Algebraic cycles on degenerate fibers

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In this paper we extend part of the theory of degeneration of Hodge structures to algebraic cycles.

The theory of limiting mixed Hodge structures leads to a proof of the local invariant cycle theorem for proper families of complex manifolds over the unit disk with semi-stable degeneration at the origin [C] [St] [S] [GN]. In the proof of this result, a key role is played by the spectral sequence coming from the weight filtration on the complex of nearby cycles. One can compute the E_1 term and the first differential d_1 of this spectral sequence. One finds that (E_1, d_1) is isomorphic to a complex (K, d) where K is a sum of cohomology groups of the different strata of the special fiber, and the differential d is defined by means of the corestriction and Gysin morphisms relating these cohomology groups [St] [GN]. Since the weight spectral sequence degenerates from E_2 on, one can prove properties of the limit Hodge structure on the cohomology of the general fiber from this computation of (E_1, d_1) and the fact that each stratum is a compact Kähler manifold ([GN], sections 3, 4, 5).

Our basic remark is that the complex (K, d) makes sense in a more general set up, for most cohomology theories and also for algebraic cycles modulo any adequate equivalence relation, given any principal reduced Cartier divisor on a scheme instead of a family of complex manifolds, even though, in general, the abutment of the weight spectral sequence needs not be defined. Furthermore, one can reproduce the proof of the local invariant cycle theorem in this more general setting, as soon as each stratum of the special fiber satisfies both the hard Lefschetz theorem and the Hodge index theorem.

Note that the second and third authors found recently another manifestation of the motivic nature of the weight spectral sequence [GS]: for any

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variety X over a field of characteristic zero, the E_1-term of the weight spectral sequence converging to the cohomology with compact supports can be computed from a complex of pure Chow motives canonically attached to X (up to homotopy). It would be of interest to express the results in the present paper in terms of these complexes of motives.

The main motivation for our study came from our attempt to develop a non archimedean analog of Arakelov theory [BGS]. If $\mathcal{X}$ is a regular proper scheme over a discrete valuation ring with special fiber Y a reduced divisor with normal crossings, we may consider Chow homology groups $\mathrm{CH_p(Y)}$ and Fulton's Chow cohomology groups $\mathrm{CH^p(Y)}$, $p \geq 0$. The inclusion $i : Y \to \mathcal{X}$ induces a group morphism $i^* i_*$ from $\mathrm{CH_p(Y)}$ to $\mathrm{CH}^{\mathrm{dim}(\mathcal{X})-\mathrm{p}}(\mathrm{Y})$, and it was shown in [BGS] that, when resolution of singularities holds, both the kernel and the cokernel of $i^* i_*$ depend only on the generic fiber $\mathcal{X} - Y$ and not on the model $\mathcal{X}$. In the analogy proposed in op.cit., these groups are analogs of $\partial\overline{\partial}$-cohomology groups of a complex manifold. We then raised the question of whether $\ker(i^* i_*)$ and $\mathrm{coker}(i^* i_*)$ could be isomorphic in appropriate degrees, the same way $\partial\overline{\partial}$-cohomology groups of Kähler manifolds coincide with the usual cohomology. Here, we prove that it is indeed the case when all strata satisfy the hard Lefschetz and Hodge index theorem (Theorem 5; see 4.4. for cases when the hypotheses hold true).

The paper is organized as follows. In section one, under general assumptions, we define a bigraded group $K^{\cdot\cdot}$ with two differentials d' and d'' and a "monodromy" operator N, which is zero or the identity map on direct summands of $K^{\cdot\cdot}$. In section two, we define using $K^{\cdot\cdot}$ several cohomology groups. These are analogs of the graded quotients of the weight filtration on cohomology in the case of limits of Hodge structures. In section 3, we assume that all strata satisfy the hard Lefschetz and Hodge index theorems, and, following [GN], we deduce that the cohomology of $(K^{\cdot\cdot}, d)$ has the structure of a bigraded polarized Hodge-Lefschetz module in the sense of Deligne and Saito (Theorem 1). This implies analogs of the fact that weight and monodromy filtrations coincide (Theorem 2), of the local invariant cycle theorem (Theorem 3) and of the Clemens-Schmidt exact sequence (Theorem 4). As a consequence, we prove in Theorems 5 and 6 that the kernel and cokernel of $i^* i_*$ coincide and enjoy properties similar to the cohomology of compact Kähler manifolds. Finally, we discuss in section 5 possible relations of our work with motivic cohomology.

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1. A bigraded complex with monodromy.

1.1. Let $\mathcal{X}$ be a regular nœtherian scheme of Krull dimension $n+1$ and Y a principal Cartier divisor with normal crossings on $\mathcal{X}$. Let $Y_1, Y_2, \ldots, Y_t$ be the irreducible components of Y. We shall assume that Y is reduced, i.e. each component has multiplicity one.

For any subset $I \subset \Sigma = \{1, \ldots, t\}$, we let $Y_I = \bigcap_{i \in I} Y_i$ if $I \neq \emptyset$, and $Y_\emptyset = \mathcal{X}$. We assume that each stratum Y_I, $I \neq \emptyset$, is a regular scheme.

The codimension of Y_I in $\mathcal{X}$ is the cardinality $r = |I|$ of I. We define also $Y^{(0)} = \mathcal{X}$,

$$Y^{(r)} = \coprod_{|I|=r} Y_I \quad \text{if} \quad 1 \leq r \leq n\,,$$

and $Y^{(r)} = \emptyset$ if $r > n$.

1.2. For every $I \subset \Sigma$ we assume given a commutative graded ring with unit $\bigoplus_{p \geq 0} A^p(Y_I)$, where $A^p(Y_I) = 0$ if $p > \dim(Y_I)$, and for every inclusion $u = u_{IJ} : Y_I \to Y_J$, corresponding to $J \subset I$, group morphisms

$$u^* : A^p(Y_J) \to A^p(Y_I)$$

and

$$u_* : A^p(Y_I) \to A^{p+|I|-|J|}(Y_J)\,.$$

The following properties are assumed to hold:

A1. When $I = J$, $u^* = u_* = \mathrm{id}$.

When $K \subset J \subset I$, let $u : Y_I \to Y_J$ and $v : Y_J \to Y_K$ be the corresponding inclusions. Then

$$(v \circ u)^* = u^* \circ v^* \quad \text{and} \quad (v \circ u)_* = v_* \circ u_*\,.$$

A2. Let I and J be two subsets in Σ. Consider the diagram of inclusions

$$\begin{array}{ccc} Y_I & \xrightarrow{v} & Y_{I \cap J} \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} \\ Y_{I \cup J} & \xrightarrow[u]{} & Y_J\,. \end{array}$$

Then $b^* \circ v_* = u_* \circ a^*$.

