W H A T   I S . . .

a Strange Attractor?

David Ruelle

Your computer will readily implement the map f sending the point $(u, v) \in \mathbf{R}^2$ to the point $(v + 1 - au^2, bu) \in \mathbf{R}^2$, where $a = 1.4$ and $b = 0.3$. Ask your computer to plot the points $x_n = f^n(0, 0)$, and you will find that they accumulate, for $n \to \infty$, on a convoluted fractal set A known as the *Hénon attractor*. This set is prototypical of what one wants to call a *strange attractor*. Such objects often arise when a diffeomorphism f stretches and folds an open set U and maps the closure $f\bar{U}$ inside U (this is a typical situation, not a necessary or a sufficient condition). The strange attractor A is visualized when a computer plots the points $x_n = f^n x_0$ with almost any initial value x_0 in U. The figures show a two-dimensional example corresponding to the Hénon attractor, and a three-dimensional example corresponding to Smale's *solenoid*. It turns out that a small change in the values of a and b can destroy the Hénon attractor, but a small change of f does not destroy the solenoid.

We give now a definition of *uniformly hyperbolic attractors*, or *Axiom A attractors*. This will cover the case of the solenoid but not the Hénon attractor. (One can also define nonattracting hyperbolic sets, called basic sets, but they will not concern us.) Let f be a diffeomorphism of the compact manifold M, and put some Riemann metric on M. Let A be a compact invariant subset of M. We say that A is a *hyperbolic* set if one can continuously choose at each point $x \in A$ a contracting (or *stable*) subspace E^s_x and an expanding (or *unstable*) subspace E^u_x of T_xM so that $(Tf)E^{s,u} = E^{s,u}$ and $TM = E^s \oplus E^u$. We thus assume that

$$||(Tf)X|| < ||X|| \qquad \text{if} \qquad 0 \neq X \in E^s$$

$$||(Tf)^{-1}X|| < ||X|| \qquad \text{if} \qquad 0 \neq X \in E^u.$$

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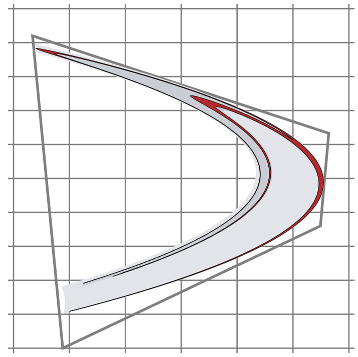
We ask that A be *attracting*, *i.e.*, that A have an open neighborhood U such that

$$\cap_{t \geq 0} f^t U = A$$

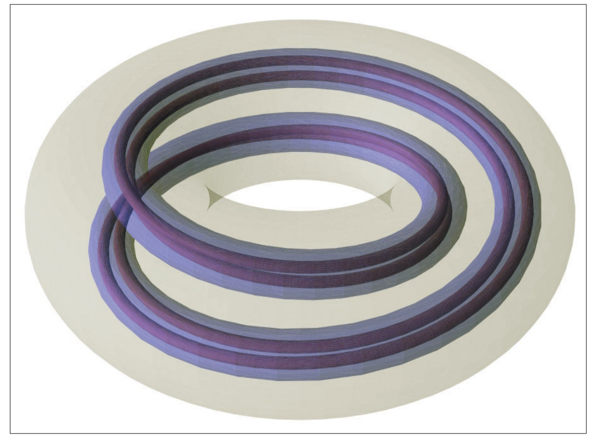
(one can then arrange that $f\bar{U} \subset U$). Smale's Axiom A also requires that f-periodic orbits be dense in A and that A contain a dense f-orbit (topological transitivity).

An Axiom A attractor is either a finite attracting periodic orbit, or it is an infinite set, and the dimension of the expanding subspaces E^u_x is > 0. In the latter case, A is a *strange* Axiom A attractor: there are points y close to x in A such that the distance between $f^n x$ and $f^n y$ grows exponentially with n until this distance becomes of the order of the diameter of A. The exponential growth of $\mathrm{dist}(f^n x, f^n y)$ expresses *chaos*, or *sensitivity to initial condition*: if there is any imprecision on x, the predictability of $f^n x$ is lost for large n. Here, f stretches the set U and necessarily also "folds" this set to put $f\bar{U}$ back in U, in agreement with the notion of a strange attractor. Careful studies have shown that various systems that occur in nature are *chaotic*: their time evolution is described by low-dimensional dynamics with sensitivity to initial condition, for which strange Axiom A attractors offer an excellent mathematical model. In particular, a proposal by Floris Takens and myself that hydrodynamic turbulence is chaotic in this sense was eventually vindicated by experiment. It is on this occasion that the name "strange attractor" seems to have been coined.

