DIFFERENTIATION OF SRB STATES: CORRECTION AND COMPLEMENTS.

by David Ruelle*

Abstract. Taking into account criticism by D. Dolgopyat and M. Jiang, we present here an improved derivation of the formula for the derivative of an SRB measure with respect to parameters.

The SRB measure ρ_f on a mixing Axiom A attractor K depends smoothly on the diffeomorphism f, and a formula for the derivative was given in [3], namely

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f \langle \mathrm{grad}(\Phi \circ f^n), X \rangle$$

$$= \sum_{n=0}^{\infty} \rho_f [\langle (\mathrm{grad}\Phi) \circ f^n, (Tf^n)X^s \rangle - (\Phi \circ f^n)\mathrm{div}^u X^u] \tag{1}$$

where $X = \delta f \circ f^{-1}$ has components X^s and X^u in the stable and unstable directions. The divergence div^u is taken along an unstable manifold with respect to the natural measure induced by ρ_f on unstable manifolds, and $\mathrm{div}^u X^u$ is a Hölder continuous function on K. This last fact, as pointed out by Dolgopyat, is not obvious, and was not clearly stated or proved in [3] (the problem is that X^u need not be smooth). Furthermore, as pointed out by Jiang, one term was omitted in the proof of the above formula (this term, see below, vanishes however in the present circumstances).

The purpose of the present note is to correct and complement the proof of the "first step" in section 3 of [3]. From this one obtains formula (1) for $\delta\rho_f$, without extra term, and with Hölder continuous $\mathrm{div}^u X^u$ as needed for applications to statistical mechanics.

The following proposition is stated for an attractor K, but an extension to general Axiom A basic sets is discussed in Remark 4 below.

1. Proposition.

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Let f be a C^r diffeomorphism ($r \geq 3$) of a compact manifold M and K a mixing Axiom A attractor. Also let $J^u_f : K \to \mathbf{R}$ be the unstable Jacobian computed with respect to some smooth volume element ω on the unstable manifolds (say the volume element associated with a Riemann metric). A change δf of f corresponds to a vector field $X = \delta f \circ f^{-1}$ on M. Let δJ^u be the corresponding first order change of J^u_f. We shall prove the formula

$$\frac{\delta J^u}{J^u_f} \sim \mathrm{div}^u_\sigma X^u \tag{2}$$

where we have used the following notation. The equivalence $\sim$ means that the integrals of both sides with respect to every f-invariant measure on K coincide. We have written $X^s + X^u$ the decomposition of X restricted to K along the stable and unstable directions. Finally, the divergence div^u_σ is computed on unstable manifolds with respect to the canonical volume element σ defined up to a multiplicative constant by $\sigma(x) = s(x)\omega(x)$ where

$$\frac{s(x)}{s(y)} = [\prod_{k=1}^{\infty} \frac{J^u_f(f^{-k}x)}{J^u_f(f^{-k}y)}]^{-1}$$

We claim that $\mathrm{div}^u_\sigma X^u$ makes sense as a Hölder continuous function on K.

The proof of the proposition will use the absolute continuity of the projection along stable directions: let $\pi : \Sigma_1 \to \Sigma_2$ be the projection along local stable manifolds between two u-dimensional manifolds Σ_1, Σ_2 transversal to stable directions (u is the unstable dimension) then π is absolutely continuous with respect to the Riemann u-volume on Σ_1, Σ_2. The corresponding Jacobian J_π is Hölder, but if one of the u-dimensional manifolds is moved, the Jacobian varies smoothly along stable manifolds (see Lemma 2 below).

We shall denote by V^s_x, V^u_x the stable and unstable subspaces of T_xM at $x \in K$.

Remember that, by structural stability, changing f to $\tilde{f}$ yields a map $j : K \to M$ such that $\tilde{f}j = jf$. Furthermore j depends smoothly on $\tilde{f}$ (and similarly for other quantities like $\tilde{V}^u_{jx}$, etc.). Our proposition is a first order calculation, where we write $\tilde{f} = f + \delta f$, $X = \delta f \circ f^{-1}$, and $jx = x + \delta x$. We also want to take $\delta J^u = \tilde{J}^u_{\tilde{f}} \circ j - J^u_f$. In order to define the unstable Jacobian $\tilde{J}^u_{\tilde{f}}$, we have to choose an unstable volume element $\tilde{\omega}$ for $\tilde{f}$. Note that changing $\tilde{\omega}$ amounts to changing $\log \tilde{J}^u_{\tilde{f}}$ by a "coboundary" term $A \circ \tilde{f} - A$. This changes $\delta J^u / J^u_f$ by $A \circ \tilde{f} \circ j - A \circ j = A \circ j \circ f - A \circ j \sim 0$. Any choice of $\tilde{\omega}$ corresponding to a continuous function A on K is thus allowed.

