# The pressure of the geodesic flow on a negatively curved manifold

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Manning [5] has identified the rate of exponential growth of the volume of a ball of radius $r$ on the universal cover of a compact manifold $M$ of negative curvature: it is the entropy of the geodesic flow on $M$. See also Sullivan [7], Chen [3], Chen and Manning [4]. Here we indicate an extension of Manning's result, where the entropy is replaced by the topological pressure $P(A)$ associated with a function $A$ on the tangent bundle. It turns out that the Riemann volume used by Manning plays no special role and may be replaced by many other measures.

Let $M$ be a compact Riemann manifold with strictly negative sectional curvatures everywhere. We denote by $\tilde{M}$ the universal cover of $M$ (with the induced metric), by $p:\tilde{M}\to M$ the canonical projection, and by $N$ a fundamental domain of finite diameter $a$. We call $B(x,r)$ the ball with center $x$ and radius $r$ in $\tilde{M}$. Let $\mu$ be a positive Radon measure on $\tilde{M}$, such that there are $\alpha, \beta, b > 0$ with

$$\alpha \leqslant \mu(B(x,b)) \leqslant \beta \tag{1}$$

for all $x \in \tilde{M}$.

We denote by $T^{(1)}M$ the unit tangent bundle and let

$$A:T^{(1)}M\to R$$

be a continuous function. For any pair $x,y \in \tilde{M}$, let $\sigma(t)$ be the point of abscissa $t \in [0,d(x,y)]$ on the unique geodesic segment $xy$ from $x$ to $y$. We define

$$A_{xy} = \int_0^{d(x,y)} A(T_t(p\sigma(t)))dt$$

and, for $0 < r_1 < r_2$,

$$Z(x,r_1,r_2) = \int_{B(x,r_2)\backslash B(x,r_1)} \mu(dy) \exp A_{x,y}.$$

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**Theorem.** *Let $c \geqslant 2(a+b)$, then*

$$\lim_{r \to \infty} \frac{1}{r} \log Z(x,r,r-c) = P(A) \tag{2}$$

*uniformly with respect to x, where $P(A)$ is the pressure of A with respect to the geodesic flow $(f^t)$ on $T^{(1)}M$.*

Our proof will closely follow that of Manning for the case $A = 0$ (see [5]).

We shall use the formulae (cf. [2])

$$P(A) = \lim_{\delta \to 0} P^{\pm}(A,\delta)$$

$$P^{\pm}(A,\delta) = \lim_{r \to \infty} \sup \frac{1}{r} Z_r^{\pm}(A,\delta)$$

$$Z_r^{+}(A,\delta) = \sup\left\{ \sum_{\xi \in S} \exp \int_0^r A(f^t \xi)dt : S \text{ is } (r,\delta) \text{ separated}\right\}$$

$$Z_r^{-}(A,\delta) = \inf\left\{ \sum_{\xi \in S} \exp \int_0^r A(f^t \xi)dt : S \text{ is } (r,\delta) \text{ spanning}\right\}$$

These formulae are easily related to those for the time 1 map $f^1$ and the function $A^1 = \int_0^1 dt\ A \cdot f^t$.

**Lemma.** *Given $\delta, \Delta > 0$ there is R such that if $\sigma, \tau : [0,r] \to \tilde{M}$ are two geodesics with $\sigma(0) = \tau(0)$, then $d(\sigma(r),\tau(r)) \leqslant \Delta$ and $r \geqslant R$ imply*

$$d(T_t\sigma(t),\ T_t\tau(t)) \leqslant \delta$$

*in $T^{(1)}\tilde{M}$ for $t \in [0,r-R]$.*

This is a form of Lemmas 1 and 2 of Manning corresponding to strictly negative curvature: geodesics diverge exponentially.

We shall use the fact, given $\epsilon > 0$, for $d(y,z) \leqslant$ constant, and sufficiently large $d(x,y)$,

$$|A_{xy} - A_{xz}| < \frac{1}{2}\, \epsilon\, d(x,y) \tag{3}$$

This follows from the lemma and the uniform continuity of $A$.

To prove (2) we first show that, for $\delta \leqslant \frac{1}{2} b$ and all $\epsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{r} \log Z(x,r,r+\delta/2) \leqslant P^+(A,\delta) + \epsilon \tag{4}$$

(uniformly in $x$). Following Manning's proof of Theorem 1 in [5] we take a maximal $2\delta$-separated set $Q \subset B(x,r+\delta/2)\backslash B(x,r)$. Then, for $r$ sufficiently large,

$$Z(x,r,r+\delta/2) \leqslant \beta \sum_{y\in Q} \exp(A_{xy} + \frac{1}{2}\epsilon(r+\delta/2)) \leqslant$$

$$\leqslant \beta\, e^{\epsilon r} \sum_{y\in Q} \exp A_{xy}$$

where we have used (1) and the fact that (3) holds for $d(y,z) \leqslant 2\delta$. The estimate (4) follows from the remark that the unit initial vectors of the geodesics $xy$ with $y \in Q$ are $(r,\delta)$ separated.