The theory of uniformly hyperbolic attractors, initiated by Dmitrii Anosov and Stephen Smale, shows that one can define stable and unstable manifolds $\mathcal{V}^S_x, \mathcal{V}^u_x \subset M$: these are nonlinear versions of the spaces $E^s_x, E^u_x \subset T_xM$, and they give a global meaning to contracting and expanding directions. One proves structural stability: if $\tilde{f}$ is close to f, then $\tilde{f}$ has an attractor $\tilde{A}$ close to A, so that $\tilde{f}|\tilde{A}$ is topologically conjugate to $f|A$. Using the density of periodic orbits, Smale proved *local product structure*: A is locally the product of a set in the contracting direction and a set in the expanding direction. (The set in the expanding direction is a manifold, while in the contracting direction it is usually a fractal, for instance a Cantor set.) Yakov Sinai obtained a global consequence of the product structure called *symbolic dynamics* (an improved later treatment was given by Rufus Bowen). Here is the result: up to well-controlled ambiguities, one can associate with each point $x \in A$ an infinite sequence of symbols $(\dots, \xi_{-1}, \xi_0, \xi_1, \dots)$. The ξ_i in the sequence are taken from a finite set F and only certain successive pairs (ξ_i, ξ_{i+1}) are allowed, but no other condition is imposed. Furthermore, replacing x by fx replaces $(\dots, \xi_{-1}, \xi_0, \xi_1, \dots)$ by the shifted sequence $(\dots, \xi_0, \xi_1, \xi_2, \dots)$. Symbolic dynamics thus replaces geometry (the diffeomorphism f) by algebra (the shift on sequences of symbols). For example, using symbolic dynamics, one can count periodic points in A (*i.e.*, give a formula for card$f^n|A$). One can also study f-invariant measures on A, in particular so-called *Gibbs* measures: they correspond to "thermal equilibrium states" for an interacting one-dimensional *spin system* as studied in statistical mechanics when the sequence $(\dots, \xi_{-1}, \xi_0, \xi_1, \dots)$ is interpreted as an infinite configuration of spins ξ_i in one dimension.



The Hénon map f sends the quadilateral U inside itself. The sets fU, f^2U, f^3U are plotted, and f^3U already looks a lot like the Hénon attractor $\cap_{t\geq 0} f^n U$.



Smale's solenoid map f sends the torus U inside itself. Shown are U, fU, f^2U, and you can imagine the solenoid $\cap_{t\geq 0} f^n U$.

For Lebesgue-almost-all points x in the neighborhood U of A, the time averages along the forward orbit $(x, fx, \dots, f^n x, \dots)$ tend to a probability measure μ_{SRB} on A, and this measure does not depend on x. This so-called SRB-measure is a Gibbs measure, introduced by Yakov Sinai, myself, and Rufus Bowen. We have a detailed understanding of SRB measures and how they depend on f.

What now about non-Axiom A attractors? To discuss them, we must introduce the wonderful idea by Yakov Pesin that the geometric condition of uniform hyperbolicity for dynamical systems can be replaced by an almost-everywhere analysis with respect to any given ergodic measure μ. Pesin theory allows one to define stable and unstable manifolds $\mathcal{V}_x^{s,u}$ for μ-almost-all x. Furthermore, those measures that can be called SRB measures have been characterized by François Ledrappier, Jean-Marie Strelcyn, and Lai-Sang Young. General SRB measures are a beautiful measure-theoretic version of Axiom A attractors, without the uniform hyperbolicity assumption; they again describe time averages for a set of positive Lebesgue measure in the manifold M. The support of a general SRB measure is typically a fractal object that deserves to be called a strange attractor and is in fact what you see when a computer plots for you, say, the Hénon attractor. We may say that, to go beyond hyperbolicity, we have replaced the geometric concept of strange attractor by the ergodic concept of SRB measure.

The great generality of Pesin theory comes at a price: it is hard to know how things change when the diffeomorphism f is replaced by a nearby diffeomorphism $\tilde{f}$. Here we reach the current frontier in the theory of smooth dynamical systems. Cvitanović has proposed a fascinating description of the "pruning front" associated with changes in the Hénon attractor. A theory of "Hénon-like" diffeomorphisms has been developed (by Benedicks-Carleson, Viana, L.-S. Young, and others) where the SRB measure is analyzed, not for all perturbations $\tilde{f}$ of f, but for a set of positive measure of such perturbations in some parameter space. Jacob Palis has proposed a beautiful set of conjectures to the effect that for most diffeomorphisms f (in some sense), and most initial points $x \in M$ (in the sense of Lebesgue measure), time averages correspond to a finite number of SRB measures (or attractors if you like). But it is also conceivable that for a large set of diffeomorphisms f and of points x, the time averages are not even defined!

The beautiful pictures above are by Bill Casselman and David Austin.

Further Reading

[1] J.-P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.* **57** (1985), 617–656.

[2] L.-S. Young, What are SRB measures, and which dynamical systems have them?, *J. Statist. Phys.* **108** (2002), 733–754.

[3] C. Bonatti, L. Diaz, and M. Viana, *Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Approach*, Springer, Berlin, 2005.