STRUCTURE AND f-DEPENDENCE OF THE A.C.I.M.
FOR A UNIMODAL MAP f OF MISIUREWICZ TYPE.

by David Ruelle*.

Abstract. *By using a suitable Banach space on which we let the transfer operator act, we make a detailed study of the ergodic theory of a unimodal map f of the interval in the Misiurewicz case. We show in particular that the absolutely continuous invariant measure ρ can be written as the sum of 1/square root spikes along the critical orbit, plus a continuous background. We conclude by a discussion of the sense in which the map $f \mapsto \rho$ may be differentiable.*

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0 Introduction.

This paper is part of an attempt to understand the smoothness of the map $f \mapsto \rho$ where (M, f) is a differentiable dynamical system and ρ an SRB measure. [For a general introduction to the problems involved, see for instance [2], [31]]. Smoothness has been established for uniformly hyperbolic systems (see [17], [21], [13], [22], [9]). In that case, one finds that the derivative of ρ with respect to f can be expressed in terms of the value at $\omega = 0$ of a *susceptibility function* $\Psi(e^{i\omega})$ which is holomorphic when the *complex frequency* ω satisfies Im $\omega > 0$, and meromorphic for Im $\omega >$ some negative constant. In the absence of uniform hyperbolicity, $f \mapsto \rho$ need not be continuous. Consider then a family $(f_\kappa)_{\kappa \in \mathbf{R}}$. A theorem of H. Whitney [29] gives general conditions under which, if ρ_κ is defined on $K \subset \mathbf{R}$, then $\kappa \mapsto \rho_\kappa$ extends to a differentiable function of κ on $\mathbf{R}$. Taking ρ_κ to be an SRB measure for f_κ, this gives a reasonable meaning to the differentiability of $\kappa \mapsto \rho_\kappa$ on K (as proposed in [24], see [20], [11] for a different application of Whitney's theorem), even though we start with a noncontinuous function $\kappa \mapsto \rho_\kappa$ on $\mathbf{R}$.

Using Whitney's theorem to study SRB states as proposed above is a delicate matter. A simple situation that one may try to analyze is when (M, f) is a unimodal map of the interval and ρ an absolutely continuous invariant measure (a.c.i.m.). [From the vast literature on this subject, let us mention [12], [15], [6], [7], [8], [28]]. A preliminary study of the Markovian case (*i.e.*, when the critical orbit is finite, see [23], [16]) shows that the susceptibility function $\Psi(\lambda)$ has poles for $|\lambda| < 1$, but is holomorphic at $\lambda = 1$. This study suggests that in non-Markovian situations Ψ may have a natural boundary separating $\lambda = 0$ (around which Ψ has a natural expansion) and $\lambda = 1$ (corresponding to $\omega = 0$). Misiurewicz [19] has studied a class of unimodal maps where the critical orbit stays away from the critical point, and he has proved the existence of an a.c.i.m. ρ for this class. This seems a good situation where one could study the dependence of ρ on f, as pointed out to the author by L.-S. Young.

To be specific, let us consider the standard example of unimodal maps f_c given by $f_c x = cx(1 - x)$ on $I = [0, 1]$. It is known that for many values of c, f_c has an a.c.i.m. ρ, and for a dense set it doesn't. So, $c \mapsto \rho$ cannot be differentiable in the usual sense. If c is restricted to a suitable set (say a set for which f_c is Misiurewicz), it might be differentiable in the sense of Whitney interpolation. Our evidence is that $c \mapsto \rho$ cannot be made differentiable in this way, but can nevertheless look differentiable numerically (and in experiments). An interesting feature will be the unexpected appearance of "acausal" singularities in the susceptibility function of unimodal maps.

A desirable starting point to study the dependence of the a.c.i.m. ρ on f is to have an operator $\mathcal{L}$ on a Banach space $\mathcal{A}$ such that $\mathcal{L}\rho = \rho$, and 1 is a simple isolated eigenvalue of $\mathcal{L}$. The main content of the present paper is the construction of $\mathcal{A}$ and $\mathcal{L}$ with the desired properties. Specifically we write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_2$ consists of *spikes*, *i.e.*, 1/square root singularities at points of the critical orbit, which are known to be present in ρ. We are thus able to prove that the a.c.i.m. ρ is the sum of a continuous background, and of the spikes (see Theorem 9, and the Remarks 16). Note that the construction of an operator $\mathcal{L}$ with a spectral gap had been achieved earlier by G. Keller and T. Nowicki [18], and by L.-S. Young [30] (our construction, in a more restricted setting, leads to stronger results).

We start studying the smoothness of the map $f \mapsto \rho$ by an informal discussion in Section 17. Theorem 19 proves the differentiability along topological conjugacy classes (which are codimension 1) and relates the derivative to the value at $\lambda = 1$ of a modified susceptibility function $\Psi(X, \lambda)$. [Following an idea of Baladi and Smania [5], it is plausible that differentiability in the sense of Whitney holds in directions tangent to a conjugacy class, see below]. Transversally to topological conjugacy classes the map $f \mapsto \rho$ is continuous, but appears not to be differentiable. While this nondifferentiability is not rigorously proved, it seems to be an unavoidable consequence of the fact that the weight of the n-th spike is roughly $\sim \alpha^{n/2}$ (for some $\alpha \in (0, 1)$) while its speed when f changes is $\sim \alpha^{-n}$. [See Section 16(c). In fact, for a smooth family (f_κ) restricted to values $\kappa \in K$ such that f_κ is in a suitable Misiurewicz class, the estimates just given for the weight and speed of the spikes suggest that $\kappa \to \rho_\kappa(A)$ for smooth A is $\frac{1}{2}$-Hölder, and nothing better, but we have not proved this]. Physically, let us remark that the spikes of high order n will be drowned in noise, so that discontinuities of the derivative of $f \mapsto \rho$ will be invisible.

Note that the susceptibility functions $\Psi(\lambda)$, $\Psi(X, \lambda)$ to be discussed may have singularities both for large $|\lambda|$ and small $|\lambda|$. [The latter singularities do not occur for uniformly hyperbolic systems, but show up for the unimodal maps of the interval in the Markovian case, as we have mentioned above. A computer search of such singularities is of interest [10]].

A study similar to that of the present paper has been made (Baladi [3], Baladi and Smania [5]) for piecewise expanding maps of the interval. In that case it is found that $f \mapsto \rho$ is not differentiable in general, but Baladi and Smania study the differentiability of $f \mapsto \rho$ along directions tangent to topological conjugacy classes (horizontal directions), not just for f restricted to a class. Note that our 1/square root spikes are replaced in the piecewise expanding case by jump discontinuities. This entails some serious differences, in particular, in the piecewise expanding case $\Psi(\lambda)$ is holomorphic for $|\lambda| < 1$.

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1 Setup.

Let I be a compact interval of $\mathbf{R}$, and $f : \mathbf{R} \to \mathbf{R}$ be real-analytic such that $fI \subset I$. We assume that there is c in the interior of I such that $f'(c) = 0$, $f'(x) > 0$ for $x < c$, $f'(x) < 0$ for $x > c$, and $f''(c) < 0$. Replacing I by a possibly smaller interval, we assume that $I = [a, b]$ where $b = fc$, $a = f^2c$, and $a < fa$.

We shall construct a *horseshoe* $H \subset (a, b)$, *i.e.*, a mixing compact hyperbolic set with a Markov partition for f. Following Misiurewicz [19] we shall assume that $fa \in H$.

Note that the existence of H with a Markov partition is a weak condition, but the Misiurewicz condition $f^N c \in H$ for some N is a strong condition. Note also that f is not assumed to have a Markov partition on $[a,b]$, which would mean that the critical point c is preperiodic ($f^N c$ periodic for some N). Our Misiurewicz condition is weaker than preperiodicity of c.

Under natural conditions to be discussed below we shall study the existence of an a.c.i.m. $\rho(x)\,dx$ for f, and its dependence on f.

Studying the smoothness of $f \mapsto \rho$ for unimodal maps f turns out to be a difficult problem. Our aim in the present paper will be to investigate new phenomena rather than to obtain very general results. In particular we make our life simpler by taking f to be real analytic rather than differentiable, and assume a Misiurewicz condition rather than Collet-Eckmann. Some other choices are made for the sake of simplicity, like $f^3 c \in H$ rather than $f^N c \in H$ with $N \geq 3$. Also we make a very geometric description of H in Section 2 in order to facilitate later discussion but , basically, Sections 2-6 just say that H is a mixing hyperbolic set with a Markov partition.

2 Construction of the set $H(u_1)$.

Let $u_1 \in [a,b]$ and define the closed set

$$H(u_1) = \{x \in [a,b] : f^n x \geq u_1 \text{ for all } n \geq 0\}$$

We have thus $fH(u_1) \subset H(u_1)$. Assuming that $H(u_1)$ is nonempty, let v be its minimum element, then $H(u_1) = H(v)$. [Since $v \in H(u_1)$ we have $v \geq u_1$, hence $H(v) \subset H(u_1)$. If $H(u_1)$ contained an element $w \notin H(v)$ we would have $H(u_1) \ni f^k w < v$ for some $k \geq 0$, in contradiction with the minimality of v]. Therefore we may (and shall) assume that $H(u_1) \ni u_1$. We shall also assume

$$a < u_1 < c, fa$$

(and $f^2 u_1 \neq u_1$, which will later be replaced by a stronger condition). There is $u_2 \in [a,b]$ such that $fu_2 = u_1$ and, since $u_1 < fa$, it follows that u_2 is unique and satisfies $c < u_2 < b$. We have $u_2 \in H(u_1)$ [because $u_2 > c > u_1$ and $fu_2 \in H(u_1)$] and if $x \in H(u_1)$ then $x \leq u_2$ [because $x > u_2$ implies $fx < u_1$]. Therefore, u_2 is the maximum element of $H(u_1)$. Let

$$V_0 = \{x \in [a,b] : fx > u_2\}$$

then $u_1 < V_0$ [because $x \leq u_1$ implies $fx \leq fu_1 \in H(u_1) \leq u_2$] and $V_0 < u_2$ [because $x \geq u_2$ implies $fx \leq fu_2 = u_1 < u_2$]. Thus we may write $V_0 = (v_1, v_2)$, with $u_1 < v_1 < c < v_2 < u_2$ [$u_1 \neq v_1$ because $f^2 u_1 \neq u_1$]. We have $v_1, v_2 \in H(u_1)$ [because $v_1, v_2 > u_1$ and $fv_1 = fv_2 = u_2 \in H(u_1)$].

Our assumptions ($H(u_1) \ni u_1$, $a < u_1 < c, fa$ and $f^2 u_1 \neq u_1$) and definitions give thus

$$H(u_1) \subset [u_1, v_1] \cup [v_2, u_2]$$

$$f[u_1, v_1] \subset [u_1, u_2] \qquad , \qquad f[v_2, u_2] = [u_1, u_2]$$

and

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = fH(u_1)$$

Let us say that the open interval $V_\alpha \subset [u_1, u_2]$ is of order n if f^n maps homeomorphically V_α onto $(v_1, v_2) = V_0$. We have thus

$$H(u_1) = [u_1, u_2] \backslash \cup \text{ all } V_\alpha$$

By induction on n we shall see that

$$[u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order } \leq n$$

is composed of disjoint closed intervals J, such that $f^n J \subset [u_1, v_1]$ or $[v_2, u_2]$ when $n > 0$, and the endpoints of $f^n J$ are u_1, u_2, v_1, v_2 or an image of these points by f^k with $k \leq n$. Assume that the induction assumption holds for n (the case of $n = 0$ is trivial) and let J be as indicated. Since $f^n J \subset [u_1, v_1]$ or $[v_2, u_2]$, f^{n+1} is monotone on J, and the endpoints of J are mapped by f^{n+1} outside of V_0 [because u_1, u_2, v_1, v_2 and their images by f^ℓ are in $H(u_1)$, hence $\notin (v_1, v_2)$]. The interval V_0 is thus either inside of $f^{n+1}J$ or disjoint from $f^{n+1}J$. Each V_α of order $n+1$ thus obtained is disjoint from other V_α of order $\leq n+1$, and the closed intervals $\tilde{J}$ in $[u_1, u_2] \backslash \cup$ the V_α of order $\leq n+1$, are such that the endpoints of $f^{n+1}\tilde{J}$ are u_1, u_2, v_1, v_2 or an image of these points by f^k with $k \leq n+1$, in agreement with our induction assumption.

We assume now that, for some $N \geq 0$, we have $f^{N+1}u_1 = u_1$ (take N smallest with this property), and we assume also that $(f^{N+1})'(u_1) > 0$. [$N = 0, 1$ cannot occur, in particular $f^2 u_1 \neq u_1$. Thus $N \geq 2$, with $f^N u_1 = u_2$, $f^{N-1}u_1 \in \{v_1, v_2\}$. Furthermore, $(f^{N-1})'(u_1) < 0$ if $f^{N-1}u_1 = v_1$, and $(f^{N-1})'(u_1) > 0$ if $f^{N-1}u_1 = v_2$, *i.e.*, $f^{N-1}(u_1+) = v_1-$ or v_2+].

Using the above assumption we now show that none of the intervals J in

$$[u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order } \leq n$$

is reduced to a point. We proceed by induction on n, assuming that $f^n J = [f^n x_1, f^n x_2]$, where $f^n x_1 < f^n x_2$ and $f^n x_1$ is of the form v_2, u_1 or $f^\ell u_1$ with $(f^\ell)'(u_1) > 0$ while $f^n x_2$ is of the form v_1, u_2 or $f^\ell u_2$ with $(f^\ell)'(u_2) > 0$. Therefore the lower limit of $f^{n+1}J$ is of the form $f^m u_1$ with $(f^m)'(u_1) > 0$ while the upper limit is of the form $f^m u_2$ with $(f^m)'(u_2) > 0$. If

$$f^{n+1}J \supset (v_1, v_2)$$

so that a new V_α of order $n+1$ is created, the set $f^{n+1}J \backslash (v_1, v_2)$ consists of two closed intervals, and one of them can be reduced to a point only if $f^m u_1 = v_1$ with $(f^m)'(u_1) > 0$ or if $f^m u_2 = v_2$ with $(f^m)'(u_2) > 0$. So, either $f^{m+2}u_1 = u_1$ with $(f^{m+2})'(u_1) < 0$, or $f^{m+1}u_2 = u_2$ with $(f^{m+1})'(u_2) < 0$ hence $f^{m+1}u_1 = u_1$ with $(f^{m+1})'(u_1) < 0$, in contradiction with our assumption that $(f^{N+1})'(u_1) > 0$.

3 Consequences.

(No isolated points)

$H(u_1)$ is obtained from $[u_1, u_2]$ by taking away successively intervals V_α of increasing order. A given $x \in H(u_1)$ will, at each step, belong to some small closed interval J, and the endpoints of J will not be removed in later steps, so that x cannot be an isolated point: $H(u_1)$ *has no isolated points*.

(Markov property)

Our assumption $f^{N+1}u_1 = u_1$ implies that, for $n = 1, \ldots, N-1$, the point $f^n u_1$ is one of the endpoints of an interval V_α of order $N-1-n$, which we call V_{N-1-n}. These open intervals V_k are disjoint, and their complement in $[u_1, u_2]$ consists of N intervals $U_1, \ldots, U_N$. Each U_i is closed, nonempty, and not reduced to a point. Furthermore, each U_i (for $i = 1, \ldots, N$) is mapped by f homeomorphically to a union of intervals U_j and V_k: this is what we call *Markov property*.