Now we show that

$$\liminf_{r\to\infty} \frac{1}{r} \log Z(x,r+a+b,\, r-a-b) \geqslant P(A) \tag{5}$$

(uniformly in $x$). Following Manning's proof of Theorem 2 in [5] we construct a maximal $2b$-separated set $Q' \subset B(x,r+a)\backslash B(x,r-a)$. Using the lemma and the estimate (3) we see that, for given $\epsilon > 0$ and large $r$,

$$Z(x,r+a+b,\, r-a-b) \geqslant \alpha \sum_{y\in Q'} \exp\,(A_{xy} - \frac{1}{2}\epsilon\, d(x,y)) \leqslant \tag{6}$$

An arbitrary geodesic segment of length $r$ in $M$ is the image by $p$ of a geodesic segment $uv$ in $\tilde{M}$, with $n \in N$ and $v \in B(x,r+a)\backslash B(x,r-a)$. There is thus a geodesic segment $xy$ with $y \in Q'$ such that $d(u,x) \leqslant a$, $d(v,y) \leqslant 2b$. Given $\delta > 0$, the lemma applied to $uv$, $uy$, then to $yu$, $yx$ yields $R > 0$ such that $d(T_t\ \sigma(t),\ T_t\ \tau_y(t)) \subset 2\delta$ in $T^{(1)}\tilde{M}$ for $t \in [R,r-R]$, where $\sigma,\tau_y$ denote the geodesics along $uv$, $xy$ suitably parametrized. If $\xi_y$ is the tangent vector to $\tau_y$ at $R$, the set $\{\xi_y \colon y \in Q'\}$ is $a(r-2R,2\delta)$ spanning set for the geodesic flow. The right-hand side of (6) is, for large $r$,

$$\geqslant \alpha \sum_{y\in Q'} \exp\,[\int_o^{r-2R} A(f^t\xi_y)dt - \epsilon r].$$

From this (5) follows.

Finally, (2) is a consequence of (4) and (5).

**Remarks.** (a) From (2) we also obtain

$$\lim_{r_2 - r_1 \to \infty, r_2/r_1 \to 1} \frac{1}{r_2} \log Z(x,r_1,r_2) = P(A)$$

$$\lim_{r_2 \to \infty, r_2 - r_1 \geqslant c} \frac{1}{r_2} \log Z(x,r_1,r_2) = P(A), \text{ if } P(A) \geqslant 0$$

$$\lim_{r_1 \to \infty, r_2, r_1 \geqslant c} \frac{1}{r_1} \log Z(x,r_1,r_2) = P(A), \text{ if } P(A) < 0$$

In particular

(7) $$\lim_{r \to \infty} \frac{1}{r} \log Z(x,0,r) = P(A), \text{ if } P(A) \geqslant 0$$

$$\lim_{r \to \infty} \frac{1}{r} \log Z(x,r,\infty) = P(A), \text{ if } P(A) < 0$$

(b) The limit does not depend on the choice of $\mu$. If one uses for $\mu$ the measure defined by the Riemann metric, and $A = 0$, then (7) is a theorem of Manning (Theorem 2 of [5]).

(c) One can take for $\mu$ the sum of a unit mass at each $y \in p^{-1}x$, so that

$$Z(x,r_1,r_2) = \sum_{y \in p^{-1}x \cap B(x,r_2) \setminus B(x,r_1)} \exp A_{xy}.$$

(d) Since the geodesic flow on $M$ is Anosov, Bowen's specification property holds (see [1] Theorem (3.8)). Therefore (as in the proof of Lemma (4.10) of [1]) one can check that $P(A)$ is the abscissa of convergence of the product

$$\zeta_A(u) = \prod_\gamma [1 - \exp \int_0^{\lambda(\gamma)} (A(f^t \xi_\gamma) - u) dt]^{-1}.$$

This product is extended over oriented closed geodesics $\gamma$, $\lambda(\gamma)$ is the length of $\gamma$, and $\xi_\gamma \in T^{(1)}M$ is any properly oriented tangent unit vector to $\gamma$. In fact, if $A$ is Hölder continuous, $P(A)$ is a simple pole of $\zeta_A$, and $\zeta_A$ is analytic in some neighborhood of $P(A)$ (see [5]).

## References

[1] R. Bowen, *Periodic orbits for hyperbolic flows*, Amer. J. Math. 94, 1-30 (1972).

[2] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. 29, 181-202 (1975).

[3] S.-S. Chen, *Entropy of geodesic flow and exponent of convergence of some Dirichlet series*, To appear.
[4] S.-S. Chen and A. Manning, *The convergence of zeta functions for certain geodesic flows depends on their pressure*, To appear.
[5] A. Manning, *Topological entropy for geodesic flows*, Ann. of Math. 110, 567-573 (1979).
[6] D. Ruelle, *Generalized zeta-functions for Axiom A basic sets*, Bull. Amer. Math. Soc. 82, 153-156 (1976).
[7] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. IHES 50, 171-202 (1979).

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