The calculation of δJ^u involves unstable manifolds $\mathcal{V}^u_x$, $\mathcal{V}^u_{fx}$, $\tilde{\mathcal{V}}^u_{jx}$, $\tilde{\mathcal{V}}^u_{jfx}$ for f and $\tilde{f}$, and there are maps $\pi : \tilde{\mathcal{V}}^u_{jx} \to \mathcal{V}^u_x, \tilde{\mathcal{V}}^u_{jfx} \to \mathcal{V}^u_{fx}$ defined by projection along the stable manifolds for f. By absolute continuity, π also defines maps $(T_{j\cdot}M)^{\wedge u} \to (V^u_\cdot)^{\wedge u}$, and we shall use the volume element $\tilde{\omega} = \omega \circ \pi$ on $\tilde{\mathcal{V}}^u_{jx}$, $f\tilde{\mathcal{V}}^u_{jx}$, $\tilde{f}\tilde{\mathcal{V}}^u_{jx}$ (note that $\tilde{\omega}$ has continuous rather than smooth density).

Let now $F(\cdot) \in (V^u_\cdot)^{\wedge u}$, $\tilde{F}(\cdot) \in (\tilde{V}^u_\cdot)^{\wedge u}$ be defined by $\langle \omega, F \rangle = 1$, $\langle \omega, \pi\tilde{F} \rangle = 1$. Write

$$\lambda(x) = \langle \omega, (T_xf)^{\wedge u}F(x) \rangle \qquad , \qquad \tilde{\lambda}(x) = \langle \omega, \pi(T_{jx}\tilde{f})^{\wedge u}\tilde{F}(jx) \rangle$$

and $\delta\lambda(x) = \tilde{\lambda}(x) - \lambda(x)$, then

$$|\lambda(x)| = J^u_f(x) \qquad , \qquad \delta\lambda(x)/\lambda(x) = \delta J^u(x)/J^u_f(x)$$

We may write

$$\delta\lambda(x) = \langle\omega, \pi(T_{x+\delta x}(f+\delta f))^{\wedge u}\tilde{F}(x+\delta x)\rangle - \langle\omega, (T_x f)^{\wedge u}F(x)\rangle$$
$$= \langle\omega, \pi(T_{x+\delta x}f)^{\wedge u}\tilde{F}(x+\delta x)\rangle - \langle\omega, (T_x f)^{\wedge u}F(x)\rangle$$
$$+\langle\omega, \pi[(T_{f(x+\delta x)}(\mathrm{id}+X))^{\wedge u} - 1](T_{x+\delta x}f)^{\wedge u}\tilde{F}(x+\delta x)\rangle$$

If we let $\delta x = \delta^u x + \delta^s x$ with $\delta^u x \in V^u_x$, $\delta^s x \in V^s_x$, we have

$$\pi(T_{x+\delta x}f)^{\wedge u}\tilde{F}(x+\delta x) = (T_{x+\delta^u x}f)^{\wedge u}F(x+\delta^u x)$$

so that

$$\langle\omega, \pi(T_{x+\delta x}f)^{\wedge u}\tilde{F}(x+\delta x)\rangle - \langle\omega, (T_x f)^{\wedge u}F(x)\rangle = \lambda(x+\delta^u x) - \lambda(x)$$

(this is the term forgotten in [3], as pointed out by Jiang). Also, to first order,

$$\langle\omega, \pi[(T_{f(x+\delta x)}(\mathrm{id}+X))^{\wedge u} - 1](T_{(x+\delta x)}f)^{\wedge u}\tilde{F}(x+\delta x)\rangle$$
$$= \langle\omega, \pi[(T_{fx}(\mathrm{id}+X))^{\wedge u} - 1](T_x f)^{\wedge u}F(x)\rangle = \langle\omega, \pi[(T_{fx}(\mathrm{id}+X))^{\wedge u} - 1]\lambda(x)F(fx)\rangle$$
$$= \lambda(x)\Phi(fx)$$

where

$$\Phi(x) = \langle\omega, \pi[(T_x(\mathrm{id}+X))^{\wedge u} - 1]F(x)\rangle$$

We claim that

$$\Phi(x) = \mathrm{div}^u_\omega X^u \tag{3}$$

The formula (3) will be obtained in this setup as a consequence of Lemma 2 below. Using (3) we have

$$\delta\lambda(x) = \lambda(x+\delta^u x) - \lambda(x) + \lambda(x)\,\mathrm{div}^u_\omega X^u(fx)$$