A3. The morphisms a^* are ring homomorphisms and the following projection formula holds:

$$u_*(u^*(x)\, y) = x\, u_*(y)\,.$$

In particular

$$u_*\, u^*(x) = x\, u_*(1)\,. \tag{1}$$

Furthermore

$$u^*\, u_*(x) = x\, u^*\, u_*(1)\,. \tag{2}$$

A4. Let $u_i : Y_i \to \mathcal{X}$, $i \in \Sigma$, be the inclusions of the components of Y into $\mathcal{X} = Y_\emptyset$. Then

$$\sum_{i=1}^{t} u_{i*}(1) = 0\,. \tag{3}$$

1.3. Examples.

1.3.1. An example of the situation described in 1.1 and 1.2 is when $\mathcal{X}$ is a smooth projective complex manifold and $A^p(Y_I) = H^{p,p}(Y_I, \mathbb{R})$ is the real cohomology of Y_I of type (p,p). If u^* is the corestriction and u_* the Gysin map, all properties A1 – A4 are easy to check. Property A2 is a consequence of the transversality of the strata.

Formula (2) holds, and $u^*\, u_*(1)$ is the Euler class of the normal bundle of Y_I in Y_J. Property (3) follows from the fact that Y is a principal divisor, hence its fundamental class $[Y] = \sum_{i=1}^{t} u_{i*}(1)$ in $H^2(\mathcal{X})$ vanishes.

This case was studied in [St], [D], [GN] by means of Hodge theory. In fact, these papers consider the full cohomology and not only the subspace of type (p,p), but we are mainly interested in classes of algebraic cycles, which are of such type (this restriction will simplify the signs coming in 3.2 below; compare [GN] §3).

1.3.2. Cohomology theories with Gysin maps, as in [BO] say, give other examples of the situation in 1.2. In [M], Mokrane studied the case of cristalline cohomology. One can also take for $A^p(Y_I)$ the Chow group $\mathrm{CH^p(Y_I)}$ of codimension p algebraic cycles on Y_I modulo linear equivalence. If all Y_I's are defined over a field or if one neglects torsion, a ring structure is known to exist on Chow groups.

Finally $A^p(Y_I)$ may also denote codimension p algebraic cycles on Y_I modulo algebraic, homological or numerical equivalence and one may replace $A^p(Y_I)$ with its tensor product over $\mathbb{Z}$ with any ring (in 3.2 below we shall assume that $A^p(Y_I)$ is a real vector space).

1.4. With the notation of 1.2 we put

$$A^p(Y^{(r)}) = \bigoplus_{|I|=r} A^p(Y_I)\,,$$

and for all $p \geq 0$ and k with $1 \leq k \leq r$, we define as follows group morphisms

$$\delta_k^* : A^p(Y^{(r)}) \to A^p(Y^{(r+1)})$$

and

$$\delta_{k*} : A^p(Y^{(r+1)}) \to A^{p+1}(Y^{(r)})\,.$$

Let $I \subset \Sigma$ be such that $|I| = r+1$. Write $I = \{i_1, \ldots, i_{r+1}\}$ with $i_1 < i_2 < \ldots < i_{r+1}$ and let $J = I - \{i_k\}$. Then the restriction of δ_{k*} to $A^p(Y_I)$ is u_{IJ*} and the component of δ_k^* in $A^{p+1}(Y_J)$ is u_{IJ}^*.
We then define morphisms

$$\rho : A^p(Y^{(r)}) \to A^p(Y^{(r+1)})$$

and

$$\gamma : A^p(Y^{(r+1)}) \to A^{p+1}(Y^{(r)})$$

by the formulae

$$\rho = \sum_{k=1}^{r+1} (-1)^{k-1}\, \delta_k^*$$

and

$$\gamma = \sum_{k=1}^{r+1} (-1)^{k-1}\, \delta_{k*}\,.$$

Now let $i, j, k \in \mathbb{Z}$ be three integers. We introduce the following notation:

$$K^{ijk} = A^{\frac{i+j-2k+n}{2}}(Y^{(2k-i+1)})$$

when $k \geq 0$, $k \geq i$ and $i+j+n \equiv 0 \pmod 2$, and

$$K^{ijk} = 0 \quad \text{otherwise.}$$

We also let $K^{ij} = \bigoplus_{\substack{k \geq 0 \\ k \geq i}} K^{ijk}$.

We now define three morphisms

$$d' : K^{ijk} \to K^{i+1,j+1,k+1},$$

$$d'' : K^{ijk} \to K^{i+1,j+1,k},$$

and

$$N : K^{ijk} \to K^{i+2,j,k+1},$$

by $d'(\alpha) = \rho(\alpha)$, $d''(\alpha) = -\gamma(\alpha)$ and $N(\alpha) = \alpha$, whenever the domain and target of these maps do not vanish, and $d'(\alpha) = 0$, $d''(\alpha) = 0$ or $N(\alpha) = 0$ otherwise. (All these definitions are inspired from [GN], especially sections 2.6 and 2.7.)

1.5. .

Here are some properties of d', d'' and N:

Lemma 1.

i) $d'^2 = d''^2 = d'\,d'' + d''\,d' = 0$.

ii) *The operator N commutes with d and d'.*

iii) *For any $i \geq 0$ and $j \in \mathbb{Z}$, the map*

$$N^i = K^{-i,j} \to K^{i,j}$$

is an isomorphism.

iv) *For any $i \geq 0$ and $j \in \mathbb{Z}$,*

$$\ker(N^{i+1}) \cap K^{-i,j} = K^{-i,j,0} = A^{\frac{j-i+n}{2}}\left(Y^{(i+1)}\right).$$

Proof. Statements ii), iii) and iv) are direct consequences of the definitions by checking degrees. In i), the equalities $d'^2 = d''^2 = 0$ follow from A1). So we are left with checking that $d'\,d'' + d''\,d' = 0$.

On K^{ijk} we have $d' \, d'' = d'' \, d' = 0$, while on K^{ijk} with $k > i$ the equality $d' \, d'' + d'' \, d' = 0$ means that

$$\rho \, \gamma + \gamma \, \rho = 0 \,. \tag{4}$$

We shall prove that (4) holds on $A^p(Y^{(r)})$ for any $r \geq 1$ and $p \geq 0$. By definition

$$\rho \, \gamma = \sum_{\ell=1}^{r} \sum_{k=1}^{r} (-1)^{k+\ell-2} \, \delta_\ell^* \, \delta_{k*}$$

and

$$\gamma \, \rho = \sum_{\ell=1}^{r+1} \sum_{k=1}^{r+1} (-1)^{k+\ell-2} \, \delta_{\ell *} \, \delta_k^* \,.$$

Fix $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_r\}$ two subsets of Σ, with $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$, and consider the component

$$\psi : A^p(Y_I) \to A^{p+1}(Y_J)$$

of the map $\rho \, \gamma + \gamma \, \rho$. First assume that $I \neq J$, i.e.

$$\{i_1, \ldots, \widehat{i}_k, \ldots, i_r\} = \{j_1, \ldots, \widehat{j}_\ell, \ldots, j_r\} = I \cap J \,,$$

and assume that $k < \ell$. Then

$$\psi = (-1)^{k+\ell-1} \, \delta_k^* \, \delta_{\ell+1,*} + (-1)^{k+\ell-2} \, \delta_{\ell *} \, \delta_k^* \,.$$

Applying A2) to I and J we get

$$\delta_k^* \, \delta_{\ell+1,*} = \delta_{\ell *} \, \delta_k^*$$

and therefore $\psi = 0$. A similar argument applies when $k > \ell$.