We impose now the following condition:

4 Hyperbolicity.

There are constants $A > 0, \alpha \in (0,1)$ *such that if* $x, fx, \ldots, f^{n-1}x \in [u_1, v_1] \cup [v_2, u_2]$, *then*

$$\left|\frac{d}{dx} f^n x\right|^{-1} < A\alpha^n$$

Note that in [19] hyperbolicity is automatic, because f is assumed to have a negative Schwarzian derivative.

We label the intervals $U_1, \ldots, U_N$ from left to right, so that u_1 is the lower endpoint of U_1, and u_2 the upper endpoint of U_N. Define also an oriented graph with vertices U_j and edges $U_j \to U_k$ when $fU_j \supset U_k$. Write $U_{j_0} \stackrel{\ell}{\Longrightarrow} U_{j_\ell}$ if $U_{j_0} \to U_{j_1} \to \cdots \to U_{j_\ell}$, and $U_j \Longrightarrow U_k$ if $U_j \stackrel{\ell}{\Longrightarrow} U_k$ for some $\ell > 0$.

5 Lemma (mixing).

(a) For each U_j *there is* $r \geq 0$ *such that* $U_j \stackrel{r+3}{\Longrightarrow} U_1$.

(b) If there is $s > 0$ *such that* $U_1 \stackrel{s}{\Longrightarrow} U_1$ *and* $U_1 \stackrel{s}{\Longrightarrow} U_N$, *then* $U_1 \stackrel{s}{\Longrightarrow} U_k$ *for* $k = 1, \ldots, N$.

(c) If there is $s > 0$ *such that* $U_j \stackrel{s}{\Longrightarrow} U_k$ *for all* $U_j, U_k \in \{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$, *then* $U_j \stackrel{s}{\Longrightarrow} U_k$ *for all* $U_j, U_k \in \{U_1 \ldots, U_N\}$, *and we say that* $H(u_1)$ *is* mixing.

(d) In particular if $N+1$ *is a prime, then* $H(u_1)$ *is mixing.*

(e) Let $u_1 < \tilde{u}_1 < c, fa$, *and suppose that* $f^{\tilde{N}+1}\tilde{u}_1 = \tilde{u}_1$, $(f^{\tilde{N}+1})'(u_1) > 0$. *Then if* $H(u_1)$ *is mixing, so is* $H(\tilde{u}_1)$.

(a) The interval U_j is contained in either $[u_1, v_1]$ or $[v_2, u_2]$. Let the same hold for the successive images up to $f^r U_j$, but $f^{r+1}U_j \ni c$ [hyperbolicity and the fact that U_j is not reduced to a point imply that r is finite]. Then $U_j \overset{r+1}{\Longrightarrow} U_k$ with $U_k \ni v_1$ or v_2, hence $U_k \overset{2}{\Longrightarrow} U_1$ and $U_j \overset{r+3}{\Longrightarrow} U_1$.

(b) The U_j such that $U_1 \overset{s}{\Longrightarrow} U_j$ form a set of consecutive intervals and, since this set contains U_1 and U_N by assumption, it contains all U_j for $j = 1, \ldots, N$.

(c) By assumption, $U_1 \overset{s}{\Longrightarrow} U_1$ and $U_1 \overset{s}{\Longrightarrow} U_N$, so that $U_1 \overset{s}{\Longrightarrow} U_k$ for $k = 1, \ldots, N$ by (b). Therefore, $\{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\} = \{U_1, \ldots, U_N\}$ by (a), and thus $U_j \overset{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_1 \ldots, U_N\}$.

(d) The *transitive* set $\{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$ decomposes into n disjoint subsets $S_0, \ldots, S_{n-1}$ such that $S_0 \overset{1}{\Longrightarrow} S_1 \overset{1}{\Longrightarrow} \cdots \overset{1}{\Longrightarrow} S_{n-1} \overset{1}{\Longrightarrow} S_0$ and there is $s > 0$ such that $U_j \overset{sn}{\Longrightarrow} U_k$ for all $U_j, U_k \in S_m$, where $m = 0, \ldots, n-1$. We may suppose that $U_1 \in S_0$, and therefore if $U_{(k)}$ denotes the interval containing $f^k u_1$ we have $U_{(k)} \in S_{(k)}$ where $(k) = k(\text{mod } n)$. Therefore $N+1$ is a multiple of n, where $n \leq N < N+1$. In particular, if $N+1$ is prime, then $n = 1$, and $U_j \overset{s}{\Longrightarrow} U_k$ for all $U_j, U_k \in \{U_j : U_1 \Longrightarrow U_j \Longrightarrow U_1\}$, so that (c) can be applied.

(e) Since $H(\tilde{u}_1)$ is a compact subset of $H(u_1)$, without isolated points, the fact that $H(u_1)$ is mixing implies that $H(\tilde{u}_1)$ is mixing. $\square$

6 Horseshoes.

Note that we have

$$H(u_1) = \{x \in [u_1, u_2] : f^n x \notin V_0 \text{ for all } n \geq 0\} = \cap_{n \geq 0} f^{-n}([u_1, u_2] \backslash V_0)$$

The sets $U_i \cap H(u_1)$ form a *Markov partition* of $H(u_1)$, *i.e.*, $f(U_i \cap H(u_1))$ is a finite union of sets $U_j \cap H(u_1)$.

A set $H = H(u_1)$ as constructed in Section 2, with the hyperbolicity and mixing conditions will be called a *horseshoe*. A horseshoe is thus a mixing hyperbolic set with a Markov partition.

Remember that the open interval $V_\alpha \subset [u_1, u_2]$ is of order n if f^n maps V_α homeomorphically onto $V_0 = (v_1, v_2)$, and let $|V_\alpha|$ be the length of V_α. Hyperbolicity has the following consequence.

7 Lemma (a consequence of hyperbolicity).

There are constants $B > 0$, $\beta \in (0, 1)$ such that

$$\sum_{\alpha : \text{order } V_\alpha = n} |V_\alpha| \leq B\beta^n$$

It suffices to prove that

$$\text{Lebesgue meas. } ([u_1, u_2] \backslash \cup \text{ the } V_\alpha \text{ of order} \leq n) \leq G\beta^n$$

[incidentally, this shows that $H(u_1)$ has Lebesgue measure 0]. The above inequality is an immediate consequence of hyperbolicity: by a smooth conjugacy we may take $A = 1$ in the definition 4 of hyperbolicity, then if we remove the V_α of successive orders $n = 1, 2, \ldots$ we loose at each step a fraction $> \gamma$ of the length for some $\gamma \in (0, 1)$. $\square$

8 Remark (the set $\tilde{H}$).

Starting from the horseshoe $H = H(u_1)$ we can, by increasing u_1 to $\tilde{u}_1$ such that $\tilde{u}_1 < c, fa$, obtain a set $\tilde{H} = H(\tilde{u}_1) \subset H$ such that $\tilde{u}_1 \in \tilde{H}$ and the distance of $\tilde{H}$ to $\{u_1, u_2, v_1, v_2\}$ is $\geq \epsilon > 0$. [In fact, using our hyperbolicity assumption we can arrange that there is $\tilde{N}$ such that $f^{\tilde{N}+1}\tilde{u}_1 = \tilde{u}_1$, $(f^{\tilde{N}+1})'(\tilde{u}_1) > 0$. In that case $\tilde{H}$ is mixing (Lemma 5(e)) and therefore again a horseshoe].

9 Theorem.

Let $H = H(u_1)$ be a horseshoe, suppose that $fa = f^2 b \in H$, and that $\{f^n b : n \geq 0\}$ has a distance $\geq \epsilon > 0$ from $\{u_1, u_2, v_1, v_2\}$. Then f has a unique a.c.i.m. $\rho(x)\,dx$. Furthermore

$$\rho(x) = \phi(x) + \sum_{n=0}^{\infty} C_n \psi_n(x)$$

The function ϕ is continuous on $[a, b]$, with $\phi(a) = \phi(b) = 0$. For $n \geq 0$ we shall choose $w_n \in \{u_1, u_2, v_1, v_2\}$ with $(w_n - c)(c - f^n b) < 0$ and let θ_n be the characteristic function of $\{x : (w_n - x)(x - f^n b) > 0\}$. Then, the above constants C_n and spikes *ψ_n are defined by*

$$C_n = \phi(c)|\frac{1}{2} f''(c) \prod_{k=0}^{n-1} f'(f^k b)|^{-1/2}$$

$$\psi_n(x) = \frac{w_n - x}{w_n - f^n b} \cdot |x - f^n b|^{-1/2}\,\theta_n(x)$$

[The condition that $\{f^n b : n \geq 0\}$ has distance $\geq \epsilon$ from $\{u_1, u_2, v_1, v_2\}$ is achieved, according to Remark 8, by taking $\epsilon \leq |u_1 - a|, |u_2 - b|$, and $f^2 b \in \tilde{H}$. Note also that $\psi_n(c) = 0$, so that $\phi(c) = \rho(c)$. Other choices of ψ_n can be useful, with the same singularity at $f^n b$, but greater smoothness at w_n and/or satisfying $\int dx\,\psi_n(x) = 0$].

10 Analysis.

We analyze the problem before starting the proof. We may define a *transfer operator* $\mathcal{L}_{(1)}$ on L^1 so that $\mathcal{L}_{(1)}\phi$ is the density of the image $f^*(\phi(x)\,dx)$ by f of $\phi(x)\,dx$ (in particular $\mathcal{L}_{(1)}\rho = \rho$). Near c we have

$$y = fx = b - A(x - c)^2 + \text{h.o.}$$

with $A = -f''(c)/2 > 0$, hence $x - c = \pm\big((b - y)/A\big)^{1/2} + O(b - y)$. Therefore, writing $U = \rho(c)/\sqrt{A}$, the density $(\mathcal{L}_{(1)}\rho)(x)$ of $f^*(\rho(x)dx)$ has, near b, a singularity

$$\frac{U}{\sqrt{(b - x)}} + O(\sqrt{b - x})$$

and, near a, a singularity

$$\frac{U}{\sqrt{-f'(b)(x-a)}} + O(\sqrt{x-a})$$

To deal with the general case of the singularity at $f^n b$, define $s_n = -\mathrm{sgn}\prod_{k=0}^{n-1} f'(f^k b)$, so that

$$\prod_{k=0}^{n-1} f'(f^k b) = -s_n U^2 C_n^{-2}$$

The density of $f^{n*}(\rho(x)dx)$ has then, near $f^n b$, a singularity given when $s_n(x - f^n b) > 0$ by

$$\frac{U}{\sqrt{(\prod_{k=0}^{n-1}|f'(f^k b)|)|x - f^n b|}} + O(\sqrt{|x - f^n b|})$$

$$= \frac{U}{\sqrt{-(x - f^n b)\prod_{k=0}^{n-1} f'(f^k b)}} + O(\sqrt{|x - f^n b|})$$

$$= \frac{C_n}{\sqrt{s_n(x - f^n b)}} + O(\sqrt{s_n(x - f^n b)})$$

and by 0 when $s_n(x - f^n b) < 0$.

We let now $w_0 = u_2$ and, for $n \geq 0$, define $w_{n+1} \in \{u_1, u_2, v_1, v_2\}$ inductively by:

$$(w_{n+1} - c)(f^{n+1}b - c) > 0 \qquad , \qquad (w_{n+1} - f^{n+1}b)(fw_n - f^{n+1}b) > 0$$

We have thus $w_0 = u_2, w_1 = u_1$, and in general

$$w_n \in \{u_1, u_2, v_1, v_2\} \quad , \quad (w_n - c)(f^n b - c) > 0 \quad , \quad s_n(w_n - f^n b) > 0 \quad , \quad |w_n - f^n b| \geq \epsilon$$

The above considerations show that the singularity expected near $f^n b$ for the density $\rho(x) = (\mathcal{L}^n_{(1)}\rho)(x)$ is also represented by

$$(1 - \frac{x - f^n b}{w_n - f^n b}) \cdot \frac{C_n}{\sqrt{s_n(x - f^n b)}}\,\theta_n(x)$$

$$= C_n\,\frac{w_n - x}{w_n - f^n b}\,|x - f^n b|^{-1/2}\,\theta_n(x) = C_n\psi_n(x)$$

in agreement with the claim of the theorem.

11 Lemma.

Write

$$f(\psi_n(x)dx) = \tilde\psi_{n+1}(x)dx \qquad , \qquad \tilde\psi_{n+1} = |f'(f^n b)|^{-1/2}\psi_{n+1} + \chi_n$$

Then, for $n \geq 0$, the χ_n are continuous of bounded variation on $[a,b]$, with $\chi_n(a) = \chi_n(b) = 0$, and the $\mathrm{Var}\,\chi_n = \int_a^b |d\chi_n/dx|dx$ are bounded uniformly with respect to n. Furthermore, if $n \geq 1$ and $V_\alpha \subset \mathrm{supp}\chi_n$, then $\chi_n|V_\alpha$ extends to a holomorphic function $\chi_{n\alpha}$ in a complex neighborhood D_α of the closure of V_α in $\mathbf{R}$ (further specified in Section 12), with the $|\chi_{n\alpha}|$ uniformly bounded.

The case $n = 0$ can be handled by inspection, and we shall assume $n \geq 1$. We let

$$I_n = \begin{cases} (fa, b) & \text{if} \quad f^n b \in [a, c) \\ (a, b) & \text{if} \quad f^n b \in (c, b) \end{cases}$$

And define $f_n^{-1} : I_n \mapsto (a,b)$ to be the inverse of f restricted respectively to (a,c) or (c,b) in the two cases above. We have then

$$\tilde{\psi}_{n+1}(x) = \frac{\psi_n(f_n^{-1}x)}{|f'(f_n^{-1}x)|}$$

Since $n \geq 1$, the region of interest $f\mathrm{supp}\psi_n \cup \mathrm{supp}\psi_{n+1}$ is $\subset [u_1, u_2] \subset (a,b)$, and we have

$$f_n^{-1}x - f^n b = (x - f^{n+1}b)A_n(x)$$

where A_n is real analytic and $A_n(f^{n+1}b) = (f'(f^n b))^{-1}$. Therefore we may write

$$\frac{1}{f_n^{-1}x - f^n b} = \frac{f'(f^n b)}{x - f^{n+1}b}(1 + (x - f^{n+1}b)\tilde{A}_n(x))$$

$$\frac{1}{f'(f_n^{-1}x)} = \frac{1}{f'(f^n b)}(1 + (x - f^{n+1}b))\tilde{B}_n(x)$$

$$\frac{w_n - f_n^{-1}x}{w_n - f^n b} = 1 + (x - f^{n+1}b)\tilde{C}_n(x)$$

and since

$$\psi_n(f_n^{-1}x) = \theta_n(f_n^{-1}x)\left|\frac{w_n - f_n^{-1}x}{w_n - f^n b}\right| \cdot |f_n^{-1}x - f^n b|^{-1/2}$$

we find

$$\tilde{\psi}_{n+1}(x) = \frac{\theta_n(f_n^{-1}x)|f'(f^n b)|^{-1/2}}{\sqrt{|x - f^{n+1}b|}}\big(1 + (x - f^{n+1}b)\tilde{D}_n(x)\big)$$

with $\tilde{D}_n$ real analytic. Note that $\tilde{\psi}_{n+1}$ and $|f'(f^n b)|^{-1/2}\psi_{n+1}$ have the same singularity at $f^{n+1}b$. It follows readily that $\tilde{\psi}_{n+1} - |f'(f^n b)|^{-1/2}\psi_{n+1}$ is a continuous function χ_n vanishing at the endpoints of its support, and bounded uniformly with respect to n. It is easy to see that $\mathrm{Var}\,\chi_n$ is bounded uniformly in n. The extension of $\chi_n|V_\alpha$ to holomorphic $\chi_{n\alpha}$ in D_α is also handled readily (see Section 12 for the description of the D_α). ∎

12 The operator $\mathcal{L}$ and the space $\mathcal{A}$.

We have $f(\rho(x)\,dx) = (\mathcal{L}_{(1)}\rho)(x)\,dx$, where the transfer operator $\mathcal{L}_{(1)}$ on $L^1(a,b)$ is defined by

$$\mathcal{L}_{(1)}\rho = \sum_{\pm} \frac{\rho \circ f_{\pm}^{-1}}{|f' \circ f_{\pm}^{-1}|}$$

and we have denoted by

$$f_{-}^{-1} : [fa, b] \mapsto [a, c] \qquad \text{and} \qquad f_{+}^{-1} : [a, b] \mapsto [c, b]$$

the branches of the inverse of f. The invariance of $\rho(x)\,dx$ under f is thus expressed by

$$\rho = \mathcal{L}_{(1)}\rho$$

We shall look for a solution of this equation in a Banach space $\mathcal{A}$ defined below. Roughly speaking, $\mathcal{A}$ consists of functions

$$\phi + \sum_{n=0}^{\infty} c_n \psi_n$$

where the ψ_n are defined in the statement of Theorem 9, and $\phi : [a, b] \to \mathbf{C}$ is a less singular rest with certain analyticity properties.