Let us now define $\sigma_n = f^n\omega$, so that

$$\sigma_n(x) = s_n(x)\omega(x) \qquad \text{with} \qquad s_n(x) = [\prod_{k=1}^n J^u_f(f^{-k}x)]^{-1}$$

and replace ω by σ_n. This replaces $J^u_f(x)$ by $J^u_f(f^{-n}x)$ so that, as $n \to \infty$, the derivative in the unstable direction of $\log J^u_f(x)$ tends to 0. In particular $\lambda(x+\delta^u x) - \lambda(x) \to 0$. Note also the identity

$$\mathrm{div}^u_{\sigma_n} X^u - \mathrm{div}^u_\sigma X^u = X^u \cdot \mathrm{grad}\log\frac{d\sigma_n}{d\sigma}$$

with the Radon-Nikodym derivative $d\sigma_n/d\sigma = s_n/s$, and

$$X^u \cdot \operatorname{grad}\log\frac{s_n}{s} = \sum_{k=n+1}^{\infty} X^u \cdot \operatorname{grad}(J_f^u \circ f^{-k})$$

When $n \to \infty$ (for fixed Hölder X^u) the above expression tends to zero in a space of Hölder continuous functions on K. Therefore, if we know that $\operatorname{div}_\omega^u X^u$ is a Hölder continuous function, so are all $\operatorname{div}_{\sigma_n}^u X^u$ and also their limit $\operatorname{div}_\sigma^u X^u$. (That $\operatorname{div}_\omega^u X^u$ is Hölder will result from 2. and 3. below). Using also $\Phi \circ f \sim \Phi$, this concludes the proof of Proposition 1 (modulo the proof of (3) in the next two Sections). □

2. Lemma.

Identify $\mathbf{R}^u$, $\mathbf{R}^s$ with subspaces of $\mathbf{R}^u + \mathbf{R}^s = \mathbf{R}^{u+s}$, and choose a chart identifying an open set of M with the sum $\mathbf{B}^u + \mathbf{B}^s$ of unit balls around 0 in $\mathbf{R}^u$ and $\mathbf{R}^s$. We assume that the stable manifolds for f are (uniformly) transversal to the affine spaces $y + \mathbf{R}^u$. Also let $0 \neq \mathrm{e} \in \mathbf{R}^s$. If $x \in \mathbf{R}^u$, let $\mathcal{V}_x^s$ be the local stable manifold through x, and let $\xi(x,s)$ be the point of intersection of $\mathcal{V}_x^s$ and $s\mathrm{e} + \mathbf{R}^u$. Thus $s \mapsto \xi(x,s)$ is smooth, $\xi(x,0) = x$, and if $\tilde{\mathrm{e}}(x) = \frac{d}{ds}\xi(x,s)|_{s=0}$, we may write $\tilde{\mathrm{e}}(x) = \mathrm{e} + \tilde{g}(x)$ where $\tilde{g}(x) \in \mathbf{R}^u$.

We claim that $x \mapsto \xi(x,s)$, $\frac{d}{ds}\xi(x,s)$ are Hölder, and that the same is true for

$$x \mapsto (T_x\xi(\cdot,s))^{\wedge u}\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u = \varpi(x,s)\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u$$

where $\mathrm{e}_1, \ldots, \mathrm{e}_u$ are the canonical basis vectors of $\mathbf{R}^u$. (We write here $(T_x\xi)^{\wedge u}$ somewhat abusively, since ξ is not differentiable in general, to indicate the well defined action on u-volume elements). Furthermore, $s \mapsto \varpi(x,s)$ is smooth, $x \mapsto \varpi'(x,s) = \frac{d}{ds}\varpi(x,s)$ is Hölder and, writing $\tilde{g}(x) = \sum_{k=1}^u \tilde{g}^k(x)\mathrm{e}_k$, we have

$$\varpi'(x,0) = \sum_{k=1}^{u} \frac{\partial}{\partial x^k}\tilde{g}^k(x)$$

in the sense of distributions (the left-hand side is a Hölder continuous function, canonically identified with a distribution on $\mathbf{R}^n$).