Now assume that $I = J = I_0$. Then we find

$$\psi = \sum_{k=1}^{r} \delta_k^* \, \delta_{k*} + \sum_{k=1}^{r+1} \delta_{k*} \, \delta_k^* \,. \tag{5}$$

Let $u : Y_{I_0} \subset \mathcal{X} = Y_\emptyset$ be the inclusion. From (1) we deduce that, for all $x \in A^p(Y_{I_0})$,

$$\sum_{k=1}^{r} \delta_k^* \, \delta_{k*}(x) = \sum_{\substack{J \supset I_0 \\ |J| = r+1}} u_{JI_0,*} \, u_{JI_0}^*(x) = x \sum_{\substack{J \supset I_0 \\ |J| = r+1}} u_{JI_0,*}(1) \,.$$

Therefore, using A2), we get

$$\sum_{k=1}^{r+1} \delta_{k*}\, \delta_k^*(x) = x\, u^* \left(\sum_{i \notin I} u_{i*}(1) \right) . \tag{6}$$

On the other hand, using (2), we get

$$\sum_{k=1}^{r} \delta_k^*\, \delta_{k*}(x) = \sum_{\substack{J \subset I_0 \\ |J| = r-1}} u_{I_0 J}^*\ u_{I_0 J,*}(x) = x \sum_{\substack{J \subset I_0 \\ |J| = r-1}} u_{I_0 J}^*\ u_{I_0 J,*}(1) .$$

Applying A2) again, we deduce that

$$\sum_{k=1}^{r} \delta_k^*\, \delta_{k*}(x) = x\, u^* \left(\sum_{i \in I} u_{i*}(1) \right) . \tag{7}$$

From (5), (6), (7) and A4 we conclude that

$$\psi(x) = x\, u^* \left(\sum_{i \in \Sigma} u_{i*}(1) \right) = 0 .$$

This ends the proof of (4) and of Lemma 1.

q.e.d.

1.6. For any integer p with $0 \leq p \leq n$, we define

$$A^p(Y) = \ker \left(\rho : A^p(Y^{(1)}) \to A^p(Y^{(2)}) \right)$$

and

$$A_p(Y) = \operatorname{coker} \left(\gamma : A^{n-p-1}(Y^{(2)}) \to A^{n-p}(Y^{(1)}) \right) .$$

For instance, when $A^p(Y_I) = \mathrm{CH^p(Y_I)}$, it is known that $A_p(Y)$ is the group of p-dimensional cycles on Y modulo linear equivalence, when $A^p(Y)$ is Fulton's operational Chow group of codimension p on Y (see [K] and [BGS] Appendix).

Notice that the maps

$$\rho = \delta_0^* : A^p(\mathcal{X}) \to A^p(Y^{(1)})$$

and

$$\gamma = \delta_{0*} : A^{p-1}(Y^{(1)}) \to A^p(\mathcal{X})$$

are such that $\rho\,\delta_0^* = \rho_{0*}\,\gamma = 0$. Therefore ρ and γ induce morphisms

$$i^* : A^p(\mathcal{X}) \to A^p(Y)$$

and

$$i_* : A_{n+1-p}(Y) \to A^p(\mathcal{X}) ,$$

where $i : Y \to \mathcal{X}$ is the inclusion.

Lemma 2. *The composite map*

$$A^{p-1}(Y^{(1)}) \to A_{n+1-p}(Y) \stackrel{i^* i_*}{\to} A^p(Y) \to A^p(Y^{(1)})$$

coincides with $-\gamma\,\rho$.

Proof. According to equation (4), the morphism $\gamma\,\rho + \rho\,\gamma$ vanishes on $A^p(Y^{(1)})$. But the map

$$\rho\,\gamma = \delta_0^*\,\delta_{0*} : A^{p-1}(Y^{(1)}) \to A^p(Y^{(1)})$$

is, by definition, the composite of the sequence of morphisms

$$A^{p-1}(Y^{(1)}) \to A_{n+1-p}(Y) \stackrel{i^* i_*}{\to} A^p(Y) \to A^p(Y^{(1)}) .$$

q.e.d.

2. Cohomology groups.

2.1. From Lemma 1 we know that $d = d' + d'' : K^{ij} \to K^{i+1,j+1}$ satisfies $d^2 = 0$ and commutes with the operator $N : K^{ij} \to K^{i+2,j}$. In particular d gives a differential on the bigraded groups

$$\ker(N)^{\cdot\cdot} = \ker(K^{\cdot\cdot} \stackrel{N}{\to} K^{\cdot+2,\cdot})$$

and

$$\operatorname{coker}(N)^{\cdot\cdot} = \operatorname{coker}(K^{\cdot\cdot} \stackrel{N}{\to} K^{\cdot+2,\cdot}) ,$$

as well as on the mapping cone of N

$$\mathrm{Cone(N)}^{\cdot\cdot} = \mathrm{Cone}(\mathrm{K}^{\cdot\cdot} \stackrel{\mathrm{N}}{\to} \mathrm{K}^{\cdot+2,\cdot}) .$$

For any $q \geq 0$ and $r \in \mathbb{Z}$, we define

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}}\,\mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) = \frac{\ker(\mathrm{d}:\mathrm{K}^{-\mathrm{r},\mathrm{q-n}} \to \mathrm{K}^{-\mathrm{r}+1,\mathrm{q-n}+1})}{\mathrm{im}(\mathrm{d}:\mathrm{K}^{-\mathrm{r}-1,\mathrm{q-n}-1} \to \mathrm{K}^{-\mathrm{r},\mathrm{q-n}})},$$

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}}\,\mathrm{H}^{\mathrm{q}}(\mathrm{X}^*) = \frac{\ker(\mathrm{d}:\mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}-1} \to \mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+2,\mathrm{q-n}})}{\mathrm{im}(\mathrm{d}:\mathrm{Cone}(\mathrm{N})^{-\mathrm{r},\mathrm{q-n}-2} \to \mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}-1})},$$

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}}\,\mathrm{H}^{\mathrm{q}}(\mathrm{Y}) = \frac{\ker(\mathrm{d}:\ker(\mathrm{N})^{-\mathrm{r},\mathrm{q-n}} \to \ker(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}+1})}{\mathrm{im}(\mathrm{d}:\ker(\mathrm{N})^{-\mathrm{r}-1,\mathrm{q-n}-1} \to \ker(\mathrm{N})^{-\mathrm{r},\mathrm{q-n}})},$$

and

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}}\,\mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathrm{X}) = \frac{\ker(\mathrm{d}:\mathrm{coker}(\mathrm{N})^{-\mathrm{r},\mathrm{q-n}-2} \to \mathrm{coker}(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}-1})}{\mathrm{im}(\mathrm{d}:\mathrm{coker}(\mathrm{N})^{-\mathrm{r}-1,\mathrm{q-n}-3} \to \mathrm{coker}(\mathrm{N})^{-\mathrm{r},\mathrm{q-n}-2})}.$$

These definitions are again inspired from the theory of variations of Hodge structures [St],[S], [GN]. But, in our general set up, the symbols X^* and $\widetilde{X}^*$ do not denote any variety, and these are purely formal definitions. However, as we shall see, the groups that we have just introduced enjoy the same properties as the graded quotients of the weight filtration on the corresponding cohomology groups in op. cit.. For instance, in the following lemma, we prove analogs of the Wang exact sequence and the standard exact sequence for cohomology with supports (see [St] (4.25)–(4.29); notice that, in loc.cit., the cohomology of Y and X coincide).