Remember that we may write

$$[a, b] = H \cup [a, u_1) \cup (u_2, b] \cup \text{the } V_\alpha \text{ of all orders} \geq 0$$

We have (see Remark 8)

$$\operatorname{clos}[a, u_1) \subset [a, \tilde{u}_1) \qquad , \qquad \operatorname{clos}(u_2, b] \subset (\tilde{u}_2, b] \qquad , \qquad \operatorname{clos} V_0 \subset \tilde{V}_0$$

where $\tilde{u}_2$ and $\tilde{V}_0 = (\tilde{v}_1, \tilde{v}_2)$, are defined for $\tilde{H}$ as u_2 and V_0 were defined for H. It is convenient to define $V_{-1} = (u_2, b]$ and $V_{-2} = [a, u_1)$ (of order -1 and -2 respectively) so that

$$[a, b] = H \cup \text{the } V_\alpha \text{ of all orders} \geq -2$$

We also define $\tilde{V}_{-1} = (\tilde{u}_2, b], \tilde{V}_{-2} = [a, \tilde{u}_1)$. We let now $\tilde{V}_\alpha$ denote the unique interval in $[a, b]\backslash\tilde{H}$ such that $V_\alpha \subset \tilde{V}_\alpha$. Note that the map $V_\alpha \mapsto \tilde{V}_\alpha$ is not injective!

For each V_α of order ≥ 0 we may choose an open set $D_\alpha \subset \mathbf{C}$ such that

$$\tilde{V}_\alpha \supset D_\alpha \cap \mathbf{R} \supset \operatorname{clos} V_\alpha$$

and, if $fV_\beta = V_\alpha$ of order ≥ 0, $fD_\beta \supset \operatorname{clos} D_\alpha$ [we have here denoted by $\operatorname{clos} V_\alpha$ the closure of V_α in $\mathbf{R}$, and by $\operatorname{clos} D_\alpha$ the closure of D_α in $\mathbf{C}$]. Let also R_a, R_b be two-sheeted Riemann surfaces, branched respectively at a, b, with natural projections $\pi_a, \pi_b : R_a, R_b \to \mathbf{C}$. We may choose open sets $D_{-1}, D_{-2} \subset \mathbf{C}$ such that, for $\alpha = -1, -2$,

$$\tilde{V}_\alpha \supset D_\alpha \cap \{x \in \mathbf{R} : a \leq x \leq b\} \supset \operatorname{clos} V_\alpha$$

and f extends to holomorphic maps $\tilde{f}_{-1} : D_0 \to R_b, \tilde{f}_{-2} : (\tilde{f}_{-1}D_0) \to R_a$ such that $\tilde{f}_{-1}D_0 \supset \pi_b^{-1}\text{clos}\, D_{-1}, \tilde{f}_{-2}\pi_b^{-1}D_{-1} \supset \pi_a^{-1}\text{clos}\, D_{-2}$. [We shall say that $\tilde{f}_{-1}$ sends (v_1, c) to the *upper* sheet of R_b and (c, v_2) to the *lower* sheet of R_b; $\tilde{f}_{-2}$ sends the upper (lower) sheet of R_b to the upper (lower) sheet of R_a].

We come now to a precise definition of the complex Banach space $\mathcal{A}$. We write $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where the elements of $\mathcal{A}_1$ are of the form (ϕ_α) and the elements of $\mathcal{A}_2$ of the form (c_n). Here the index set of the ϕ_α is the same as the index set of the intervals V_α (of order ≥ -2); the index n of the $c_n \in \mathbf{C}$ takes the values $0, 1, \ldots$ [the c_n should not be confused with the critical point c]. We assume that ϕ_α is a holomorphic function in D_α when V_α is of order ≥ 0, while ϕ_{-1}, ϕ_{-2} are holomorphic on $\pi_b^{-1}D_{-1}, \pi_a^{-1}D_{-2}$ and, for all α, $||\phi_\alpha|| = \sup_{z \in D_\alpha} |\phi_\alpha(z)| < \infty$.

[We shall later consider a function $\phi : [a, b] \to \mathbf{C}$ such that $\phi|V_\alpha = \phi_\alpha|V_\alpha$ when V_α is of order ≥ 0. For $x \in V_{-1}$ we shall require $\phi(x) = \Delta\phi(x) = \phi_{-1}(x^+) - \phi_{-1}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_b on the upper (lower) sheet of $\pi_b^{-1}D_{-1}$; for $x \in V_{-2}$ we shall require $\phi(x) = \Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_a on the upper (lower) sheet of $\pi_a^{-1}D_{-2}$. But at this point we discuss an operator $\mathcal{L}$ on $\mathcal{A}$ instead of the transfer operator $\mathcal{L}_{(1)}$ acting on functions $\phi + \sum_n c_n\psi_n$].

Let γ, δ be such that $1 < \gamma < \beta^{-1}, 1 < \delta < \alpha^{-1/2}$ with β as in Lemma 7 and α as in the definition of hyperbolicity (Section 4). We write

$$||(\phi_\alpha)||_1 = \sup_{n \geq -2} \gamma^n \sum_{\alpha:\text{order}\, V_\alpha = n} |V_\alpha|.||\phi_\alpha|| \qquad , \qquad ||(c_n)||_2 = \sup_{n \geq 0} \delta^n |c_n|$$

and, for $\Phi = ((\phi_\alpha), (c_n))$, we let $||\Phi|| = ||(\phi_\alpha)||_1 + ||(c_n)||_2$. We let then $\mathcal{A}_1, \mathcal{A}_2$ be the Banach spaces of sequences $(\phi_\alpha), (c_n)$ as above , such that the norms $||(\phi_\alpha)||_1, ||(c_n)||_2$ are finite. We shall define $\mathcal{L}$ on $\mathcal{A}$ such that $\mathcal{L}\Phi = \tilde{\Phi}$. We first describe what contribution each ϕ_α or c_n gives to $\tilde{\Phi}$ and then we shall check that this is a consistent description of an element $\tilde{\Phi}$ of $\mathcal{A}$.

(i) $\phi_\beta \Rightarrow \hat{\phi}_{\beta\alpha} = \dfrac{\phi_\beta}{|f'|} \circ (f|D_\beta)^{-1}$ in D_α if order $\beta > 0$ and $fV_\beta = V_\alpha$

[we have here denoted by $|f'|$ the holomorphic function $\pm f'$ such that $\pm f' > 0$ for real argument, we shall use the same notation in (ii)-(vi) below].

(ii) $\phi_0 \Rightarrow \left(\hat{c}_0 = C_0\phi_0(c)\, , \, \hat{\phi}_{-1} = \pm\dfrac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - C_0\phi_0(c)(\pm\dfrac{1}{2}\psi_0 \circ \pi_b) \quad \text{in } \pi_b^{-1}D_{-1}\right)$

where the signs $\pm$ correspond to the upper/lower sheet of $\pi_b^{-1}D_{-1}$. We claim that $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1}D_{-1}$ as the difference of two meromorphic functions with a simple pole at the branch point b, with the same residue. To see this we uniformize $\pi_b^{-1}D_{-1}$ by the map $u \mapsto b - u^2$. We have thus to express $\pm\dfrac{\phi_0}{|f'|}(c + x) = \dfrac{\phi_0}{f'}(c + x)$ in terms of u where $c + x = \tilde{f}_{-1}^{-1}(b - u^2)$ or $u = \sqrt{b - \tilde{f}_{-1}(c + x)}$ which gives a meromorphic function with a simple pole $1/2\sqrt{A}u$. Since $\pm C_0\phi_0(c)\psi_0(b - u^2)$ is meromorphic with the same simple pole, $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1}D_{-1}$.

(iii) $\phi_{-1} \Rightarrow \hat{\phi}_{-2} = \dfrac{\phi_{-1}}{|f'|} \circ \tilde{f}_{-2}^{-1} \qquad \text{in } \pi_a^{-1} D_{-2}$.

(iv) $\phi_{-2} \Rightarrow \hat{\phi}_\alpha = \dfrac{\Delta\phi_{-2}}{f'} \circ f^{-1} \qquad \text{in } D_\alpha \text{ if } f(a,u_1) \supset V_\alpha \text{ , } 0 \text{ otherwise}$

[we have written $\Delta\phi_{-2}(x) = \phi_{-2}(x^+) - \phi_{-2}(x^-)$ where $x^+(x^-)$ is the preimage of x by π_a on the upper (lower) sheet of $\pi_a^{-1}D_{-2}$].

(v) $c_0 \;\Rightarrow\; \Big(\hat{c}_1 \;=\; |f'(b)|^{-1/2} c_0\,,\; \chi_0 \;=\; \pm\dfrac{1}{2} c_0 \big(\dfrac{\psi_0}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2}\psi_1 \circ \pi_a\big)$ in $\pi_a^{-1}D_{-2}\Big)$ where the sign $\pm$corresponds to the upper/lower sheet of $\pi_a^{-1}D_{-2}$.

(vi) $c_n \Rightarrow \Big(\hat{c}_{n+1} = |f'(f^n b)|^{-1/2} c_n\,,\; \chi_{n\alpha} = c_n \big[\dfrac{\psi_n}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2}\psi_{n+1}\big]$

$\quad$ in D_α if $V_\alpha \subset \{x : \theta_n(f_n^{-1}x) > 0\}\,, 0 \text{ otherwise}\Big)$

if $n \geq 1$.

We may now write

$$\tilde{\Phi} = ((\tilde{\phi}_\alpha), (\tilde{c}_n))$$

where

$\tilde{\phi}_{-2} = \hat{\phi}_{-2} + \chi_0 \qquad$ (see (iii),(v))
$\tilde{\phi}_{-1} = \hat{\phi}_{-1} \qquad$ (see(ii))
$\tilde{\phi}_\alpha = \displaystyle\sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_\alpha + \sum_{n\geq 1} \chi_{n\alpha}$ if order $\alpha \geq 0 \qquad$ (see (i),(iv),(vi))
$\tilde{c}_0 = \hat{c}_0 \qquad$ (see (ii))
$\tilde{c}_1 = \hat{c}_1 \qquad$ (see (v))
$\tilde{c}_n = \hat{c}_n \qquad$ for $n > 1 \qquad$ (see (vi))

Note that, corresponding to the decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, we have

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_0 + \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$$

where

$\mathcal{L}_0(\phi_\alpha) = (\sum_{\beta: fV_\beta = V_\alpha} \hat{\phi}_{\beta\alpha})$
$\mathcal{L}_1(\phi_\alpha) = (\hat{\phi}_\alpha)$
$\mathcal{L}_2(c_n) = (\chi_0, (\sum_{n\geq 1} \chi_{n\alpha})_{\alpha > -1})$
$\mathcal{L}_3(\phi_\alpha) = (\hat{c}_0, (0)_{n>0})$
$\mathcal{L}_4(c_n) = (0, (\hat{c}_n)_{n>0})$

Holomorphic functions in D_α are defined by (i),(iv),(vi) when order $\alpha \geq 0$, and in $\pi_b^{-1}D_{-1}$, $\pi_a^{-1}D_{-2}$ by (ii),(iii),(v). Using Lemma 7, one sees that $\mathcal{L}_0, \mathcal{L}_1$ are bounded $\mathcal{A}_1 \to \mathcal{A}_1$. Using Lemma 11, one sees that $\mathcal{L}_3$ is bounded $\mathcal{A}_2 \to \mathcal{A}_1$. It is also readily seen that $\mathcal{L}_2, \mathcal{L}_4$ are bounded, so that $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is bounded.

13 Theorem (structure of $\mathcal{L}$).

With our definitions and assumptions, the bounded operator $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is a compact perturbation of $\mathcal{L}_0 \oplus \mathcal{L}_4$; its essential spectral radius is $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$.

Since $fa \in \tilde{H}$, we may assume that $f(a, u_1) \supset V_\alpha$ implies $f(D_{-2}\backslash\text{negative reals}) \supset \operatorname{clos} D_\alpha$. Therefore, $\phi_{-2} \mapsto \hat{\phi}_\alpha | D_\alpha$ is compact. For N positive integer, define the operator $\mathcal{L}_{N1}$ such that

$$\mathcal{L}_{N1}(\phi_\alpha) = \frac{\Delta\phi_{-2}}{f'} \circ f^{-1} \qquad \text{in } D_\alpha \text{ if } f(a, u_1) \supset V_\alpha \text{ and order } \alpha > N \text{ , 0 otherwise}$$

Then $\mathcal{L}_1$ is a perturbation of $\mathcal{L}_{N1}$ by a compact operator and, using Lemma 7, we see that

$$||\mathcal{L}_{N1}(\phi_\alpha)||_1 \leq C \sup_{n>N} \gamma^n \beta^n \to 0 \qquad \text{when } N \to \infty$$

We can write $\mathcal{L}_2 = \mathcal{L}_{N2} +$ finite range, where

$$\mathcal{L}_{N2}(c_n) = (0, 0, (\sum_{n \geq N} \chi_{n\alpha})_{\alpha \geq 0})$$

Using Lemma 11 we find a bound $|| \sum_{n \geq N} \chi_{n\alpha} || \leq C' \delta^N$ and, using Lemma 7,

$$||\mathcal{L}_{N2}||_{\mathcal{A}_2 \to \mathcal{A}_1} \leq C'' \delta^N \to 0 \qquad \text{when } N \to \infty$$

The operator $\mathcal{L}_3$ has one-dimensional range. Therefore $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are compact operators, and the essential spectral radius of $\mathcal{L}$ is the max of the essential spectral radius of $\mathcal{L}_0$ on $\mathcal{A}_1$ and $\mathcal{L}_4$ on $\mathcal{A}_2$.

