Hilbert space, to appear.