Lemma 3. *For all $i \in \mathbb{Z}$, there are exact sequences*

(8)
$$\cdots \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}}(\mathrm{X}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \xrightarrow{\mathrm{N}} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}-2}\,\mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}+1}(\mathrm{X}^*) \to \cdots$$

and

(9)
$$\cdots \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}}(\mathrm{Y}) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}}(\mathrm{X}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}+1}_{\mathrm{Y}}(\mathrm{X}) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\,\mathrm{H}^{\mathrm{q}+1}(\mathrm{Y}) \to \cdots.$$

Proof. By definition the differential on

$$\mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}-1} = \mathrm{K}^{-\mathrm{r},\mathrm{q-n}} \oplus \mathrm{K}^{-\mathrm{r}+1,\mathrm{q-n}-1}$$

maps (x, y) to $(-dx, Nx + dy) \in \mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+2,\mathrm{q-n}}$. In particular there are exact sequences

$$0 \to K^{-r+1,q-n-1} \to \mathrm{Cone}(\mathrm{N})^{-\mathrm{r}+1,\mathrm{q-n}-1} \to \mathrm{K}^{-\mathrm{r},\mathrm{q-n}} \to 0$$

and, when $i = q + r$ is fixed, these give rise to a short exact sequence of complexes. Its associated long exact sequence of cohomology groups is (8). Similarly there are inclusions

$$\ker(N)^{-r,q-n} \hookrightarrow K^{-r,q-n} \hookrightarrow \mathrm{Cone(N)}^{-r+1,q-n-1}$$

and projections

$$\mathrm{Cone(N)}^{-r+1,q-n-1} \to \mathrm{K}^{-r+1,q-n-1} \to \mathrm{Coker(N)}^{-r+1,q-n-1}$$

which induce maps of complexes for any fixed value of $i = q + r$. The total complex of the double complex

$$\begin{aligned} 0 \to (\ker(N)^{-r,q-n}, d) \to (\mathrm{Cone(N)}^{-r+1,q-n-1}, \mathrm{d}) \\ \to (\mathrm{Coker(N)}^{-r+1,q-n-1}, -\mathrm{d}) \to 0\,, \end{aligned}$$

$q+r = i$, is acyclic, and (9) is the associated long exact sequence of cohomology groups.

q.e.d.

2.2. Now we give formulae for some of the groups defined in 2.1.

Lemma 4.

i) *The group* $\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}(\mathrm{Y})$ *is zero unless* $r \leq 0$ *and* $q + r$ *is even, in which case*

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}(\mathrm{Y}) = \frac{\ker\left(\rho : \mathrm{A}^{\frac{q+r}{2}}(\mathrm{Y}^{(-r+1)}) \to \mathrm{A}^{\frac{q+r}{2}}(\mathrm{Y}^{(-r+2)})\right)}{\mathrm{im}\left(\rho : \mathrm{A}^{\frac{q+r}{2}}(\mathrm{Y}^{(-r)}) \to \mathrm{A}^{\frac{q+r}{2}}(\mathrm{Y}^{(-r+1)})\right)}$$

when $r < 0$ *and*

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \mathrm{H}^{\mathrm{2p}}(\mathrm{Y}) = \mathrm{A}^{\mathrm{p}}(\mathrm{Y})\,.$$

ii) *The group* $\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathrm{X})$ *is zero unless* $r \geq 0$ *and* $q + r$ *is even, in which case*

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathrm{X}) = \frac{\ker\left(\gamma : \mathrm{A}^{\frac{q-r-2}{2}}(\mathrm{Y}^{(r+1)}) \to \mathrm{A}^{\frac{q-r}{2}}(\mathrm{Y}^{(r)})\right)}{\mathrm{im}\left(\gamma : \mathrm{A}^{\frac{q-r-4}{2}}(\mathrm{Y}^{(r+2)}) \to \mathrm{A}^{\frac{q-r-2}{2}}(\mathrm{Y}^{(r+1)})\right)}$$

when $r > 0$ *and*

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \mathrm{H}^{\mathrm{2p}}_{\mathrm{Y}}(\mathrm{X}) = \mathrm{A}_{\mathrm{n+1-p}}(\mathrm{Y})\,.$$

iii) *The map* $\mathrm{gr}_i^W \mathrm{H}_Y^q(X) \to \mathrm{gr}_i^W \mathrm{H}^q(X)$ *in the exact sequence* (9) *is zero unless q is even and $i = q$. On* $\mathrm{gr}_{2p}^W \mathrm{H}_Y^{2p}(X)$ *this map coincides with the morphism*

$$-i^* \, i_* : A_{n+1-p}\,(Y) \to A^p(Y)\,.$$

Proof.

i) The map

$$N : K^{ijk} \to K^{i+2,j,k+1}$$

is the identity unless $k = i \geq 0$, in which case $N = 0$. Therefore

$$(\ker N)^{-r,q-n} = K^{-r,q-n,-r} = A^{\frac{q+r}{2}}\,(Y^{(=r+1)})$$

where $r \leq 0$ and $q + r$ is even, and $(\ker N)^{-r,q-n} = 0$ otherwise. The differential

$$d'' : K^{-r,q-n,-r} \to K^{-r+1,q-n+1,-r} = 0$$

vanishes, therefore $d = d'$ on $\ker(N)^{\cdot\cdot}$. The formula for $\mathrm{gr}_{q+r}^W \mathrm{H}^q(Y)$ follows from this. In particular

$$\mathrm{gr}_{2p}^W \mathrm{H}^{2p}(Y) = \ker\left(\mathrm{A}^p(Y^{(1)}) \xrightarrow{\rho} \mathrm{A}^p(Y^{(2)})\right) = \mathrm{A}^p(Y)$$

by definition (see 1.6).

ii) The full group K^{ijk} lies in the image of N unless $k = 0$. Therefore the composite map

$$K^{-r,q-n,0} \to K^{-r,q-n} \to \mathrm{coker}(N)^{-r,q-n}$$

is an isomorphism when $r \geq 0$ and $\mathrm{coker}(N)^{-r,q-n} = 0$ when $r < 0$. Furthermore $\mathrm{im}(d') \cap K^{-r+1,q-n+1,0} = \{0\}$, therefore we get a commutative diagram

$$\begin{array}{ccc} K^{-r,q-n,0} & \xrightarrow{d''} & K^{-r+1,q-n+1,0} \\ \downarrow & & \downarrow \\ \mathrm{coker}(N)^{-r,q-n} & \xrightarrow{d} & \mathrm{coker}(N)^{-r+1,q-n+1} \end{array}$$

where the vertical maps are isomorphisms.