The spectral radius of $\mathcal{L}_4$ is

$$\leq ||\mathcal{L}_4^N||^{1/N} \leq \left(\delta^N C''' \sup_{\ell \geq 0} \prod_{k=0}^{N-1} |f'(f^{k+\ell} b)|^{-1/2}\right)^{1/N} \qquad \text{with limit} < \delta\alpha^{1/2} \text{ when } N \to \infty$$

The essential spectral radius of $\mathcal{L}_0$ is

$$\leq \lim_{N \to \infty} \frac{\sup_{n \geq N} \gamma^n \sum_{\alpha : \text{order} V_\alpha = n} |V_\alpha|.|| \sum_{\beta : f V_\beta = V_\alpha} \hat{\phi}_{\beta\alpha}||}{\sup_{n \geq N} \gamma^{n+1} \sum_{\beta : \text{order} V_\beta = n+1} |V_\beta|.||\phi_\beta||}$$

$$\leq \gamma^{-1} \lim_{\text{order} V_\alpha \to \infty} \frac{|V_\alpha|.|| \sum_{\beta : f V_\beta = V_\alpha} \hat{\phi}_{\beta\alpha}||}{\sum_{\beta : f V_\beta = V_\alpha} |V_\beta|.||\phi_\beta||} = \gamma^{-1}$$

In fact, no eigenvalue of $\mathcal{L}_0$ can be $> \gamma^{-1}$, so the spectral radius of $\mathcal{L}_0$ acting on $\mathcal{A}_1$ is $\leq \gamma^{-1}$. The essential spectral radius of $\mathcal{L}$ is thus $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$. $\square$

[Note also that when $\gamma \to \beta^{-1}, \delta \to 1$, we have $\max(\gamma^{-1}, \delta\alpha^{1/2}) \to \max(\beta, \alpha^{1/2})$].

14 The eigenvalue 1 of $\mathcal{L}$.

Let the map $\Delta : \mathcal{A}_1 \to L^1(a,b)$ be such that $\Delta(\phi_\alpha)|(a,u_1) = \Delta\phi_{-2}$, $\Delta(\phi_\alpha)|(u_2,b) = \Delta\phi_{-1}$, and $\Delta(\phi_\alpha)|V_\beta = \phi_\beta$ if order $\beta \geq 0$. We also define $w : \mathcal{A} \to L^1(a,b)$ by $w((\phi_\alpha),(c_n)) = \Delta(\phi_\alpha) + \sum_{n=0}^{\infty} c_n\psi_n$ and check readily that

$$w\mathcal{L}\Phi = \mathcal{L}_{(1)}w\Phi$$

If $\lambda^0 \neq 0$ is an eigenvalue of $\mathcal{L}$, and $\Phi^0 = ((\phi^0_\alpha),(c^0_n))$ is an eigenvector to this eigenvalue, we have $w\Phi^0 \neq 0$ [because $w\Phi^0 = 0$ implies $\phi^0_0 = 0$, hence $\phi^0_{-1} = 0, \phi^0_{-2} = 0$, and $(c^0_n) = 0$; then $\Delta(\phi^0_\alpha) = 0$, so $\phi^0_\alpha = 0$ when order $\alpha \geq 0$, *i.e.*, $\Phi_0 = 0$]. Therefore

$$\lambda^0 w\Phi^0 = \mathcal{L}_{(1)}(w\Phi^0)$$

$$|\lambda^0|\int_a^b |w\Phi^0| = \int_a^b |\mathcal{L}_{(1)}(w\Phi^0)| \leq \int_a^b \mathcal{L}_{(1)}|w\Phi^0| = \int_a^b |w\Phi^0|$$

hence $|\lambda^0| \leq 1$.

If $c^0_0 = 0$, then $(c^0_n) = 0$, and λ^0 is thus an eigenvalue of $\mathcal{L}_0$ acting on $\mathcal{A}_1$, so that $|\lambda^0| \leq \gamma^{-1}$ (see Section 13). Therefore $|\lambda^0| > \gamma^{-1}$ implies $c^0_0 \neq 0, c^0_1 \neq 0$, hence $\Delta\phi_{-1} + c_0\psi_0 \neq 0$, $\Delta\phi_{-2} + c_1\psi_1 \neq 0$. Note that, by analyticity, $\Delta\phi_{-2} + c_1\psi_1$ is nonzero almost everywhere in (a,u_1). The image $f(a,u_1)$ contains some (small) interval $U_{i_0} \cap f^{-1}(U_{i_1} \cap f^{-1}(U_{i_2} \ldots))$ on which the image of $\Delta\phi_{-2} + c_1\psi_1$ by $\mathcal{L}_{(1)}$ does not vanish, and therefore (by mixing),

$$\int_a^b |\mathcal{L}_{(1)}w\Phi^0| < \int_a^b \mathcal{L}_{(1)}|w\Phi^0|$$

when $w\Phi^0/|w\Phi^0|$ is not constant on (a,b). Thus either (after multiplication of Φ^0 by a suitable constant $\neq 0$), $w\Phi^0 \geq 0$, or

$$|\lambda^0|\int_a^b |w\Phi^0| < \int_a^b |w\Phi^0| \qquad (*)$$

i.e., $|\lambda^0| < 1$. Thus 1 is the only possible eigenvalue λ^0 with $|\lambda^0| = 1$, but 1 is an eigenvalue, otherwise the spectral radius of $\mathcal{L}$ would be < 1 [contradicting the fact that $\int_a^b w\mathcal{L}^n\Phi = \int_a^b w\Phi > 0$ when $w\Phi > 0$]. $(*)$ also implies that if $\mathcal{L}\Phi^1 = \Phi^1$, then $w\Phi^1$ is proportional to $w\Phi^0$, hence ϕ^1_0 is proportional to ϕ^0_0, hence Φ^1 is proportional to Φ^0. Furthermore, the generalized eigenspace to the eigenvalue 1 contains only the multiples of Φ_0 [otherwise there would exist Φ^1 such that $\mathcal{L}^n\Phi^1 = \Phi^1 + n\Phi^0$, contradicting $\int_a^b w\mathcal{L}\Phi^1 = \int_a^b w\Phi^1$]. We have proved the first part of the following

15 Proposition.

(a) Apart from the simple eigenvalue 1*, the spectrum of* $\mathcal{L}$ *has radius* < 1*. The eigenvector* Φ^0 *to the eigenvalue* 1 *(after multiplication by a suitable constant* $\neq 0$*) satisfies* $w\Phi^0 \geq 0$.

(b) Write $\Phi^0 = ((\phi^0_\alpha),(c^0_n))$ and $\Delta(\phi^0_\alpha) = \phi^0$, then ϕ^0 is continuous, of bounded variation, and $\phi^0(a) = \phi^0(b) = 0$.

The interval $[u_1, u_2]$ is divided into N closed intervals $W_1, \ldots, W_N$ by the points $f^n u_1$ for $n = 1, \ldots, N-1$. The intervals $W_1, \ldots, W_N$ are ordered from left to right, by doubling the common endpoints we make the W_j disjoint. Define $\gamma^0 = (\gamma^0_j)_{j=1}^N$ by $\gamma^0_j = \phi^0|W_j \in L^1(W_j)$. Then, the equation $\Phi^0 = \mathcal{L}\Phi^0$ implies

$$\gamma^0 = \mathcal{L}_*\gamma^0 + \eta \qquad (*)$$

or

$$\gamma^0_j = \sum_k \mathcal{L}_{jk}\gamma^0_k + \eta_j$$

where $\mathcal{L} = (\mathcal{L}_{jk})$ is a transfer operator defined as follows. Letting $(f^{-1})_{kj} : W_j \to W_k$ be such that $f \circ (f^{-1})_{kj}$ is the identity on W_j we write

$$\mathcal{L}_{jk}\gamma_k = \begin{cases} \frac{\gamma_k \circ (f^{-1})_{kj}}{|f' \circ (f^{-1})_{kj}|} & \text{if } fW_k \supset W_j \\ 0 & \text{otherwise} \end{cases}$$

[the term $\mathcal{L}_*\gamma^0$ in $(*)$ comes from (i) in Section 12]. We let

$$\eta_j = \sum_{n=0}^{\infty} \eta_{jn}$$

Here

$$\eta_{j0}(x) = \frac{\Delta\phi^0_{-2}(y)}{f'(y)}$$

if $f(a, u_1) \cap W_j$ contains more than one point, and $y \in (a, u_1)$, $fy = x \in W_j$; we let $\eta_{j0}(x) = 0$ otherwise [this term comes from (iv) in Section 12]. For $n \geq 1$, we let $\eta_{jn} = C_n\chi_n|W_j$ where $\chi_n = (\psi_n/|f'|) \circ f_n^{-1} - |f'(f^n b)|^{-1/2}\psi_{n+1}$ [this term comes from (vi) in Section 12].

Because fu_1 is one of the division points between the intervals W_j, the function η_{j0} is continuous on W_j; the η_{jn} for $n \geq 1$ are also continuous. Furthermore, η_{j0} and the η_{jn} for $n \geq 1$ are uniformly of bounded variation. If $\mathcal{H}_j$ denotes the Banach space of continuous functions of bounded variation on W_j we have thus $\eta_j \in \mathcal{H}_j$ for $j = 1, \ldots, N$. We shall now obtain an upper bound on the essential spectral radius of $\mathcal{L}_*$ acting on $\mathcal{H} = \oplus_1^N \mathcal{H}_j$ by studying $||\mathcal{L}_*^n - F_n||$, where F_n has finite-dimensional range (we use here a simple case of an argument due to Baladi and Keller [4]). Define

$$W_{i_n \cdots i_0} = \{x \in W_{i_n} : fx \in W_{i_{n-1}}, \ldots, f^n x \in W_{i_0}\}$$

when $fW_{i_k} \supset W_{i_{k-1}}$ for $k = n, \ldots, 1$. For $\eta = (\eta_j) \in \mathcal{H}$, we let $\pi_n\eta = (\pi_{jn}\eta_j)$ where $\pi_{jn}\eta_j$ is a piecewise affine function on W_j such that $(\pi_{jn}\eta_j)(x) = \eta_j(x)$ whenever x is an endpoint of W_j or of an interval $W_{ji_{n-1}\cdots i_0}$, and is affine between all such endpoints. Then $F_n = \mathcal{L}_*^n\pi_n$ has finite rank (*i.e.*, finite-dimensional range), and $\mathcal{L}_*^n - F_n = \mathcal{L}_*^n(1-\pi_n)$ maps

$\mathcal{H}$ to $\mathcal{H}$. Let $\mathrm{Var}\,\gamma = \sum_1^N \mathrm{Var}_j\gamma_j$ where Var_j is the total variation on W_j. Let also $||\cdot||_0$ denote the sup-norm and $||\cdot|| = \max\{\mathrm{Var}\,\cdot, ||\cdot||_0\}$ be the bounded variation norm. We have

$$\mathrm{Var}(\gamma - \pi_n\gamma) \leq 2\mathrm{Var}\,\gamma$$

$$\sum_{i_0\cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}||_0 \leq \mathrm{Var}\,\gamma$$

[the second inequality follows from the first because $\gamma - \pi_n\gamma$ vanishes at the endpoints of $W_{i_n\cdots i_0}$]. Since $\mathcal{L}_*^n(1-\pi_n)\gamma$ vanishes at the endpoints of the W_j, we have

$$||(\mathcal{L}_*^n - F_n)\gamma|| = \mathrm{Var}((\mathcal{L}_*^n - F_n)\gamma)$$

$$= \mathrm{Var} \sum_{i_0\cdots i_n} ((\gamma - \pi_n\gamma)_{i_n} \circ \tilde{f}_{i_n\cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n\cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0})$$

where we have written

$$\tilde{f}_{i_\ell\cdots i_0} = (f^{-1})_{i_\ell i_{\ell-1}} \circ \cdots (f^{-1})_{i_1 i_0}$$

and

$$\tilde{f}' = \frac{1}{|f'|}$$

hence

$$||(\mathcal{L}_*^n - F_n)\gamma|| \leq \sum_{i_0\cdots i_n} \mathrm{Var}[((\gamma - \pi_n\gamma)_{i_n} \circ \tilde{f}_{i_n\cdots i_0})(\tilde{f}' \circ \tilde{f}_{i_n\cdots i_0}) \cdots (\tilde{f}' \circ \tilde{f}_{i_1 i_0})]$$

$$= \sum_{i_0\cdots i_n} \mathrm{Var}[((\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}) \prod_{\ell=0}^{n-1} (\tilde{f}' \circ (f^\ell|W_{i_n\cdots i_0}))]$$

The right-hand side is bounded by a sum of $n+1$ terms where Var is applied to $(\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}$ or a factor $\tilde{f}' \circ (f^\ell|W_{i_n\cdots i_0}))$, and the other factors are bounded by their $||\cdot||_0$-norm. Thus, using the hyperbolicity condition of Section 4, we have

$$||(\mathcal{L}_*^n - F_n)\gamma||$$

$$\leq \mathrm{Var}(\gamma - \pi_n\gamma).A\alpha^n + \sum_{\ell=0}^{n-1} \sum_{i_0\cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}||_0.A\alpha^\ell.\mathrm{Var}(\tilde{f}'|W_{i_{n-\ell}\cdots i_0}).A\alpha^{n-\ell-1}$$

$$\leq 2A\alpha^n\mathrm{Var}\,\gamma + nA^2\alpha^{n-1}\mathrm{Var}\,\tilde{f}' \sum_{i_0\cdots i_n} ||(\gamma - \pi_n\gamma)|W_{i_n\cdots i_0}||_0$$

$$\leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n\mathrm{Var}\,\gamma \leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n||\gamma||$$

so that

$$||\mathcal{L}_*^n - F_n|| \leq (2A + nA^2\alpha^{-1}\mathrm{Var}\,\tilde{f}')\alpha^n$$

and therefore $\mathcal{L}_*$ has essential spectral radius $\leq \alpha < 1$ on $\mathcal{H}$. Suppose that there existed an eigenfunction $\gamma \in \mathcal{H}$ to the eigenvalue 1 of $\mathcal{L}_*$; the fact that γ is continuous and $\neq 0$ on some W_j would imply

$$\int (\mathcal{L}_*^n|\gamma|)(x)\,dx < \int |\gamma|(x)\,dx$$

[because, for some n, $\mathcal{L}_*^n$ sends "mass" into V_0]. But this is in contradiction with

$$\int |\gamma|(x)\,dx = \int |\mathcal{L}_*^n\gamma|(x)\,dx \leq \int (\mathcal{L}_*^n|\gamma|)(x)\,dx$$

Therefore, 1 cannot be an eigenvalue of $\mathcal{L}_*$, and there is $\gamma = (1-\mathcal{L}_*)^{-1}\eta \in H$ such that

$$\gamma = \mathcal{L}_*\gamma + \eta$$

Since γ^0 satisfies the same equation in L^1, we have $\gamma^0 - \gamma = \mathcal{L}_*(\gamma^0 - \gamma)$ hence $\gamma^0 - \gamma = 0$ by the same argument as above [$|\gamma^0 - \gamma|$ is in L^1, with "mass" in some V_α because $H(u_1)$ has measure 0, and this is sent to V_0 by $\mathcal{L}_*^n$ for some n]. Thus γ^0 is continuous of bounded variation on the intervals W_j for $j = 1, \ldots, N$, and ϕ^0 has bounded variation on $[a, b]$, with possible discontinuities only at $f^n u_1$ for $n = 0, \ldots, N$, and $\phi^0(a) = \phi^0(b) = 0$. We have

$$\mathcal{L}_{(1)}\phi^0 - c_0^0\psi_0 + \sum_{n=0}^{\infty} c_n^0\chi_n = \phi^0$$

Therefore, hyperbolicity along the periodic orbit of u_1 shows that ϕ^0 cannot have discontinuities, and this proves part (b) of Proposition 15. ∎

This also concludes the proof of Theorem 9. ∎

16 Remarks.

(a) Theorem 9 shows that the density $\rho(x)$ of the unique a.c.i.m. $\rho(x)\,dx$ for f can be written as the sum of spikes $\approx |x - f^n b|^{-1/2}\theta_n(x)$ (where θ_n vanishes unless $x > f^n b$ or $x < f^n b$) and a continuous background $\phi(x)$. In fact, one can also write $\rho(x)$ as the sum of singular terms $\approx |x - f^n b|^{-1/2}\theta_n(x), |x - f^n b|^{1/2}\theta_n(x)$ and a background $\phi(x)$ which is now differentiable. This result is discussed in Appendix A. It seems clear that one could write $\rho(x)$ as a sum of terms $|x - f^n b|^{k/2}\theta_n(x)$ with $k = -1, 1, \ldots, \frac{2\ell-1}{2}$ and a background $\phi(x)$ of class C^ℓ, but we have not written a proof of this.