It is well known that $x \mapsto \xi(x,s)$, $\frac{d}{ds}\xi(x,s)$ are Hölder. To prove the "absolute continuity" result that $x \mapsto \varpi(x,s)$ is Hölder we use the formula

$$\lim_{n\to\infty} \frac{||(T_x f^n)^{\wedge u}\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u||}{||(T_{\xi(x,s)} f^n)^{\wedge u}\varpi(x,s)\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u||} = 1$$

(where the norm $||\cdot||$ is based, say, on a Riemann metric on M). If we write $F_0 = \mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u$, $\tau_0 = 1$, and

$$F_k(x,s) = (T_{f^{k-1}\xi(x,s)} f)^{\wedge u} F_{k-1}(x,s)/\tau_{k-1}(x,s)$$

$$\tau_k(x,s) = ||F_k(x,s)||$$

we find

$$\log \varpi(x,s) = -\sum_{k=1}^{\infty}[\log \tau_k(x,s) - \log \tau_k(x,0)]$$

where the sum converges exponentially fast (by hyperbolicity and the fact that $\xi(x,s)$ is in the stable direction and the F_k tend to the unstable direction) and is a Hölder continuous function by the usual argument. Also, $\varpi'(x,s) = \frac{d}{ds}\varpi(x,s)$ is given by a convergent series and is a Hölder continuous function of x.

Let $\mathcal{F}^{(0)}$ be a smooth foliation of a neighborhood of K, which is transversal to the stable manifolds $\subset K$. Let $\mathcal{F}^{(n)} = f^{-n}\mathcal{F}^{(0)}$ (restricted to a neighborhood of K). Then $\mathcal{F}^{(n)}$ tends to the "Hölder foliation" by stable manifolds: $\mathcal{F}_x^{(n)} \to \mathcal{V}_x^s$ in C^r, uniformly in x. Define $\xi^{(n)}(x,s)$, $\tilde{\mathrm{e}}^{(n)}(x)$, $\varpi^{(n)}(x,s)$ in the same way as $\xi(x,s)$, $\tilde{\mathrm{e}}(x)$, $\varpi(x,s)$, but with stable manifolds replaced by leaves of $\mathcal{F}^{(n)}$. Then, there is $\alpha \in (0,1)$ such that $\xi^{(n)}(x,\cdot) \to \xi(x,\cdot)$ in $\mathrm{C}^r((-\alpha,\alpha))$ uniformly for $x \in \alpha\mathbf{B}^u$.

A 3ϵ argument also shows that $\varpi^{(n)}(x,s) \to \varpi(x,s)$, $\varpi^{(n)\prime}(x,s) \to \varpi'(x,s)$ uniformly in $\alpha\mathbf{B}^u \times (-\alpha,\alpha)$, but since $\mathcal{F}^{(n)}$ is a smooth foliation

$$\varpi^{(n)\prime}(x,s) = \sum_{k=1}^{u} \frac{\partial}{\partial x^k}\frac{d}{ds}\xi^{(n)k}(x,s)$$

where we have written $\xi^{(n)} = \sum_{k=1}^{u}\xi^{(n)k}\mathrm{e}_k$. Taking $n \to \infty$, then $s = 0$ we obtain

$$\varpi'(x,0) = \sum_{k=1}^{u}\frac{\partial}{\partial x^k}\tilde{g}^k(x)$$

where the left-hand side is a Hölder continuous function of x, and the right-hand side a sum of distributional derivatives, as announced. $\square$

3. Proof of (3).

Note that both sides of (3) are defined independently of a coordinate system. We choose coordinates $x^1,\ldots,x^u,x^{u+1},\ldots,x^{u+s}$ such that $\mathcal{V}_x^u$ satisfies $x^{u+1} = \ldots = x^{u+s} = 0$, and $\omega = dx^1 \wedge \ldots \wedge dx^u$, and X is constant $=$ e. We then see that

$$\Phi(x) = \langle \omega, \pi[(T_x(\mathrm{id}+X))^{\wedge u} - 1]F(x)\rangle = \langle \omega, [(T_x(\mathrm{id} - \frac{d}{ds}\xi(x,s)|_{s=0})^{\wedge u} - 1]\mathrm{e}_1 \wedge \ldots \wedge \mathrm{e}_u\rangle$$

$$= -\varpi'(x,0) = -\sum_{k=1}^{u}\frac{\partial}{\partial x^k}\tilde{g}^k(x) = \mathrm{div}_\omega^u X^u$$

(the second equality is geometrically clear, and the last follows from $X^u = -\tilde{g}$). $\square$

4. Remark.

If the mixing Axiom A basic set K is not an attractor the local stable manifolds do not cover a neighborhood of K. The proof of Proposition 1 carries over to this situation, except that $\varpi'(x,0)$ cannot in a straightforward manner be interpreted as a divergence. We do not pursue this topic here in spite of its interest (it is related to "escape from quasi-attractors").