Since $d'' = \gamma$ on $K^{-r,q-n,0} = A^{\frac{q-r}{2}}(Y^{(r+1)})$ if $r \geq 0$ and $q-r$ is even, our formula for $\mathrm{gr}^W_{q+r} \mathrm{H}^q_Y(X)$ follows. In particular

$$\mathrm{gr}^W_{2p} \mathrm{H}^{2p}_Y(X) = \mathrm{coker}\left(\mathrm{A}^{p-2}(\mathrm{Y}^{(2)}) \xrightarrow{\gamma} \mathrm{A}^{p-1}(\mathrm{Y}^{(1)})\right) = \mathrm{A}_{n+1-p}(\mathrm{Y})$$

(see 1.6).

iii) The vanishing of the map

$$\mathrm{gr}^W_{q+r} \mathrm{H}^q_Y(X) \to \mathrm{gr}^W_{q+r} \mathrm{H}^q(Y)$$

when $r \neq 0$ is clear, since, by i) and ii), one of these groups is zero.

The map

$$\mathrm{gr}^W_{2p} \mathrm{H}^{2p}_Y(X) \to \mathrm{gr}^W_{2p} \mathrm{H}^{2p}(Y)$$

can be described explicitly as follows. Let $x \in \mathrm{coker}(N)^{0,2p-n-2}$ be a closed representative of $[x] \in \mathrm{gr}^W_{2p} \mathrm{H}^{2p}_Y(X)$. Denote by

$$\widetilde{x} \in K^{0,2p-n-2,0} = A^{p-1}(Y^{(1)})$$

the element mapping to x by the isomorphism

$$K^{0,2p-n-2,0} \to \mathrm{coker}(N)^{0,2p-n-2,0} .$$

Since $d''(\widetilde{x}) = 0$ in $K^{1,2p-n-1}$ we get

$$d(\widetilde{x}) = d'(\widetilde{x}) = \rho(\widetilde{x}) \in K^{1,2p-n-1,1} .$$

Therefore $d(\widetilde{x}) = N(y)$ for a unique element

$$y \in K^{-1,2p-n-1,0} = A^{p-1}(Y^{(2)}) .$$

The pair $(-y, \widetilde{x}) \in K^{-1,2p-n-1} \oplus K^{0,2p-n-2} = \mathrm{Cone(N)}^{0,2p-n-2}$ maps to $x \in \mathrm{coker}(N)^{0,2p-n-1}$ by the canonical projection. Its boundary in $\mathrm{Cone(N)}^{\cdot\cdot}$ is equal to

$$d(-y, \widetilde{x}) = (dy, 0)$$

and dy lies in the kernel of N:

$$N\, d(y) = d\, N(y) = d \circ d(\widetilde{x}) = 0 .$$

By construction of the exact sequence (9), the class of $d(y)$ in $\mathrm{gr}^W_{2p} \mathrm{H}^{2p}(Y)$ is the image of $[x]$ we are looking for. But

$$d(y) = d''(y) \in K^{0,2p-n,0}$$

is nothing but

$$d''(y) = -\gamma(y) = -\gamma\, \rho(\widetilde{x}) \in A^p(Y^{(1)}) .$$

By Lemma 2, its image in $A^p(Y) = \mathrm{gr}^W_{2p} \mathrm{H}^{2p}(Y)$ coincides with $-i^*\, i_*([x])$.

q.e.d.

3. A result by Saito and Deligne.

3.1. Fix an element L in $A^1(Y)$. For any nonempty subset $I \subset \Sigma$ and any integer $p \geq 0$, we get a *Lefschetz operator*

$$\ell : A^p(Y_I) \to A^{p+1}(Y_I)$$

by mapping any element x to its product by the image of L in $A^1(Y_I)$. Similarly, there are morphisms

$$\ell : A^p(Y^{(r)}) \to A^{p+1}(Y^{(r+1)})$$

and

$$\ell : K^{ijk} \to K^{i,j+2,k}\,.$$

Lemma 5.

i) *The following identities hold*

$$[\ell, N] = [\ell, d'] = [\ell, d''] = 0\,.$$

ii) *For any integers i and j, the group*

$$K^{-i,-j} \cap \ker(\ell^{j+1}) \cap \ker(N^{j+1})$$

coincides with the set

$$\ker\left(\ell^{j+1} : A^{\frac{-i-j+n}{2}}(Y^{(i+1)}) \to A^{\frac{-i+j+n+2}{2}}(Y^{(i+1)})\right)$$

(primitive cycles).

Proof. The identities in i) are easy consequences of the definitions and of the projection formula in A3). Statement ii) follows from Lemma 1 iv).

q.e.d.

We shall say *all strata satisfy the hard Lefschetz theorem* (HLT) when, given any nonempty subset $I \subset \Sigma$ and $p \geq 0$ an integer with $n \geq 2p + r - 1$, where $r = |I|$, the map

$$\ell^{n-2p-r-1} : A^p(Y_I) \to A^{n-r+1-p}(Y_I)$$

is an isomorphism. This means that

(10) $$\ell^j : K^{i,-j} \to K^{i,j}$$

is an isomorphism for all $i \in \mathbb{Z}$ and $j \geq 0$. For cases where this hypothesis is known to hold, see 4.4 below.

3.2. From now on, we assume that all rings $\underset{p\geq 0}{\oplus} A^p(Y_I)$, $I \subset \Sigma$, are graded $\mathbb{R}$-algebras, that the maps u^* and u_* in 1.2 are $\mathbb{R}$-linear and that, for any nonempty $I \subset \Sigma$, there exists a *trace map*

$$\mathrm{tr} : A^{n+1-|I|}(Y_I) \to \mathbb{R}$$

such that $\mathrm{tr}(u_*(x)) = \mathrm{tr}(x)$ for any $u : Y_I \to Y_J$ as in 1.2 (in the examples of 1.3, when Y is proper, such a trace map is given by the degree of 0-dimensional cycles). We can then define a pairing

$$\psi : K^{-i,-j,k} \otimes K^{i,j,k+i} \to \mathbb{R}$$

by the formula

$$\psi(x \otimes y) = (-1)^{\frac{i+j-n}{2}} \ \mathrm{tr}(xy)$$

where

$$x \in K^{-i,-j,k} = A^{\frac{-i-j-2k+n}{2}}(Y^{(2k+i+1)})$$

and

$$y \in K^{i,j,k+i} = A^{\frac{-i+j-2k+n}{2}}(Y^{(2k+i+1)}) .$$

We extend ψ by zero to get a pairing

$$\psi : K^{\cdot\cdot} \otimes K^{\cdot\cdot} \to \mathbb{R}$$

(which vanishes on $K^{ijk} \otimes K^{i'j'k'}$ unless $i' + i = j' + j = k' + i - k = 0$).

Lemma 6. *The following identities hold*

$$\psi(y, x) = (-1)^n \, \psi(x, y) ,$$

$$\psi(Nx, y) + \psi(x, Ny) = 0 ,$$

$$\psi(\ell x, y) + \psi(x, \ell y) = 0 ,$$

$$\psi(d'x, y) = \psi(x, d''y) ,$$

and

$$\psi(d''x, y) = \psi(x, d'y) .$$

Proof. The first three equalities are direct consequences of the definitions. To check the fourth one, let $x \in K^{-i-1,-j-1,k-1}$ and $y \in K^{i,j,k+i}$. Then we get

$$\psi(d'x, y) = (-1)^{\frac{i+j-n}{2}} \operatorname{tr}(\rho(x)y) = (-1)^{\frac{i+j-n}{2}} \sum_{k\geq 1} (-1)^{k-1} \operatorname{tr}(\delta_k^*(x)y) .$$

But for all k

$$\operatorname{tr}(\delta_k^*(x)y) = \operatorname{tr}(\delta_{k*}(\delta_k^*(x)y)) = \operatorname{tr}(x\, \delta_{k*}(y))$$

(by the projection formula A3)). Therefore

$$\psi(d'x, y) = (-1)^{\frac{i+j-n}{2}} \operatorname{tr}(x\, \rho(y)) = -(-1)^{\frac{i+j-n}{2}} \operatorname{tr}(x\, d''(y)) = \psi(x, d''\, y) .$$

The last identity in Lemma 6 follows from this by using the first one.

q.e.d.