(b) Let $u \in (-\infty, u_1) \cup (u_1, v_1) \cup (v_2, u_2) \cup (u_2, \infty)$ and choose $w \in \{u_1, u_2, v_1, v_2\}$ such that w is an endpoint of the interval containing u. If $\pm(w - u) > 0$ and $\theta_\pm$ is the characteristic function of $\{x : (w - x)(x - u) > 0\}$ we define

$$\psi_{(u\pm)}(x) = \frac{w - x}{w - u} \cdot |x - u|^{-1/2}\theta_\pm(x)$$

or a similar expression with the same singularity at u, greater smoothness at w, and/or $\int \psi_{(u\pm)} = 0$. [Note that the ψ_n are of this form]. Claim: if $u \in \tilde{H}$, there exists a unique $(\phi_\alpha) \in \mathcal{A}_1$ such that $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$ for all α; furthermore $||(\phi_\alpha)||_1$ has a bound independent of $u\pm$. These results are proved in Appendix B (assuming $\gamma < \alpha^{-1/2}$).

Note that if $((\phi_\alpha),(c_n)) \in \mathcal{A}$ and $c_0 = c_1 = 0$, there is $(\tilde{\phi}_\alpha) \in \mathcal{A}_1$ such that $\Delta(\tilde{\phi}_\alpha) = w((\phi_\alpha),(c_n))$. It seems thus that we might have replaced $\mathcal{A}$ by $\mathcal{A}_1$ in our earlier discussions. However, separating the spikes (c_n) from the background (ϕ_α) was needed in the spectral study of $\mathcal{L}$.

(c) The eigenvector Φ^0 of $\mathcal{L}$ corresponding to the eigenvalue 1 (with $w\Phi^0 \geq 0, \int w\Phi^0 = 1$) depends continuously on f. To make sense of this statement we may consider a one-parameter family (f_κ) such that $f_0 = f$. We let $H_\kappa, \tilde{H}_\kappa$ (hyperbolic sets) and $\mathcal{A}_{1\kappa}$ (Banach space) reduce to $H, \tilde{H}$ and $\mathcal{A}_1$ when $\kappa = 0$. We restrict κ to a compact set K such that $f_\kappa^3 c_\kappa \in \tilde{H}_\kappa$ (where c_κ is the critical point of f_κ). The intervals $V_{\kappa\alpha}$ associated with H_κ can be mapped to the V_α associated with H, providing an identification $\eta_\kappa : \mathcal{A}_{\kappa 1} \to \mathcal{A}_1$. There are natural definitions of $\mathcal{L}_\kappa : \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2 \to \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$ and the eigenvector Φ_κ^0 reducing to $\mathcal{L}$ and Φ^0 when $\kappa = 0$. We claim that $\kappa \mapsto \Phi_\kappa^\times = (\eta_\kappa, \mathbf{1})\Phi_\kappa^0$ is a continuous function $K \to \mathcal{A}_1 \oplus \mathcal{A}_2$. This result is proved in Appendix C. It implies that, if A is smooth, $\kappa \to \langle \Phi_{f_\kappa}^0, A\rangle$ is continuous on K. The weight of the n-th spike is $C_0 \prod_{k=1}^n |f_\kappa'(f_\kappa^{k-1} b_\kappa)|^{-1/2}$ and its speed is

$$\frac{d}{d\kappa} f_\kappa^n b_\kappa = \prod_{k=1}^n f_\kappa'(f_\kappa^{k-1} b_\kappa) \frac{db_\kappa}{d\kappa} + \sum_{\ell=1}^n \prod_{k=\ell+1}^n f_\kappa'(f_\kappa^{k-1} b_\kappa) f_\kappa^*(f_\kappa^{\ell-1} b_\kappa) \quad \text{with} \quad f_\kappa^* = \frac{df_\kappa}{d\kappa}$$

The weight may be roughly estimated as $\sim \alpha^{n/2}$ and the speed as $\sim \alpha^{-n}$ for some $\alpha \in (0,1)$, suggesting that $\kappa \to \langle \Phi_{f_\kappa}^0, A\rangle$ is $\frac{1}{2}$-Hölder on K.

17 Informal study of the differentiability of $f \mapsto \langle \Phi_f^0, A\rangle$.

Writing Φ_f^0 instead of Φ^0 we want to study the change of $\langle \Phi_f^0, A\rangle = \int dx\,(w\Phi_f^0)(x)A(x)$ when f is replaced by $\hat{f}$ close to f (and the critical orbit $\hat{f}^k\hat{c}$ for $k \geq 3$ is in the perturbed hyperbolic set $\hat{\tilde{H}}$). Writing $g = \text{id} - \hat{f}(\hat{c}) + f(c)$, we see that $\hat{f}$ is conjugate to $g \circ \hat{f} \circ g^{-1}$, which has maximum $f(c)$ at $g(\hat{c})$. With proper choice of the inverse f^{-1} we have $f^{-1} \circ (g \circ \hat{f} \circ g^{-1}) = h$ close to id, hence $g \circ \hat{f} \circ g^{-1} = f \circ h$ and $(h \circ g) \circ \hat{f} \circ (h \circ g)^{-1} = h \circ f$, *i.e.*, $\hat{f}$ is conjugate to $h \circ f$ and we may write

$$\langle \Phi_{\hat{f}}^0, A\rangle = \langle \Phi_{h\circ f}^0, A \circ h \circ g\rangle$$

The differentiability of $\hat{f} \mapsto A \circ h \circ g$ is trivial, and we concentrate on the study of $h \mapsto \langle \Phi_{h\circ f}^0, A\rangle$. Writing $h = \text{id} + X$, where X is analytic, we see that the change $\delta(w\Phi_f^0)$ when f is replaced by $(\text{id} + X) \circ f$ is, to first order in X, formally

$$(1 - \mathcal{L})^{-1}\mathcal{D}(-X\Phi_f^0)$$

where $\mathcal{D}$ denotes differentiation. [The above formula is standard first order perturbation calculation, and we have omitted the w map from our formula].

Writing $\Phi_f^0 = ((\phi_\alpha^0), (C_n))$, we can identify $\mathcal{D}(-X((\phi_\alpha^0), 0))$ with an element $\Phi^\times$ of $\mathcal{A}$ (so that $w\Phi^\times = \mathcal{D}(Xw((\phi_\alpha^0), 0))$ and $\int dx\, w\Phi^\times(x) = 0$, use Appendix A) which is easy to study, and we are left to analyze the singular part $\mathcal{D}(-X(0, (C_n)))$. To study this singular part we shall write $(0, (C_n)) = \sum_{n=0}^\infty C_n \psi_{(f^n b)}$, and use the equivalence $\sim$ modulo the elements of $\mathcal{A}$. We extend the domain of definition of $\mathcal{L}$ so that $\mathcal{L}\psi_{(u)} \sim |f'(u)|^{-1/2}\psi_{(fu)}$, where we use the notation $\psi_{(u\pm)}$ of Section 16(b), but omit the $\pm$, and we assume that $\int \psi_{(u)} = 0$. We have thus

$$\mathcal{D}(-X(0, (C_n))) \sim -\sum_{n=0}^\infty C_n X(f^n b)\mathcal{D}\psi_{(f^n b)} \sim \sum_{n=0}^\infty C_n X(f^n b)\frac{d}{du}\psi_{(u)}\Big|_{u=f^n b}$$

$$= \sum_{n=0}^\infty C_n X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\frac{d}{db}\psi_{(f^n b)} \sim \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\frac{d}{db}\mathcal{L}^n C_0\psi_{(b)}$$

We may thus write (introducing $(1-\lambda\mathcal{L})^{-1}$ instead of $(1-\mathcal{L})^{-1}$)

$$(1-\lambda\mathcal{L})^{-1}\mathcal{D}(-X(0, (C_n))) \sim \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}(1-\lambda\mathcal{L})^{-1}(\lambda\mathcal{L})^n C_0\psi_{(b)}$$

$$= \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}(1-\lambda\mathcal{L})^{-1}C_0\psi_{(b)} - Z$$

where

$$Z = \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\lambda^{-n}\frac{d}{db}\sum_{\ell=0}^{n-1}(\lambda\mathcal{L})^\ell C_0\psi_{(b)}$$

$$\sim \sum_{n=0}^\infty X(f^n b)[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}|\prod_{k=0}^{\ell-1} f'(f^k b)|^{-1/2}\frac{d}{db}C_0\psi_{(f^\ell b)}$$

$$= \sum_{n=0}^\infty X(f^n b)\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}[\prod_{k=\ell}^{n-1} f'(f^k b)]^{-1}|\prod_{k=0}^{\ell-1} f'(f^k b)|^{-1/2}\frac{d}{du}C_0\psi_{(u)}\Big|_{u=f^\ell b}$$

$$\sim -\mathcal{D}\sum_{n=0}^\infty X(f^n b)\sum_{\ell=0}^{n-1}\lambda^{-n+\ell}[\prod_{k=\ell}^{n-1} f'(f^k b)]^{-1}C_\ell\psi_\ell$$

$$= -\mathcal{D}\sum_{r=1}^\infty\sum_{\ell=0}^\infty X(f^{\ell+r}b)\lambda^{-r}[\prod_{k=0}^{r-1} f'(f^{\ell+k}b)]^{-1}C_\ell\psi_\ell$$

$$= -\mathcal{D}\sum_{\ell=0}^\infty C_\ell\psi_\ell\sum_{r=1}^\infty\lambda^{-r}[\prod_{k=0}^{r-1} f'(f^{\ell+k}b)]^{-1}X(f^{\ell+r}b)$$

We have thus an (informal) proof of the following result

For $\ell = 0, 1, \ldots$, define

$$F_\ell(X) = \sum_{n=1}^{\infty} \lambda^{-n} [\prod_{k=0}^{n-1} f'(f^{k+\ell}b)]^{-1} X(f^{n+\ell}b)$$

which are holomorphic functions of λ when $|\lambda| > \alpha$. Then the susceptibility *function*

$$\Psi(\lambda) = \langle (1 - \lambda\mathcal{L})^{-1}\mathcal{D}(-X\Phi_f^0), A\rangle$$

has the form

$$\Psi(\lambda) \sim (X(b) + F_0(X))\frac{d}{db}\langle (1 - \lambda\mathcal{L})^{-1}C_0\psi_{(b)}, A\rangle - \sum_{\ell=0}^{\infty} F_\ell(X)C_\ell\langle \psi_\ell, \mathcal{D}A\rangle$$

The derivative $\frac{d}{db}\langle (1 - \lambda\mathcal{L})^{-1}C_0\psi_{(b)}, A\rangle$ exists as a distribution, but is in principle a divergent quantity for given b. The corresponding term disappears however if $X(b) + F_0(X) = 0$, and we are then left with a finite expression, meromorphic in λ for $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$ and holomorphic when $\alpha < |\lambda| \leq 1$.

Note that in writing the equivalence $\sim$ we have omitted terms with the singularities of $(1 - \lambda\mathcal{L})^{-1}$; this explains the meromorphic contributions for $|\lambda| > 1$. The condition $X(b) + F_0(X) = 0$ for $\lambda = 1$ is known as *horizontality* (see the discussion in Section 19 below).

18 A modified susceptibility function $\Psi(X, \lambda)$.

At this point we extend the definition of the operator $\mathcal{L}$ to $\mathcal{L}^\sim$ acting on a larger space. Remember that $\mathcal{L}$ was obtained from the transfer operator $\mathcal{L}_{(1)}$ by separating the spikes ψ_n from the background in order to obtain better spectral properties. We now also introduce derivatives ψ'_n of spikes, so that the transfer operator sends ψ'_n to

$$\frac{f'(f^n b)}{|f'(f^n b)|^{1/2}}\psi'_{n+1} + \text{a term in } w(\mathcal{A}_1 + \mathcal{A}_2)$$

The coefficients of ψ'_n form an element of $\mathcal{A}_3 = \{(Y_n) : ||(Y_n)||_3 = \sup_n \delta^n |Y_n| < \infty\}$. We define $\mathcal{L}^\sim$ on $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$ so that

$$\mathcal{L}^\sim = \begin{pmatrix} \mathcal{L}_0 + \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_5 \\ \mathcal{L}_3 & \mathcal{L}_4 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix}$$

where we omit the explicit definition of $\mathcal{L}_5$, $\mathcal{L}_6$, and let

$$\mathcal{L}_7\Big(\frac{Z_n}{\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big) = \Big(\frac{\tilde{Z}_n}{\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)$$

with $\tilde{Z}_0 = 0$, $\tilde{Z}_n = f'(f^{n-1}b)Z_{n-1}$ for $n > 0$. Since

$$\begin{pmatrix} 0 & 0 & \mathcal{L}_5 \\ 0 & 0 & \mathcal{L}_6 \\ 0 & 0 & \mathcal{L}_7 \end{pmatrix} \mathcal{L} = 0$$

we have

$$\mathcal{L}^{\sim n} = \mathcal{L}^n + \sum_{k=1}^{n} \mathcal{L}^{k-1}(\mathcal{L}_5 + \mathcal{L}_6)\mathcal{L}_7^{n-k} + \mathcal{L}_7^n$$

and formally

$$(\mathbf{1} - \lambda\mathcal{L}^{\sim})^{-1} = (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1} + (\mathbf{1}_3 - \lambda\mathcal{L}_7)^{-1} + (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1}\lambda(\mathcal{L}_5 + \mathcal{L}_6)(\mathbf{1}_3 - \lambda\mathcal{L}_7)^{-1}$$

where $\mathbf{1}_{12}$ and $\mathbf{1}_3$ denote the identity on $\mathcal{A}_1 \oplus \mathcal{A}_2$ and $\mathcal{A}_3$ respectively.

For λ close to 1, $(\mathbf{1}_3 - \lambda\mathcal{L}_7)^{-1}$ and thus $(\mathbf{1} - \lambda\mathcal{L}^{\sim})^{-1}$ are not well defined. But there is a natural definition of a left inverse $\mathcal{L}_{7L}^{-1}$ of $\mathcal{L}_7$ where

$$\mathcal{L}_{7L}^{-1}\Big(\frac{Z_n}{\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big) = \Big(\frac{\tilde{Z}_n}{\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)$$

with $\tilde{Z}_n = f'(f^n b)^{-1}Z_{n+1}$ for $n \geq 0$. The spectral radius of $\mathcal{L}_{7L}^{-1}$ is thus $\leq \alpha^{1/2}/\delta$. This gives natural left inverses

$$(\mathbf{1}_3 - \lambda\mathcal{L}_7)_L^{-1} = -\sum_{n=1}^{\infty} \lambda^{-n}\mathcal{L}_{7L}^{-n}$$

for $|\lambda| > \alpha^{1/2}/\delta$, and

$$(\mathbf{1} - \lambda\mathcal{L}^{\sim})_L^{-1} = (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1} + (\mathbf{1}_3 - \lambda\mathcal{L}_7)_L^{-1} + (\mathbf{1}_{12} - \lambda\mathcal{L})^{-1}\lambda(\mathcal{L}_5 + \mathcal{L}_6)(\mathbf{1}_3 - \lambda\mathcal{L}_7)_L^{-1}$$

when $|\lambda| > \alpha^{1/2}/\delta$ and $(\mathbf{1}_{12} - \lambda\mathcal{L})^{-1}$ exists. This gives a modified susceptibility function

$$\Psi_L(\lambda) = \langle(\mathbf{1} - \lambda\mathcal{L}^{\sim})_L^{-1}\mathcal{D}(-X\Phi_f^0), A\rangle$$

meromorphic in λ for $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$ and holomorphic for $\alpha < |\lambda| \leq 1$.