5. Calculation of $\delta\rho_f$.

The fact that the attractor K and the SRB measure ρ_f depend smoothly on f have been noted by various authors at various levels of generality. To the references in [3] one should add Bakhtin [1]. For recent results see also [2]. The specific formula (1) goes however beyond the proof of differentiability. As indicated in [3], $\delta\rho_f$ is a sum of two terms: $\delta^{(1)}\rho_f$ which takes into account the change (2) in $\log J^u_f$, and $\delta^{(2)}\rho_f$ which takes into account the change $x \mapsto x + \delta x$ associated with structural stability of K. One has (with $\Phi \in \mathrm{C}^2(M)$)

$$\delta^{(1)}\rho_f(\Phi) = \sum_{k\in\mathbf{Z}}[\rho_f((\Phi\circ f^k).(-\mathrm{div}^u_\sigma X^u)) - \rho_f(\Phi).\rho_f(-\mathrm{div}^u_\sigma X^u)]$$

$$\delta^{(2)}\rho_f(\Phi) = \rho_f[\sum_{n=0}^{\infty}\langle\mathrm{grad}(\Phi\circ f^n), X^s\rangle - \sum_{n=1}^{\infty}\langle\mathrm{grad}(\Phi\circ f^{-n}), X^u\rangle]$$

But since ρ_f has conditional measure σ on the unstable manifolds, the integral of div^u_σ vanishes: $\rho_f(-\mathrm{div}^u_\sigma X^u) = 0$ and also

$$\rho_f[\langle\mathrm{grad}(\Phi\circ f^{-n}), X^u\rangle + (\Phi\circ f^{-n}).\mathrm{div}^u_\sigma X^u] = 0$$

[To see this it is convenient to use a Markov partition $\{R\}$. We may disintegrate ρ_f in each rectangle R with respect to the partition into local unstable manifolds, writing $\rho_f|R$ as an integral of measures σ. The integral of $\mathrm{div}^u_\sigma X^u$ with respect to σ on a piece of unstable manifold reduces to boundary terms. And the the boundary terms of different rectangles cancel when we sum over R. Therefore $\rho_f(\mathrm{div}^u_\sigma\cdot) = 0$].

In conclusion

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty}\rho_f[\langle\mathrm{grad}(\Phi\circ f^n), X^s\rangle + (\Phi\circ f^n).(-\mathrm{div}^u_\sigma X^u)]$$

as announced.

6. Calculation of $\delta(\rho_f(-\log J^u_f))$.

The expression $\delta(\rho_f(-\log J^u_f))$ cannot be directly evaluated from (1) (indeed $J^u_f = |\lambda|$ depends on f, and is defined on K, not on M). Remembering that $\delta\lambda = \tilde\lambda - \lambda$ where $|\tilde\lambda|$ is the unstable Jacobian for $\tilde f(x)$ estimated at $x + \delta x$, we see that in fact

$$\delta(\rho_f(-\log\lambda)) = (\delta^{(1)}\rho_f)(-\log\lambda) - \rho_f(\lambda^{-1}\delta\lambda)$$

$$= \sum_{k\in\mathbf{Z}}[\rho_f((\log\lambda\circ f^k).\mathrm{div}^u_\sigma X^u) - \rho_f(\lambda).\rho_f(\mathrm{div}^u_\sigma X^u)] - \rho_f(\mathrm{div}^u_\sigma X^u\circ f)$$

$$= \sum_{k \in \mathbf{Z}} \rho_f((\log \lambda \circ f^k).\mathrm{div}^u_\sigma X^u)$$

Therefore

$$\delta(\rho_f(-\log J^u_f)) = \sum_{k \in \mathbf{Z}} \rho_f((\log J^u_j \circ f^k).\mathrm{div}^u_\sigma X^u)$$

If the unstable volume element ω can be chosen such that the function J^u_f is constant on K, then $\delta(\rho_f(-\log J^u_f)) = 0$ (but this is not the case in general).

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