We shall say that *all strata satisfy the Hodge index theorem* (HIT) when, given any nonempty subset $I \subset \Sigma$, $p \geq 0$ with $n \geq 2p + r - 1$, where $r = |I|$, and $x \in A^p(Y_I)$ such that $\ell^{n-r+2-2p}(x) = 0$, then

$$(-1)^p \operatorname{tr}(x\, \ell^{n-r+1-2p}\, (x)) \geq 0 ,$$

with equality if and only if $x = 0$. By Lemma 5 ii) this means that the bilinear form

$$Q(x, y) = \psi(x, \ell^j\, N^i\, y) \tag{11}$$

is positive definite on the real vector space

$$K^{-i,-j} \cap \ker(\ell^{j+1}) \cap \ker(N^{i+1}) .$$

3.3. From now on we shall assume that all strata satisfy both HLT and HIT. Then the Lemma 1, 5 and 6 say that $K = \bigoplus_{i,j} K^{ij}$, together with the pairing ψ define a differential polarized Hodge-Lefschetz bigraded module in the sense of [GN] (4.1)–(4.3); to fit exactly with the situation in [GN], the group K^{ijk} is given a real pure Hodge structure entirely of type (p, p) with $p = \frac{j-i+n}{2}$, hence the Weil operator C is the identity map. We further assume that K is a finite dimensional real vector space, i.e. $A^p(Y_I)$ is finite dimensional for every $p \geq 0$ and I not empty. We can then apply all the results of [GN] to our situation. Namely, by Lemma 1 iii) and (10) the operators N and ℓ induce an action on K of the Lie algebra $s\ell_2(\mathbb{R}) \times s\ell_2(\mathbb{R})$, hence of the Lie group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Consider the matrices $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{R})$ and

$$w_2 = (w, w) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R})\,.$$

The pairing

$$Q(x, y) = \psi(x, w_2\, y)$$

is then symmetric and positive definite on K; this follows from (11) as in [GN], (4.3), Proposition. The differential $d : K \to K$ has an adjoint $d^* = w_2^{-1}\, dw_2$ for Q (this follows from Lemma 6), and we may consider the Laplace operator

$$\Box = dd^* + d^*d\,.$$

The main result of [GN], due to Saito [S] 4.2.2., and Deligne (unpublished), is then the following

Theorem 1.

i) *The Laplace operator* $\Box$ *on* K *commutes with the action of* $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

ii) *The cohomology* $H^*(K, d)$ *of* K*, when equipped with* N*,* ℓ*, and* ψ*, is a polarized bigraded Hodge-Lefschetz module in the sense of* [GN], (4.1)–(4.3).

Proof. See [GN] (4.5). Statement ii) follows from i) by identifying $H^*(K, d)$ with the group of harmonic elements $\ker \Box$. It means that both operators N and ℓ on the bigraded module $H^*(K, d)$ satisfy hard Lefschetz theorems (as in Lemma 1 iii) and (10)), and that the bilinear form $\psi(x, w_2\, y)$ on this module is symmetric and positive definite.

4. Consequences.

4.1. We keep the assumption that all strata satisfy both HLT and HIT and that K is finite dimensional. One can then deduce from Theorem 1 analogs of several classical results about limits of Hodge structures.

First, the assertion that the operator N on the cohomology of K satisfies the hard Lefschetz theorem can be rephrased as follows:

Theorem 2. *For any $q \geq 0$ and $r \geq 0$, the composition of r copies of the operator N induces an isomorphism*

$$N^r : \mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \stackrel{\sim}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{q-r}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*)\,.$$

From this and Lemma 3, as in [St] (5.12), one gets the following invariant cycle theorem. Define the specialization map sp to be the composite of the morphisms

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\mathrm{Y}) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\mathrm{X}^*) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*)$$

from (9) and (8) in Lemma 2. Then we have:

Theorem 3. *For all $q \geq 0$ and $i \in \mathbb{Z}$, the sequence*

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\mathrm{Y}) \stackrel{\mathrm{sp}}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \stackrel{\mathrm{N}}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i-2}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*)$$

is exact.

Finally, let λ be the composite of the morphisms

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{i-2}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q+1}}(\mathrm{X}^*) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q+2}}_{\mathrm{Y}}(\mathrm{X})$$

in the sequences (8) and (9). From Lemma 3 and Theorem 2, one gets, by a diagram chasing described in [Sc] and [GNPP] IV (7.14), the following Clemens-Schmidt exact sequence:

Theorem 4. *For all $i \in \mathbb{Z}$ there is a long exact sequence*

$$\cdots \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathrm{X}) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\mathrm{Y}) \stackrel{\mathrm{sp}}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*)$$
$$\stackrel{N}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i-2}}\, \mathrm{H}^{\mathrm{q}}(\widetilde{\mathrm{X}}^*) \stackrel{\lambda}{\rightarrow} \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q+2}}_{\mathrm{Y}}(\mathrm{X}) \rightarrow \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}\, \mathrm{H}^{\mathrm{q+2}}(\mathrm{Y}) \stackrel{\mathrm{sp}}{\rightarrow} \cdots\,.$$

4.2. Recall from 1.6 that there is a morphism

$$i^*\, i_* : A_{n+1-p}(Y) \rightarrow A^p(Y)$$

for all $p \geq 0$. In [BGS] §2 we considered the kernel and the cokernel of $i^*\, i_*$, which played a role similar to $\partial\bar{\partial}$-cohomology groups of complex manifolds. Under the assumption that all strata satisfy HLT and HIT we shall see that these groups coincide. For any $p \geq 0$, denote by

$$\alpha : A^p(Y) \rightarrow A_{n-p}(Y)$$

the composite of the morphisms

$$A^p(Y) \to A^p(Y^{(1)}) \to A_{n-p}(Y) .$$

Theorem 5. *The following long sequence is exact*

$$\cdots \to A_{n+1-p}(Y) \stackrel{i^* i_*}{\to} A^p(Y) \stackrel{\alpha}{\to} A_{n-p}(Y) \stackrel{i^* i_*}{\to} A^{p+1}(Y) \to \dots .$$

In other words, α induces an isomorphism

$$\tau : \operatorname{coker}(i^* \, i_* : A_{n+1-p}(Y) \to A^p(Y)) \stackrel{\sim}{\to} \ker\left(i^* \, i_* : A_{n-p}(Y) \to A^{p+1}(Y)\right) .$$

Proof. (Compare [BGS] (6.4).) According to Lemma 4 iii) we can identify $-i^* \, i_*$ on $A_{n+1-p}(Y)$ with the map