Note that the $\mathcal{A}_3$ part of $\mathcal{D}(-X\Phi_f^0)$ is

$$(Y_n) = \Big(\frac{-X(f^n b)}{\frac{1}{2}|f''(c)|^{1/2}\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)_{n\geq 0}$$

where $\sup_n |X(f^n b)| < \infty$. Therefore, for small $|\lambda|$,

$$(\mathbf{1}_3 - \lambda\mathcal{L}_7)^{-1}(Y_n) = \Big(\frac{-\sum_{k=0}^{n}\lambda^k(\prod_{\ell=1}^{k} f'(f^{n-\ell}b))X(f^{n-k}b)}{\frac{1}{2}|f''(c)|^{1/2}\prod_{k=0}^{n-1}|f'(f^k b)|^{1/2}}\Big)_{n\geq 0}$$

because the right-hand side is in $\mathcal{A}_3$. Note that the right-hand side is also in $\mathcal{A}_3$ under the condition

$$\sum_{n=0}^{\infty} \lambda^{-n} (\prod_{k=0}^{n-1} f'(f^k b))^{-1} X(f^n b) = 0 \tag{$*$}$$

because this condition implies

$$-\sum_{k=0}^{n} \lambda^{-k} (\prod_{\ell=0}^{k-1} f'(f^\ell b))^{-1} X(f^k b) = \sum_{k=n+1}^{\infty} \lambda^{-k} (\prod_{\ell=0}^{k-1} f'(f^\ell b))^{-1} X(f^k b)$$

hence, multiplying by $\lambda^n \prod_{\ell=0}^{n-1} f'(f^\ell b)$,

$$-\sum_{k=0}^{n} \lambda^{n-k} (\prod_{\ell=k}^{n-1} f'(f^\ell b)) X(f^k b) = \sum_{k=n+1}^{\infty} \lambda^{n-k} (\prod_{\ell=n}^{k-1} f'(f^\ell b))^{-1} X(f^k b)$$

or

$$-\sum_{k=0}^{n} \lambda^{k} (\prod_{\ell=1}^{k} f'(f^{n-\ell} b)) X(f^{n-k} b) = \sum_{k=1}^{\infty} \lambda^{-k} (\prod_{\ell=0}^{k-1} f'(f^{n+\ell} b))^{-1} X(f^{n+k} b)$$

for each n, provided $|\lambda| > \alpha$. We have proved that:

Under the condition $(*)$, *a resummation of the series defining*

$$\langle (\mathbf{1} - \lambda \mathcal{L}^{\sim})^{-1} \mathcal{D}(-X\Phi_f^0), A \rangle$$

yields $\Psi_L(\lambda)$.

It is then natural to define a modified susceptibility function $\Psi(X, \lambda)$ by

$$(X, \lambda) \mapsto \Psi(X, \lambda) = \Psi_L(\lambda) \quad \text{on} \quad \{(X, \lambda) : (*) \text{ holds}\}$$

Note that the left-hand side of $(*)$ is equal to the quantity $X(b) + F_0(X)$ met in Section 17, and that $(*)$ with $\lambda = 1$ reduces to the horizontality condition.

We shall conclude with a rigorous result agreeing in part with the informal study in Section 17, in part with a conjecture of Baladi [3], Baladi and Smania [5].

First we recall the definition of *topological conjugacy*. If $I \subset X$, $\tilde{I} \subset \tilde{X}$, we say that $f : I \to X$, $\tilde{f} : \tilde{I} \to \tilde{X}$ are topologically conjugate if there exists a homeomorphism $h : X \to \tilde{X}$ such that $h \circ f = \tilde{f} \circ h$ on I. Topological conjugacy classes for unimodal maps have been analyzed in [1]; those called *hybrid classes* are Banach codimension 1 manifolds and their tangent vectors are characterized by a general horizontality condition. We shall be interested in the case where topological conjugacy classes are restricted to Misiurewicz diffeomorphisms, and might thus be called *Misiurewicz classes* (Misiurewicz classes are a special case of hybrid classes). We shall in fact not be concerned with the conjugacy h on I, but with a consequence of a special case of conjugacy (namely $f_\kappa^3 c = \xi_\kappa f^3 c$, corresponding to a Misiurewicz class). We shall also use the infinitesimal form of conjugacy, *viz.* horizontality. For a general discussion of conjugacy classes for unimodal maps, under somewhat different conditions than those used here, we must refer to [1], which also gives precise conditions under which a topological class contains a quadratic element f_c (with $f_c x = cx(1-x)$).

19 Theorem (differentiability along topological conjugacy classes).

Let $f_\kappa = h_\kappa \circ f$ where the h_κ are real analytic, depend smoothly on κ, and $f_\kappa^3 c = \xi_\kappa f^3 c$ identically in κ. [This last condition expresses that f_κ belongs to a conjugacy class, and $\xi_k : H \to H_\kappa$ is the conjugacy defined in Appendix C]. Then, if A is smooth, $\kappa \mapsto \langle \Phi^0_\kappa, A\rangle = \int dx\, (w\Phi^0_{f_k})(x)A(x)$ is continuously differentiable. Furthermore

$$\frac{d}{d\kappa}\langle \Phi^0_{f_k}, A\rangle\Big|_{\kappa=0} = \Psi(X,1)$$

where $\Psi(X,\lambda)$ is defined in Section 18 with $X = \frac{d}{d\kappa}h_\kappa|_{\kappa=0}$, and $\Psi(X,\lambda)$ is holomorphic for $\alpha < |\lambda| \leq 1$, meromorphic for $\alpha < |\lambda| < \min(\beta^{-1}, \alpha^{-1/2})$.

[The value $\kappa = 0$ plays no special role, and is chosen for notational simplicity in the formulation of the theorem].

Our notion of topological conjugacy class is a special case of that discussed in [1].

Note that $\xi_0 = \mathrm{id}$, and that ξ_κ depends differentiably on κ. Since $f_\kappa^3 c = \xi_\kappa f^3 c$ and $f_\kappa \xi_\kappa = \xi_\kappa f$ on H, we have $f_\kappa^n c = \xi_\kappa f^n c$ for $n \geq 3$ and by differentiation (writing $\xi' = \frac{d}{d\kappa}\xi_\kappa|_{\kappa=0}$):

$$\sum_{k=1}^{n}[\prod_{\ell=k}^{n-1} f'(f^\ell c)]X(f^k c) = \xi'(f^n c)$$

or

$$\sum_{k=1}^{n}[\prod_{\ell=1}^{k-1} f'(f^\ell c)]^{-1}X(f^k c) = [\prod_{\ell=1}^{n-1} f'(f^\ell c)]^{-1}\xi'(f^n c)$$

and letting $n \to \infty$:

$$\sum_{k=1}^{\infty}[\prod_{\ell=1}^{k-1} f'(f^\ell c)]^{-1}X(f^k c) = 0 \qquad \text{or} \qquad \sum_{n=0}^{\infty}[\prod_{k=0}^{n-1} f'(f^k b)]^{-1}X(f^n b) = 0$$

This is the horizontality condition derived much more generally in [1].

The proof of the theorem will be based on Appendices A, B, C, and use particularly the notation of Appendix C. We write $\Phi^0_{f_\kappa} = \Phi^0_\kappa$ and recall that the expression

$$\langle \Phi^0_\kappa, A\rangle_\kappa = \int dx\, (w_\kappa \Phi^0_\kappa)(x)A(x) = \sum_\alpha \int_{V_{\kappa\alpha}} \phi^0_{\kappa\alpha} A(x)dx + \sum_n c^0_{\kappa n}\int \psi_{\kappa n}(x)A(x)dx$$

depends explicitly on the intervals $V_{\kappa\alpha}$ and the points $f_\kappa^k c$ for $k \geq 1$. We shall first prove the existence of $\frac{d}{d\kappa}\langle\Phi_\kappa^0, A\rangle_\kappa|_{\kappa=0} = \lim_{\kappa\to 0}\frac{1}{\kappa}\int(w_\kappa\Phi_\kappa^0 - w\Phi^0)A$ and give an expression involving only the intervals V_α and the points $f^k c$ (corresponding to $\kappa = 0$). Then we shall transform the expression obtained to the form $\Psi(X, 1)$.

Since the map $\xi_\kappa : H \to H_\kappa$ depends smoothly on κ (in particular $f'_\kappa(f_\kappa^k b_\kappa) = f'_\kappa(\xi_\kappa f^k b)$ is continuous uniformly in k), it is easily seen that the operator $\mathcal{L}_\kappa^\times$ defined in Appendix C now depends continuously and even differentiably on κ.

We may write

$$\langle\Phi_\kappa^0, A\rangle_\kappa = \sum_\alpha \int_{V_{\kappa\alpha}} \phi_{\kappa\alpha}^0(x)A(x)\,dx + \sum_n c_{\kappa n}^0 \int \psi_{\kappa n}(x)A(x)\,dx$$

$$= \langle((\phi_{\kappa\alpha}^0), (c_{\kappa n}^0)), ((A|V_{\kappa\alpha}), A)\rangle_\kappa$$

$$= \langle\Phi_\kappa^0, ((A|V_{\kappa\alpha}), 0)\rangle_\kappa + \langle\Phi_\kappa^0, (0, (c_{\kappa n}^0))\rangle_\kappa$$

For notational simplicity we study the derivative of this quantity at $\kappa = 0$ but the proof will show that the derivative depends continuously on κ. We have

$$\frac{1}{\kappa}\Big[\langle\Phi_\kappa^0, A\rangle_\kappa - \langle\Phi^0, A\rangle\Big] = I + II$$

where

$$II = \frac{1}{\kappa}\sum_n \int [c_{\kappa n}^0\psi_{\kappa n}(x) - c_n^0\psi_n(x)]A(x)\,dx$$

$$\to \sum_n \int [\frac{dc_{\kappa n}^0}{d\kappa}\psi_n(x) + c_n^0\frac{d}{d\kappa}\psi_{\kappa n}(x)]A(x)\,dx\Big|_{\kappa=0}$$

[$\frac{d}{d\kappa}\psi_{\kappa n}$ is a distribution with singular part $\frac{d}{d\kappa}|x - f_\kappa^n b_\kappa|^{-1/2}$; integrating by part over x, and using $f_\kappa^n b_\kappa = \xi_\kappa f^n b$ for $k \geq 2$, we see that the right-hand side makes sense, and is the limit of the left-hand side when $\kappa \to 0$].

We also have

$$\langle\Phi_\kappa^0, ((A|V_{\kappa\alpha}), 0)\rangle_\kappa = \langle\Phi_\kappa^\times, ((A_{\kappa\alpha}), 0)\rangle$$

where $A_{\kappa\alpha} = (A|V_{\kappa\alpha}) \circ \tilde{\eta}_{\kappa\alpha}^{-1}$, so that

$$I = \langle\frac{\Phi_\kappa^\times - \Phi_0^\times}{\kappa}, ((A_{\kappa\alpha}), 0)\rangle + \langle\Phi_0^\times, ((\frac{A_{\kappa\alpha} - A_{0\alpha}}{\kappa}), 0)\rangle$$

and the second term is readily seen to tend to a limit when $\kappa \to 0$. In the first term remember that for $\kappa = 0$ we have $\Phi_\kappa^\times = \Phi_0^\times = \Phi^0$, and $\mathcal{L}_\kappa^\times = \mathcal{L}_0^\times = \mathcal{L}$. Also

$$(\mathbf{1} - \mathcal{L})(\Phi_\kappa^\times - \Phi_0^\times) = (\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times)\Phi_\kappa^\times$$

hence

$$\Phi_\kappa^\times - \Phi_0^\times = (\mathbf{1} - \mathcal{L})^{-1}(\mathcal{L}_\kappa^\times - \mathcal{L}_0^\times)\Phi_\kappa^\times$$

Since $(\mathbf{1}-\mathcal{L})^{-1}$ is bounded and $\kappa \mapsto \mathcal{L}_\kappa^\times$ differentiable, we have

$$\langle \frac{\Phi_\kappa^\times - \Phi_0^\times}{\kappa}, ((A_{\kappa\alpha}), 0)\rangle \to \langle (\mathbf{1}-\mathcal{L})^{-1}(\frac{d}{d\kappa}\mathcal{L}_\kappa^\times\Big|_{\kappa=0})\Phi^0, ((A_{0\alpha}), 0)\rangle$$

when $\kappa \to 0$, proving that $\kappa \mapsto \langle \Phi_\kappa^0, A\rangle$ is differentiable.

If we replace in the above calculation the Banach space $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ by $\mathcal{A}' = \mathcal{A}_1' \oplus \mathcal{A}_2'$ as in Appendix A, we obtain an expression of $\frac{d}{d\kappa}\langle \Phi_\kappa^0, A\rangle_\kappa|_{\kappa=0}$ that can be re-expressed in terms of the ψ_n', ψ_n and an element of $\mathcal{A}_1$. We may thus write

$$\frac{d}{d\kappa}\langle \Phi_\kappa^0, A\rangle_\kappa\Big|_{\kappa=0} = \langle \tilde{\Phi}, A\rangle^\sim$$

where $\tilde{\Phi} \in \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. The part $\tilde{\Phi}_3$ of $\tilde{\Phi}$ in $\mathcal{A}_3$ is uniquely determined by $A \mapsto \langle \tilde{\Phi}, A\rangle^\sim$; the calculation of II above shows that n-th component of $\tilde{\Phi}_3$ is

$$-\frac{d}{d\kappa} f_\kappa^n b_\kappa\Big|_{\kappa=0} c_n^0 = -\frac{d}{d\kappa} f_\kappa^{n+1} c\Big|_{\kappa=0} c_n^0$$

$$= -\sum_{k=1}^{n+1} X(f^k c)\big(\prod_{\ell=k}^{n} f'(f^\ell c)\big) c_n^0 = -\sum_{k=0}^{n} X(f^k b)\big(\prod_{\ell=k}^{n-1} f'(f^\ell b)\big) c_n^0$$

and as a result

$$(\mathbf{1}-\mathcal{L}_7)\tilde{\Phi}_3 = (-X(f^n b) C_n^0)_{n\geq 0}$$

$$\tilde{\Phi}_3 = (\mathbf{1}-\mathcal{L}_7)_L^{-1}(-X(f^n b) C_n^0)_{n\geq 0}$$

The part Φ^* of $\tilde{\Phi}$ in $\mathcal{A}_1 \oplus \mathcal{A}_2$ is not uniquely determined (because of the ambiguity discussed in Appendix B); this part satisfies $\int w\Phi^* = 0$.