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}_{\mathrm{Y}}(\mathrm{X}) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}(\mathrm{Y})$$

in the exact sequence (9). By the Clemens-Schmidt exact sequence (Theorem 4) this means that the group

$$H^{pp}_{II} = \operatorname{coker}(i^* \, i_* : A_{n+1-p}(Y) \to A^p(Y))$$

coincides with

$$\ker\left(N : \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p-2}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*)\right) ,$$

and the group

$$H^{pp}_{I} = \ker\left(i^* \, i_* : A_{n-p}(Y) \to A^{p+1}(Y)\right)$$

coincides with

$$\operatorname{coker}\left(N : \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p+2}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*)\right) .$$

Now, by Theorem 2, the map

$$N^2 : \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p+2}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p-2}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*)$$

is an isomorphism, therefore H^{pp}_{II} is canonically isomorphic to H^{pp}_{I}. The isomorphism is induced by the composite of the maps

$$A^p(Y) = \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}(\mathrm{Y}) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}} \, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p+1}} \, \mathrm{H}^{\mathrm{2p+1}}(\mathrm{X}^*)$$
$$\to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p+2}} \, \mathrm{H}^{\mathrm{2p+2}}_{\mathrm{Y}}(\mathrm{X}) = \mathrm{A}_{\mathrm{n-p}}(\mathrm{Y})$$

in Lemma 3. This composite comes from the projection

$$\ker(N)^{0,2p-n} = A^p(Y^{(1)}) \to \mathrm{coker}(N)^{0,2p-n} = A^p(Y^{(1)}) \,,$$

hence it coincides with α.

q.e.d.

4.3. Let $M = \mathcal{X} - Y$ and $p \geq 0$ an integer. We shall denote by $H^{pp}(M)$ the group $\ker(i^* \, i_* : A_{n-p}(Y) \to A^{p+1}(Y))$ considered in Theorem 5. This notation is justified by [BGS] Theorem 2.3.1 iii), where it is proved that this group depends only on M and not on $\mathcal{X}$ when resolution of singularities holds (conditions M1) and M2) in op.cit. (1.1)) and when A^p of a blow up can be computed in a standard fashion (Remark (6.2.7) in [BGS]).

We shall now describe $H^{pp}(M)$ as a subspace of $A^p(Y^{(1)})$. The operator $(d')^* = w_2^{-1} \, d'' w_2$ is an adjoint to d' for the scalar product Q, as follows from Lemma 6, and similarly $(d'')^* = w_2^{-1} \, d' w_2$ is an adjoint to d''. In particular the map

$$\rho : A^p(Y^{(1)}) = K^{0,2p-n,0} \to A^p(Y^{(2)})$$

has an adjoint ρ^* and the map

$$\gamma : A^p(Y^{(2)}) = K \to A^p(Y^{(1)})$$

has an adjoint γ^*. These adjoint morphisms are unique since, by assumption, Q is positive definite. On the other hand, we let

$$\ell : H^{pp}(M) \to H^{p+1,p+1}(M)$$

be the map induced by the product by the class of $L \in A^p(Y^{(1)})$ and

$$\psi : H^{pp}(M) \otimes H^{n-p,n-p}(M) \to \mathbb{R}$$

be the pairing induced by

$$A^p(Y^{(1)}) \otimes A^{n-p}(Y^{(1)}) \to A^n(Y^{(1)}) \xrightarrow{\mathrm{tr}} \mathbb{R} \,.$$

Theorem 6.

i) *The Lefschetz operator induces isomorphisms*

$$\ell^{n-2p} : H^{pp}(M) \to H^{n-p,n-p}(M)$$

whenever $n \geq p$.

ii) *If $x \in H^{pp}(M)$ is such that $\ell^{n-2p+1}(x) = 0$, then*

$$(-1)^p\, \psi(x\, \ell^{n-2p}(x)) \geq 0\,,$$

with equality if and only if $x = 0$.

iii) *$H^{pp}(M)$ is isomorphic to the kernel of $\rho\rho^* + \gamma\gamma^*$ acting on $A^p(Y^{(1)})$.*

Proof. From the proof of Theorem 5 we know that $H^{pp}(M) = H_I^{pp}$ is canonically isomorphic to the kernel of

$$N : \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p}}\, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*) \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{2p-2}}\, \mathrm{H}^{\mathrm{2p}}(\widetilde{\mathrm{X}}^*)\,.$$

By Theorem 1, $\Box$ commutes with N, so $H^{pp}(M)$ is isomorphic to the group of harmonic elements

$$\mathcal{H}^{pp}(M) = \ker \Box \cap \ker(N)^{0,2p-n}$$

in $\ker(N)^{0,2p-n} \subset K^{0,2p-n}$. In particular, since Q is positive definite on K (by HIT), we get ii), i.e. $H^{pp}(M)$ satisfies the Hodge index theorem. Since ℓ and N commute it is also clear that ℓ^{n-2p} is injective on $\mathcal{H}^{pp}(M)$ when $n \geq 2p$. Conversely, if $x = \ell^{n-2p}(y)$ lies in $\mathcal{H}^{n-p,n-p}(M)$ with y in $K^{\cdot\cdot}$, we have

$$\ell^{n-2p}(\Box\, y) = 0$$

and

$$\ell^{n-2p}(Ny) = 0\,,$$

therefore $\Box\, y = Ny = 0$ and y lies in $\mathcal{H}^{pp}(M)$. This shows that ℓ^{n-2p} is surjective, and that $H^{pp}(M)$ satisfies the hard Lefschetz theorem (property i)).

To prove iii), notice that $d'' = (d')^* = 0$ on $\ker(N)^{0,2p-n}$ for degree reasons. Therefore, on this group we have

$$\begin{aligned} \Box &= (d' + d'')(d'^* + d''^*) + (d'^* + d''^*)(d' + d'') \\ &= d''(d'')^* + (d')^*\, d' + d'(d'')^* + (d'')^*\, d'\,. \end{aligned}$$

The map $d'(d'')^* + (d'')^*\, d'$ sends $A^p(Y^{(1)})$ into $A^p(Y^{(3)})$. But $\Box$ preserves bidegrees in $K^{\cdot\cdot}$ and it commutes with N, therefore $\Box$ maps $A^p(Y^{(1)}) =$

$\ker(N)^{0,2p-n}$ into itself. It follows that $d'(d'')^* + (d'')^* d' = 0$ on $A^p(Y^{(1)})$, and $\square = d'(d')^* + (d'')^* d'' = \rho\rho^* + \gamma\gamma^*$ on this group. This proves i).

q.e.d.

4.4. Recall that our proof of Theorems 1 to 6 depends on the assumption that all strata satisfy both HLT and HIT. In [BGS], Theorem 6.3.1 and 6.4.1, Theorem 5 is shown to hold unconditionally for ℓ-adic cohomology and Betti cohomology of complex manifolds (using the Weil conjectures and Hodge theory respectively).

Apart from the cohomology of complex varieties (as in 1.3.1 above), the assumption that all strata satisfy both HLT and HIT and that K is finite dimensional is satisfied when $A^p(Y_I) = \mathrm{CH^p(Y_I)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ are Chow groups, all strata are smooth projective varieties over a field k contained in the algebraic closure of a finite field, and $n \leq 2$. Indeed in that case $A^0(Y_I) = \mathbb{R}$, $A^{\dim(Y_I)}(Y_I) = \mathbb{R}$, and, by Weil, when Y_I is a surface, the Hodge index theorem is known to hold for $A^1(Y_I) = \mathrm{CH^1(Y_I)} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \mathrm{NS(Y_I)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ (the hard Lesfchetz theorem is also known to hold in this context for some varieties of higher dimension, see [So] Thm. 7). See also [M] for a similar result in cristalline cohomology. Under these assumptions, one could expect that HLT and HIT are still true for all strata when $n \geq 3$.

5. Relations with Motivic Homology.

One of the potential applications of the ideas above concerns mixed characteristic degenerations and relations with motivic homology. Assume, that is, that $\mathcal{X}$ is regular and flat and proper over the ring of integers in a mixed characteristic local field. Let X and Y denote the generic and closed fibres of $\mathcal{X}$. We take $A^p(Y_I) := CH^p(Y_I) \otimes \mathbb{Q}$. Unfortunately there are serious technical problems dealing with cycles in mixed characteristic. However, if we make some plausible conjectures about localization in mixed characteristic (i.e. that it behaves like localization in equal characteritic), and the behavior of motivic cohomology for smooth proper varieties over finite fields (cf. Conjecture below), we obtain an identification between the groups $\mathrm{gr_i^W\, H^j_Y}(\mathcal{X})$ in 2.1 and motivic homology of the closed fibre Y. The ranks of these groups and of the groups $H_I^{pp} \cong H_{II}^{pp}$ can then be related, assuming standard conjectures like the Tate conjecture for smooth varieties over finite fields, to orders of zeroes of local factors of L-functions at integer values of s. Results concerning

L-function zeroes are due to Consani [Co]. We sketch here the relation with motivic homology.

Let T be a smooth, proper variety over a field k. We define motivic cohomology in terms of higher Chow groups

$$H^p_{\mathcal{M}}(T, \mathbb{Z}(q)) := CH^q(T, 2q-p).$$

The latter groups are defined as follows. Let $\Delta^n := \mathrm{Spec}\big(\mathrm{k}[\mathrm{t}_0, \ldots, \mathrm{t}_\mathrm{n}]/(\sum \mathrm{t}_\mathrm{i} - 1)\big)$, and let $\mathbb{Z}^q(T,n)$ be the free abelian group generated by irreducible subvarieties $V \subset T \times \Delta^n$ of codimension q meeting faces (defined by $\{t_{i_1} = \ldots = t_{i_r} = 0\}$ for $\{i_1 \ldots, i_r\} \subset \{0, \ldots, n\}$) properly. Restriction to faces (together with evident degeneracy maps) makes

$$\mathbb{Z}^q(T, \cdot) := \cdots \mathbb{Z}^q(T,2) \overset{\to}{\underset{\to}{\to}} \mathbb{Z}^q(T,1) \rightrightarrows \mathbb{Z}^q(T,0)$$

a simplicial abelian group. By definition

$$CH^q(T,n) := \pi_n(\mathbb{Z}^q(T, \cdot)) \cong H_n(\mathbb{Z}^q(T, \cdot)).$$

Note $CH^q(T,0) \cong CH^q(T)$ is the usual Chow group.

Conjecture. *Let T be smooth and proper over a finite field k. Then*
i) $H^p_{\mathcal{M}}(T, \mathbb{Q}(q)) = (0)$ for $\mathrm{p} \neq 2\mathrm{q}$.
ii) $H^{2q}_{\mathcal{M}}(T, \mathbb{Q}(q)) \hookrightarrow H^{2q}_{\acute{e}t}(T_{\overline{k}}, \mathbb{Q}_\ell(q))$.

Now let $Y = \bigcup_{i=1}^{i=t} Y_i \subset \mathcal{X}$ be as above. Assume Y is reduced, and the Y_i meet transversally.

Proposition 1

Assume (i) of the above conjecture. Then

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathcal{X}) \cong \mathrm{CH}^{\frac{\mathrm{q+r-2}}{2}}(\mathrm{Y}, \mathrm{r}) \otimes \mathbb{Q}$$

Proof. Consider the double complex in negative cohomological degrees

$$\mathcal{A}^{\cdot,\cdot}(q) \ : \ \mathbb{Z}^{q-t+1}(Y^{(t)}, \cdot)_{\mathbb{Q}} \xrightarrow{\gamma} \ldots \xrightarrow{\gamma} \mathbb{Z}^q(Y^{(1)}, \cdot)_{\mathbb{Q}},$$

with γ as in 1.4 and $\mathcal{A}^{p,r}(q) = \mathbb{Z}^{q+p}(Y^{(1-p)}, -r)_{\mathbb{Q}}$. It is easy to check that the projection $Y^{(1)} \to Y$ induces a quasi-isomorphism

$$\mathrm{tot}\mathcal{A}^{\cdot,\cdot}(\mathrm{q}) \to \mathbb{Z}^{\mathrm{q}}(\mathrm{Y}, \cdot)_{\mathbb{Q}},$$

where tot denotes the simple complex associated to a double complex. On the other hand, the above conjecture says that the column $\mathcal{A}^{p,\cdot}(q) \simeq CH^{p+q}(Y^{(1-p)})_{\mathbb{Q}}[0]$. It follows easily from Lemma 4 (ii) that

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}_{\mathrm{Y}}(\mathcal{X}) \cong \mathrm{H}^{-\mathrm{r}}(\mathcal{A}^{\cdot,\cdot}(\mathrm{q})) \cong \mathrm{CH}^{\frac{\mathrm{q+r-2}}{2}}(\mathrm{Y},\mathrm{r}) \otimes \mathbb{Q},$$

which proves the proposition.

Example. Suppose Y is geometrically connected and $n = \dim Y = 1$, i.e. $\mathcal{X}$ is a degenerating curve. We have

$$H^{0,0}_I = \ker(i^* i_* : CH_1(Y)_{\mathbb{Q}} \to CH^0(Y)_{\mathbb{Q}}) \cong \mathbb{Q}$$

(negative semi-definiteness for intersection of components Y_i) and

$$H^{1,1}_I = \ker(i^* i_* : CH_0(Y)_{\mathbb{Q}} \to CH^2(Y)_{\mathbb{Q}}) \cong \mathbb{Q}$$

(Y is connected). In the arithmetic case, these correspond to the zero of the L-factor associated to $H^0(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ at $s = 0$ (resp. $H^2(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ at $s = 1$). In addition, the group

$$\mathrm{gr}^{\mathrm{W}}_4 \mathrm{H}^3_{\mathrm{Y}}(\mathcal{X}) = \ker(\mathrm{CH}^0(\mathrm{Y}^{(2)}) \to \mathrm{CH}^1(\mathrm{Y}^{(1)}))$$

can be non-zero, e.g. for an elliptic curve with split multiplicative reduction. The rank of this group equals the order of zero at $s = 0$ of the L-factor associated to the inertia invariants on H^1.

Remark A motivic understanding of $\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}(\mathrm{Y})$ (i.e. a construction of motivic cohomology for singular varieties) and of $\mathrm{gr}^{\mathrm{W}}_{\mathrm{q+r}} \mathrm{H}^{\mathrm{q}}(\mathrm{X})$ (motivic "Tate cohomology") would be very useful.

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