If $\mathcal{L}_{(1)\kappa}$ is the transfer operator corresponding to f_κ, we have $\mathcal{L}_{(1)\kappa} w_\kappa \Phi_\kappa^0 = w_\kappa \Phi_\kappa^0$, hence

$$(\mathbf{1}-\mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w\Phi^0) = (\mathcal{L}_{(1)\kappa} - \mathcal{L}_{(1)}) w_\kappa \Phi_\kappa^0$$

Therefore (using the fact that we may let $\mathcal{L}_{(1)}$ act on A) we have

$$\langle (\mathbf{1}-\mathcal{L}^\sim)\tilde{\Phi}, A\rangle^\sim = \lim_{\kappa\to 0} \int A \frac{1}{\kappa}(\mathbf{1}-\mathcal{L}_{(1)})(w_\kappa \Phi_\kappa^0 - w\Phi^0)$$

$$= \lim_{\kappa\to 0} \int A \frac{1}{\kappa}(\mathcal{L}_{(1)\kappa} - \mathcal{L}_{(1)}) w_\kappa \Phi_\kappa^0 = \lim_{\kappa\to 0} \int A \frac{1}{\kappa}(\mathrm{id}^* - h_{-\kappa}^*) w_\kappa \Phi_\kappa^0 = \lim_{\kappa\to 0} \int A \frac{1}{\kappa}(h_\kappa^* - \mathrm{id}^*) w\Phi^0$$

where h^* denotes the direct image of a measure (here a L^1 function) under h, and the last equality uses the existence of a continuous derivative for $\kappa \mapsto \langle \Phi_\kappa^0, A\rangle_\kappa$. According to Appendix A we may write $w\Phi^0$ as a sum of terms $C_n^{(0)}\psi_n^{(0)}$, $C_n^{(1)}\psi_n^{(1)}$, and a differentiable background. Corresponding to this we may identify $\lim_{\kappa\to 0} \frac{1}{\kappa}(h_\kappa^* - \mathrm{id}^*)\Phi^0$ with a naturally defined element $\mathcal{D}(-X\Phi^0)$ of $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, where $\mathcal{D}$ denotes differentiation. We write

$\mathcal{D}(-X\Phi^0) = (D^*, D_3)$ with $D^* \in \mathcal{A}_1 \oplus \mathcal{A}_2, D_3 \in \mathcal{A}_3$. Since the coefficient of ψ'_n in $\mathcal{D}(-X\Phi^0)$ is $-X(f^n b)c^0_n$, we have $D_3 = (\mathbf{1} - \mathcal{L}_7)\tilde{\Phi}_3$. With $\tilde{\Phi} = (\Phi^*, \tilde{\Phi}_3)$ we have thus

$$\langle(\mathbf{1} - \mathcal{L}^\sim)(\Phi^*, \tilde{\Phi}_3), A\rangle^\sim = \langle\mathcal{D}(-X\Phi^0), A\rangle^\sim$$

and

$$\langle(\mathbf{1} - \mathcal{L})\Phi^*, A\rangle = \langle\mathcal{D}(-X\Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3), A\rangle$$

In particular $\int w[\mathcal{D}(-X\Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)] = 0$ and we may define

$$\Phi = (\mathbf{1} - \mathcal{L})^{-1}[\mathcal{D}(-X\Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)] \in \mathcal{A}$$

We have then $\langle(\mathbf{1} - \mathcal{L})(\Phi^* - \Phi), A\rangle = 0$, hence

$$w(\mathbf{1} - \mathcal{L})(\Phi^* - \Phi) = 0$$

hence

$$w(\Phi^* - \Phi) = \mathcal{L}_{(1)}w(\Phi^* - \Phi)$$

with $\int w(\Phi^* - \Phi) = 0$, so that $w(\Phi^* - \Phi) = 0$, and

$$\langle\Phi^*, A\rangle = \langle\Phi, A\rangle = \langle(\mathbf{1} - \mathcal{L})^{-1}[\mathcal{D}(-X\Phi^0) - (\mathbf{1} - \mathcal{L}^\sim)(0, \tilde{\Phi}_3)], A\rangle$$

$$= \langle(\mathbf{1} - \mathcal{L})^{-1}[D^* + \mathcal{L}_5 + \mathcal{L}_6)\tilde{\Phi}_3], A\rangle = \langle(\mathbf{1} - \mathcal{L})^{-1}[D^* + \mathcal{L}_5 + \mathcal{L}_6)(\mathbf{1} - \mathcal{L}_7)_L^{-1} D_3], A\rangle$$

$$= \langle(\mathbf{1} - \mathcal{L}^\sim)_L^{-1}(D^*, D_3), A\rangle - \langle(\mathbf{1} - \mathcal{L}_7)_L^{-1} D_3, A\rangle = \Psi(X, 1) - \langle(0, \tilde{\Phi}_3), A\rangle^\sim$$

so that finally

$$\frac{d}{d\kappa}\langle\Phi^0_\kappa, A\rangle_\kappa|_{\kappa=0} = \langle\tilde{\Phi}, A\rangle^\sim = \Psi(X, 1)$$

as announced. □

Note that in [5], Baladi and Smania study (in the case of piecewise expanding maps) the more difficult problem of differentiability in horizontal directions (*i.e.*, directions tangent to a topological class). It appears likely that this could be done here also (as conjectured in [5]) , but we have not tried to do so.

20 Discussion.

The codimension 1 condition $X(b) + F_0(X) = 0$ for $\lambda = 1$ expresses that X is a *horizontal* perturbation, which means that it is tangent to a topological class of unimodal maps (see [1] and references given there). In our case, a family (f_κ) is in a topological conjugacy class if $f^3_\kappa c_\kappa = \xi_\kappa f^3 c$ in the notation of Appendix C. The informal result obtained in Section 17 and the formal proof of differentiability along a topological conjugacy class given by Theorem 19 support the conjecture by Baladi and Smania [5] that the map $f \mapsto \langle\Phi^0_f, A\rangle$ is differentiable (in the sense of Whitney) in horizontal directions, *i.e.*, along a curve tangent to a topological conjugacy class. Our theorem 19 also relates the derivative along a topological conjugacy class to a naturally defined susceptibility function. It seems unlikely that a derivative (in the sense of Whitney) exists in nonhorizontal directions. Note however that if $f \mapsto \langle \Phi_f^0, A \rangle$ is nondifferentiable, it will be in a mild way: the "nondifferentiable" contribution to $\Psi(\lambda)$ is, as we saw above, proportional to

$$\frac{d}{db}\langle (1-\lambda\mathcal{L})^{-1}\psi_{(b)}, A\rangle \sim \sum_n \lambda^n \frac{d}{db}\langle \mathcal{L}^n \psi_{(b)}, A\rangle$$

where $\langle \mathcal{L}^n \psi_{(b)}, A\rangle$ decreases exponentially with n, while $\frac{d}{db}\langle \mathcal{L}^n \psi_{(b)}, A\rangle$ increases exponentially. Therefore, if one does not see the small scale fluctuations of $b \mapsto \langle (1-\lambda\mathcal{L})^{-1}\psi_{(b)}, A\rangle$, this function will seem differentiable. But the singularities with respect to λ (with $|\lambda| < 1$) may remain visible. In conclusion, a physicist may see singularities with respect to λ of a derivative (with respect to f or b) while this derivative may not exist for a mathematician.

A Appendix (proof of Remark 16(a)).

We return to the analysis in Section 10, and note that by an analytic change of variable $x \mapsto \xi(x)$ we can get $y = fx = b - \xi^2$ [we have indeed $b-y = A(x-c)^2(1+\beta(x).(x-c))$ with β analytic, and we can take $\xi = (x-c)A^{1/2}(1+\beta(x).(x-c))^{\frac{1}{2}}$]. Write $\rho(x)\,dx = \tilde{\rho}(\xi)\,d\xi$ (where $\tilde{\rho}$ is analytic near 0). The density of the image $\delta(y)\,dy$ by f of $\rho(x)\,dx = \tilde{\rho}(\xi)\,d\xi$ is, near b,

$$\delta(y) = \frac{1}{2\sqrt{y-b}}(\tilde{\rho}(\sqrt{y-b}) + \tilde{\rho}(-\sqrt{y-b})) = \frac{\hat{\rho}(y-b)}{\sqrt{y-b}}$$

where $\hat{\rho}$ is analytic near 0. Therefore, near b,

$$\delta(x) = \frac{U}{\sqrt{b-x}} + U'\sqrt{b-x} + \dots$$

where $U = \rho(c)/\sqrt{A}$, and U' is linear in $\rho(c), \rho'(c), \rho''(c)$ with coefficients depending on the derivatives of f at c. Near a we find

$$\delta(x) = U|f'(b)|^{-1/2}\frac{1}{\sqrt{x-a}} + (U'|f'(b)|^{-3/2} - \frac{3}{4}Uf''(b)|f'(b)|^{-5/2})\sqrt{x-a}$$

Writing $s_n = -\mathrm{sgn}\prod_{k=0}^{n-1} f'(f^k b)$, $t_n = |\prod_{k=0}^{n-1} f'(f^k b)|^{-1/2}$, we claim that near $f^n b$ we have a singularity given for $s_n(x - f^n b) < 0$ by 0, and for $s_n(x - f^n b) > 0$ by

$$\delta(x) = \frac{Ut_n}{\sqrt{s_n(x-f^n b)}} + (U't_n^3 - \frac{3}{4}Ut_n\sum_{k=0}^{n-1} s_{k+1}\frac{f''(f^k b)}{|f'(f^k b)|}\frac{t_n^2}{t_k^2})\sqrt{s_n(x-f^n b)}$$

[to prove this we use induction on n, and the fact that, when $f : x \mapsto y$ for x close to $f^n b$ we have:

$$s_n(x - f^n b) = \frac{s_{n+1}(y - f^{n+1}b)}{|f'(f^n b)|}[1 - \frac{f''(f^n b)}{2|f'(f^n b)|^2}(y - f^{n+1}b)]$$

$$dx = \frac{dy}{|f'(f^n b)|}[1 - \frac{f''(f^n b)}{|f'(f^n b)|^2}(y - f^{n+1}b)] \qquad].$$

Define now

$$\psi_n^{(0)}(x) = \big(1 - (\frac{x - f^n b}{w_n - f^n b})^2\big)\frac{\theta_n(x)}{\sqrt{s_n(x - f^n b)}}$$

$$\psi_n^{(1)}(x) = \big(1 - (\frac{x - f^n b}{w_n - f^n b})^2\big)\theta_n(x)\sqrt{s_n(x - f^n b)}$$

for $s_n(x - f^n b) > 0$, 0 otherwise. Then, the expected singularity of δ near $f^n b$ is given by

$$Ut_n\psi_n^{(0)} + (U't_n^3 - \frac{3}{4}Ut_n\sum_{k=0}^{n-1} s_{k+1}\frac{f''(f^k b)}{|f'(f^k b)|}\frac{t_n^2}{t_k^2})\psi_n^{(1)} = C_n^{(0)}\psi_n^{(0)} + C_n^{(1)}\psi_n^{(1)}$$

where $C_0^{(0)} = U$, $C_0^{(1)} = U'$, and

$$C_{n+1}^{(0)} = |f'(f^n b)|^{-1/2} C_n^{(0)}$$

$$C_{n+1}^{(1)} = |f'(f^n b)|^{-3/2} C_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) C_n^{(0)}$$
$$= |f'(f^n b)|^{-3/2} (C_n^{(1)} - \frac{3}{4} s_{n+1} \frac{f''(f^n b)}{|f'(f^n b)|} C_n^{(0)})$$

Let

$$f(\psi_n^{(0)}(x)\,dx) = \tilde{\psi}_{n+1}^{(0)}(x)\,dx \qquad , \qquad f(\psi_n^{(1)}(x)\,dx) = \tilde{\psi}_{n+1}^{(1)}(x)\,dx$$

and write

$$\tilde{\psi}_{n+1}^{(0)} = |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) \psi_{n+1}^{(1)} + \chi_n^{(0)}$$

$$\tilde{\psi}_{n+1}^{(1)} = |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)} + \chi_n^{(1)}$$

The density of $f(C_n^{(0)} \psi_n^{(0)}(x)\,dx + C_n^{(1)} \psi_n^{(1)}(x)\,dx)$ is then

$$C_{n+1}^{(0)} \psi_{n+1}^{(0)} + C_{n+1}^{(1)} \psi_{n+1}^{(1)} + C_n^{(0)} \chi_n^{(0)} + C_n^{(1)} \chi_n^{(1)}$$

The functions $\chi_n^{(0)}, \chi_n^{(1)}$ have been constructed such that they and their first derivatives $\chi_n^{(0)\prime}, \chi_n^{(1)\prime}$ have the properties of Lemma 11. Namely, $\chi_n^{(0)}, \chi_n^{(1)}, \chi_n^{(0)\prime}, \chi_n^{(1)\prime}$ are continuous with bounded variation on $[a,b]$ uniformly in n, they vanish at a, b, and if $n \geq 1$ they extend to holomorphic functions on the appropriate D_α, with uniform bounds.

Let $\mathcal{A}_1' \subset \mathcal{A}_1$ consist of the (ϕ_α) such that the derivatives ϕ_{-1}', ϕ_{-2}' of ϕ_{-1}, ϕ_{-2} vanish at $\pi_b^{-1} b$ and $\pi_a^{-1} a$ respectively. Let also $\mathcal{A}_2'$ consist of the sequences $(c_n^{(0)}, c_n^{(1)})$, with $c_n^{(0)}, c_n^{(1)} \in \mathbf{C}$, $n = 0, 1, \ldots$ such that

$$||(c_n^{(0)}, c_n^{(1)})||_2' = \sup_{n \geq 0} \delta^n (|c_n^{(0)}| + |c_n^{(1)}|) < \infty$$

If $\Phi' = ((\phi_\alpha), (c_n^{(0)}, c_n^{(1)})) \in \mathcal{A}' = \mathcal{A}_1' \oplus \mathcal{A}_2'$ we let $||\Phi'||' = ||(\phi_\alpha)||_1 + ||(c_n^{(0)}, c_n^{(1)})||_2'$, making $\mathcal{A}'$ into a Banach space. We may now proceed as in Section 12, replacing $\mathcal{A}$ by $\mathcal{A}'$, and defining $\mathcal{L}' : \mathcal{A}' \mapsto \mathcal{A}'$ in a way similar to $\mathcal{L} : \mathcal{A} \mapsto \mathcal{A}$, but with (ii), (v), (vi) replaced as follows:

(ii) $\phi_0 \Rightarrow \Big((\hat{c}_0^{(0)}, \hat{c}_0^{(1)}) = (U, U')\,,\ \hat{\phi}_{-1} = \pm \frac{\phi_0}{|f'|} \circ \tilde{f}_{-1}^{-1} - U(\pm \frac{1}{2} \psi_0^{(0)} \circ \pi_b) - U'(\pm \frac{1}{2} \psi_0^{(1)} \circ \pi_b) \Big)$
so that $\hat{\phi}_{-1}$ is holomorphic in $\pi_b^{-1} D_{-1}$ with vanishing derivative at $\pi_b^{-1} b$

(v) $(c_0^{(0)}, c_0^{(1)}) \Rightarrow \Big((\hat{c}_1^{(0)}, \hat{c}_1^{(1)}) = (|f'(b)|^{-1/2} c_0^{(0)}, |f'(b)|^{-3/2} c_0^{(1)} - \frac{3}{4} |f'(b)|^{-5/2} f''(b) c_0^{(0)}),$
$\chi_0 = \pm \frac{1}{2} c_0^{(0)} (\frac{\psi_0^{(0)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-1/2} \psi_1^{(0)} \circ \pi_a + \frac{3}{4} |f'(b)|^{-5/2} f''(b) \psi_1^{(1)} \circ \pi_a)$
$\pm \frac{1}{2} c_0^{(1)} (\frac{\psi_0^{(1)}}{|f'|} \circ \pi_b \circ \tilde{f}_{-2}^{-1} - |f'(b)|^{-3/2} \psi_1^{(1)} \circ \pi_a)$ in $\pi_a^{-1} D_{-2} \Big)$

(vi) $(c_n^{(0)}, c_n^{(1)}) \Rightarrow$
$\Big((\hat{c}_{n+1}^{(0)}, \hat{c}_{n+1}^{(1)}) = (|f'(f^n b)|^{-1/2} c_n^{(0)}, |f'(f^n b)|^{-3/2} c_n^{(1)} - \frac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) c_n^{(0)}),$

$\chi_{n\alpha} = c_n^{(0)} [\frac{\psi_n^{(0)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-1/2} \psi_{n+1}^{(0)} + \dfrac{3}{4} s_{n+1} |f'(f^n b)|^{-5/2} f''(f^n b) |\psi_{n+1}^{(1)}]$

$+ c_n^{(1)} [\frac{\psi_n^{(1)}}{|f'|} \circ f_n^{-1} - |f'(f^n b)|^{-3/2} \psi_{n+1}^{(1)}]$ in D_α if $V_\alpha \subset \{x : \theta_n(f_n^{-1} x) > 0\}$, 0 otherwise$\Big)$
if $n \geq 1$.

We write then

$$\mathcal{L}' \Phi' = \tilde{\Phi}' = ((\tilde{\phi}_\alpha), (\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}))$$

where

$$\tilde{\phi}_{-2} = \hat{\phi}_{-2} + \chi_0 \qquad , \qquad \tilde{\phi}_{-1} = \hat{\phi}_{-1}$$

$$\tilde{\phi}_\alpha = \sum_{\beta : f V_\beta = V_\alpha} \hat{\phi}_{\beta\alpha} + \hat{\phi}_\alpha + \sum_{n \geq 1} \chi_{n\alpha} \qquad \text{if order } \alpha \geq 0$$

$$(\tilde{c}_n^{(0)}, \tilde{c}_n^{(1)}) = (\hat{c}_n^{(0)}, \hat{c}_n^{(1)}) \qquad \text{for } n \geq 0$$

With the above definitions and assumptions we find, by analogy with Theorem 13, that $\mathcal{L}' : \mathcal{A}' \to \mathcal{A}'$ has essential spectral radius $\leq \max(\gamma^{-1}, \delta\alpha^{1/2})$. There is (see Proposition 15) a simple eigenvalue 1, and the rest of the spectrum has radius < 1. It is convenient to denote by $\Phi^0 = ((\phi_\alpha^0), (c_n^{0(0)}, c_n^{0(1)}))$ the eigenfunction to the eigenvalue 1. We find again that $\phi^0 = \Delta(\phi_\alpha^0)$ is continuous, of bounded variation, and satisfies $\phi^0(a) = \phi^0(b) = 0$, but we can say more. Using the notation in the proof of Proposition 15, we have again

$$\gamma_j^0 = \sum_k \mathcal{L}_{jk} \gamma_k^0 + \eta_j$$

with $\eta_j = \sum_{n=0}^\infty \eta_{jn}$, but now $\eta_{jn} = c_n^{0(0)} \chi_n^{(0)} + c_n^{0(1)} \chi_n^{(1)} | W_j$ for $n \geq 1$, so that the η_j have derivatives $\eta_j' \in \mathcal{H}_j$. The derivatives $\gamma_j^{0\prime}$of the γ_j^0 are measures satisfying

$$\gamma_j^{0\prime} = \sum_k \mathcal{L}'_{jk} \gamma_k^{0\prime} + \eta_j^*$$

The operator $\mathcal{L}'_{jk}$ has the same form as $\mathcal{L}_{jk}$, but with an extra denominator $f' \circ (f^{-1})_{kj}$, and therefore $\mathcal{L}'_* = (\mathcal{L}'_{jk})$ acting on measures has spectral radius $\leq \alpha < 1$. The term η_j^* is the sum of η_j' and a term$\sum_k \mathcal{L}_{kj}^{\sim} \gamma_k^0$ where $\mathcal{L}_{kj}^{\sim}$ involves the derivative of $|f' \circ (f^{-1})_{kj}|^{-1}$ so that $\eta_j^* \in \mathcal{H}_j$. The operator $\mathcal{L}'_*$ also maps $\mathcal{H}$ to $\mathcal{H}$ and, by the same argument as for $\mathcal{L}_*$, has essential spectral radius < 1 on $\mathcal{H}$. Furthermore, 1 cannot be an eigenvalue since $\mathcal{L}'_*$ has spectral radius < 1 on measures. It follows that $(\gamma^{0\prime}) = (\gamma_j^{0\prime}) = (1 - \mathcal{L}'_*)^{-1}(\eta_j^*) \in \mathcal{H}$. Therefore, the derivative $\phi^{0\prime}$ of ϕ^0 may have discontinuities only on the orbit of u_1, and hyperbolicity again shows that this cannot happen. In conclusion, ϕ^0 and its derivative $\phi^{0\prime}$ are both of bounded variation, continuous, and vanishing at a, b.

A discussion similar to the above shows that the equation $\gamma = (1 - \mathcal{L}'_*)^{-1} \eta^*$ also defines γ with finite norm in $\mathcal{A}_1$, and this γ must coincide with $(\gamma^{0\prime})$ as a measure. Therefore the family of derivatives $(\phi_\alpha^{0\prime})$ is an element of $\mathcal{A}_1$. [For simplicity, we have written $\phi_{-1}^{0\prime}$, $\phi_{-2}^{0\prime}$ for the functions which, under application of Δ, give the derivative of $\Delta\phi_{-1}^0$, $\Delta\phi_{-2}^0$]. $\square$

B Appendix (proof of Remark 16(b)).

If $u \in \tilde{H}$ and $\psi_{(u\pm)}$ is defined as in Remark 16(b), we want to show that there is a unique (ϕ_α) in $\mathcal{A}_1$ such that $\phi_\alpha = \psi_{(u\pm)}|V_\alpha$ for all α. Furthermore $||(\phi_\alpha)||_1$ is bounded uniformly for $u \in \tilde{H}$, provided we assume $1 < \gamma < \min(\beta^{-1}, \alpha^{-1/2})$.

Note that uniqueness is automatic, and that $\phi_\alpha = 0$ unless order $V_\alpha > 0$. Omitting the $\pm$ we let

$$f(\psi_{(f^n u)}(x)\, dx) = [|f'(f^n u)|^{-1/2}\psi_{(f^{n+1}u)}(x) + \chi_{(f^n u)}(x)]\, dx$$

For $n \geq 0$ there is a unique ω_{un} such that $f^{n+1}(\omega_{un}\, dx) = \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}\chi_{(f^n u)}\, dx$ and $[f^k u - c] \times [\text{supp}\, f^k(\omega_{un}(x)\, dx) - c] > 0$ for $0 \leq k \leq n$. Furthermore $\psi_{(u)} = \sum_{n=0}^{\infty} \omega_{un}$ where the sum restricted to each V_α is finite. If $[\chi_{(f^n u)}]$ denotes the element of $\mathcal{A}_1$ corresponding to $\chi_{(f^n u)}$, we find that $||[\chi_{(f^n u)}]||_1$ is bounded uniformly in n and u. Also note that we obtain ω_{un} from $\prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}\chi_{(f^n u)}$ by multiplying with $\prod_{k=0}^{n-1} |f'(f^k u)|$ (up to a factor bounded uniformly in n because of hyperbolicity) and composing with f^{n+1} (restricted to a small interval J such that $f^{n+1}|J$ is invertible). We have thus

$$||[\omega_{un}]||_1 \leq \text{const}\, \gamma^n \prod_{k=0}^{n-1} |f'(f^k u)|^{-1/2}$$

where $[\omega_{un}]$ is the element of $\mathcal{A}_1$ corresponding to ω_{un} [This is because the replacement of $|V_\alpha|$ by $|(f|J)^{-n-1}V_\alpha|$ in the definition of $||\cdot||_1$ is compensated up to a multiplicative constant by the factor $\prod_{k=0}^{n-1} |f'(f^k u)|$]. Thus

$$||[\omega_{un}]||_1 \leq \text{const}\, (\gamma\alpha^{1/2})^n$$

Since $\gamma < \alpha^{-1/2}$ we find that $\sum_n ||[\omega_{un}]||_1 <$ constant independent of u. Therefore, since $(\phi_\alpha) = \sum_n [\omega_{un}]$, we see that $||(\phi_\alpha)||_1$ is bounded independently of u. $\square$

C Appendix (proof of Remark 16(c)).

We consider a one-parameter family (f_κ) of maps, reducing to $f = f_0$ for $\kappa = 0$. We assume that $(\kappa, x) \mapsto f_\kappa x$ is real-analytic. For κ close to 0, f_κ has a critical point c_κ and maps $[a_\kappa, b_\kappa]$ to itself, with $b_\kappa = f_\kappa c_\kappa, a_\kappa = f_\kappa^2 c_\kappa$. There is (by hyperbolicity of H with respect to f) a homeomorphism $\xi_\kappa : H \to H_\kappa$ where H_κ is an f_κ-invariant hyperbolic set for f_κ and $f_\kappa \circ \xi_\kappa = \xi_\kappa \circ f$ on H. We shall consider a compact set K of values of κ such that $f_\kappa a_\kappa \in \tilde{H}_\kappa$; we let $K \ni 0$, K of small diameter, and assume now $\kappa \in K$. We may in a natural way define a Banach space $\mathcal{A}_\kappa = \mathcal{A}_{\kappa 1} \oplus \mathcal{A}_2$ and an operator $\mathcal{L}_\kappa : \mathcal{A}_\kappa \to \mathcal{A}_\kappa$ associated with f_κ so that $\mathcal{A}_\kappa, \mathcal{L}_\kappa$ reduce to $\mathcal{A}, \mathcal{L}$ for $\kappa = 0$. Note that, since $\kappa \in K$ is close to 0, we may assume that the constants A, α in the definition (Section 4) of hyperbolicity, and the constants B, β (Section 7) are uniform in κ.

Let $\eta_{\kappa,-2}$ be a biholomorphic map of the complex neighborhood D_{-2} of $[a, u_1]$ to the complex neighborhood $D_{\kappa,-2}$ of the corresponding interval $[a_\kappa, u_{\kappa 1}]$, and lift $\eta_{\kappa,-2}$ to a holomorphic map $\tilde{\eta}_{\kappa,-2} : \pi_a^{-1} D_{-2} \to \pi_{a_\kappa}^{-1} D_{\kappa,-2}$. We also lift $\eta_{\kappa,-1} = f_\kappa^{-1} \circ \eta_{\kappa,-2} \circ f$ to

$$\tilde{\eta}_{\kappa,-1} = \tilde{f}_{\kappa,-2}^{-1} \circ \eta_{\kappa,-2} \circ \tilde{f}$$

where the notation is that of Section 12, with obvious modification. We write

$$\tilde{\eta}_{\kappa 0} = \tilde{f}_{\kappa,-1}^{-1} \circ \tilde{\eta}_{\kappa,-1} \circ \tilde{f}_{-1}$$

and

$$\tilde{\eta}_{\kappa\beta} = (f_\kappa | V_{\kappa\beta})^{-1} \circ \tilde{\eta}_{\kappa\alpha} \circ f | V_\beta$$

if order $\beta > 0$ and $f V_\beta = V_\alpha$. We have defined $\eta_{\kappa\alpha}$ above for $\alpha = -1, -2$, and we let $\eta_{\kappa\alpha} = \tilde{\eta}_{\kappa\alpha}$ when order $\alpha \geq 0$.

We introduce a map $\eta_\kappa : \mathcal{A}_{\kappa 1} \to \mathcal{A}_1$ by

$$\eta_\kappa(\phi_{\kappa\alpha}) = ((\phi_{\kappa\alpha} \circ \tilde{\eta}_{\kappa\alpha}).\eta'_{\kappa\alpha})$$

so that $\mathcal{L}_\kappa^\times = (\eta_\kappa, \mathbf{1}) \mathcal{L}_\kappa (\eta_\kappa^{-1}, \mathbf{1})$ acts on $\mathcal{A}$. Using the decomposition

$$\mathcal{L}_\kappa = \begin{pmatrix} \mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1} & \mathcal{L}_{\kappa 2} \\ \mathcal{L}_{\kappa 3} & \mathcal{L}_{\kappa 4} \end{pmatrix}$$

as in Section 12, we define $L_\kappa^\times$ on $\mathcal{A}_1$ by

$$L_\kappa^\times(\phi_\alpha) = \eta_\kappa(\mathcal{L}_{\kappa 0} + \mathcal{L}_{\kappa 1}) \eta_\kappa^{-1}(\phi_\alpha) + (\eta_\kappa^{-1}\phi_\alpha)_0(c_\kappa).\eta_\kappa \mathcal{L}_{\kappa 2}(|\frac{1}{2} f_\kappa''(c_\kappa) \prod_{k=0}^{n-1} f_\kappa'(f_\kappa^k b_k)|^{-1/2})$$

$$= \mathcal{L}_0(\phi_\alpha) + \eta_\kappa \mathcal{L}_{\kappa 1} \eta_\kappa^{-1}(\phi_\kappa) + \eta'_{\kappa 0}(c_\kappa)^{-1} \phi_0(c_\kappa).\eta_\kappa \mathcal{L}_{\kappa 2}(|\frac{1}{2} f_\kappa''(c_\kappa) \prod_{k=0}^{n-1} f_\kappa'(f_\kappa^k b_k)|^{-1/2})$$

$L_\kappa^\times$ is a compact perturbation of $\mathcal{L}_{\kappa 0}$, and has therefore essential spectral radius $\leq \gamma^{-1}$. If (ϕ_α) is a (generalized) eigenfunction of $L_\kappa^\times$ to the eigenvalue μ, then

$$((\phi_\alpha), \eta_{\kappa 0}(c_\kappa)^{-1}\phi_0(c_\kappa).(|\frac{1}{2}f_\kappa''(c_\kappa)\prod_{k=0}^{n-1}f_\kappa'(f_\kappa^k b_k)|^{-1/2}))$$

is a (generalized) eigenfunction of $\mathcal{L}_\kappa^\times$ to the same eigenvalue μ. We have thus a multiplicity-preserving bijection of the eigenvalues μ of $L_\kappa^\times$ and $\mathcal{L}_\kappa^\times$ when $|\mu| > \max(\gamma^{-1}, \delta\alpha^{1/2})$. In particular, 1 is a simple eigenvalue of $L_\kappa^\times$ for the values of κ considered (a compact neighborhood K of 0).

The operator $L_\kappa^\times$ acting on $\mathcal{A}_1$ depends continuously on κ. [This is because $\hat{\phi}_{\kappa\alpha}$, $\chi_{\kappa 0}$, $\chi_{\kappa n\alpha}$ depend continuously on κ (in particular, the $\chi_{\kappa n\alpha}$ for large n are uniformly small). Note however that $\mathcal{L}_\kappa^\times$ does not depend continuously on κ because the continuity of $f_\kappa'(f_\kappa^k b_\kappa)$ is not uniform in k]. There is $\epsilon > 0$ such that $L_\kappa^\times$ has no eigenvalue μ_κ with $|\mu_\kappa - 1| < \epsilon$ except the simple eigenvalue 1 [otherwise the continuity of $\kappa \to L_\kappa^\times$ would imply that 1 has multiplicity > 1 for some κ]. Therefore, the 1-dimensional projection corresponding to the eigenvalue 1 of $L_\kappa^\times$ depends continuously on κ, and so does the eigenvector $\Phi_\kappa^\times = (\eta_\kappa, 1)\Phi_\kappa^0$ of $\mathcal{L}_\kappa^\times$, where Φ_κ^0 denotes the eigenvector the the eigenvalue 1 of $\mathcal{L}_\kappa$ normalized so that $w_\kappa\Phi_\kappa^0 \geq 0$ and $\int w_\kappa\Phi_\kappa^0 = 1$, with the obvious definition of w_κ (involving the spikes $\psi_{\kappa n}$ associated with f_κ).

Note that a number of results have been obtained earlier on the continuous dependence of the a.c.i.m. ρ on parameters. I am indebted to Viviane Baladi for communicating the references [25], [27], [14], and also [26].

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