SPECIAL SUBVARIETIES OF NON-ARITHMETIC BALL QUOTIENTS AND HODGE THEORY

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ABSTRACT. Let $\Gamma \subset \mathrm{PU}(1,n)$ be a lattice, and S_Γ the associated ball quotient. We prove that, if S_Γ contains infinitely many maximal complex totally geodesic subvarieties, then Γ is arithmetic. We also prove an Ax-Schanuel Conjecture for S_Γ, similar to the one recently proven by Mok, Pila and Tsimerman. One of the main ingredients in the proofs is to realise S_Γ inside a period domain for polarised integral variations of Hodge structures and interpret totally geodesic subvarieties as unlikely intersections.

CONTENTS

1. Introduction 1
2. Notation, terminology and recollections 5
3. Real Shimura data and general properties 8
4. $\mathbb{Z}$-Hodge theory on non-arithmetic quotients and rigidity 16
5. Comparison between the Γ-special and $\mathbb{Z}$-special structures 23
6. Finiteness of special subvarieties 30
7. Proof of the non-arithmetic Ax-Schanuel conjecture 37
8. Special points and their Zariski closure 41
References 45

1. INTRODUCTION

1.1. **Motivation.** The study of lattices of semisimple Lie groups G is a field rich in open questions and conjectures. Complex hyperbolic lattices and their finite dimensional representations are certainly far from being understood. A lattice $\Gamma \subset G$ is archimedean superrigid if for any simple noncompact Lie group G' with trivial centre, every homomorphism $\Gamma \to G'$ with Zariski dense image extends to a homomorphism $G \to G'$. Thanks to the work of Margulis [55], Corlette [18] and Gromov–Schoen [42], all lattices in simple Lie groups are archimedean superrigid unless G is $\mathrm{SO}(1,n)$ or $\mathrm{PU}(1,n)$ for some $n \geq 1$. Real hyperbolic lattices are known to be softer and more flexible than their complex counterpart and non-arithmetic lattices in $\mathrm{SO}(1,n)$ can be constructed for every n [41]. Non-arithmetic complex hyperbolic lattices have been constructed only in $\mathrm{PU}(1,n)$, for $n = 1, 2, 3$ [62, 24].

Date. July 15, 2020.
2020 Mathematics Subject Classification. 14G35, 22E40, 03C64, 14P10, 32H02, 32M15.
Key words and phrases. Non-arithmetic lattices, Rigidity, Hodge theory and period domains, Functional transcendence, Unlikely intersections.

In both cases one can consider the quotient by Γ of the symmetric space X associated to G. In the complex hyperbolic case we obtain a ball quotient S_Γ which has a natural structure of a quasi-projective variety, as proven by Baily-Borel [5] in the arithmetic case, and by Mok [24] in general. By the commensurability criterion for arithmeticity of Margulis [55] we can decide whether Γ is arithmetic or not by looking at modular correspondences in the product $S_\Gamma \times S_\Gamma$. That is Γ is arithmetic if and only if S_Γ admits infinitely many totally geodesic correspondences. In this paper, by totally geodesic subvarieties we always mean complex subvarieties of S_Γ, whose smooth locus is totally geodesic with respect to the canonical Kähler metric. Even if Γ is arithmetic, S_Γ may have no strict totally geodesic subvarieties. However, if it contains one totally geodesic subvariety then it contains countably many of such. The aim of this paper is to study totally geodesic subvarieties of S_Γ from multiple point of views, ultimately explaining how often and why they appear.

1.2. **Main results.** Let G be $\mathrm{PU}(1,n)$ for some $n>1$ and $\Gamma \subset G$ a lattice. Let X be the Hermitian symmetric space associated to G and S_Γ be the quotient of X by Γ. We first look at maximal totally geodesic subvarieties of S_Γ, that is totally geodesic subvarieties that are not strictly contained in any totally geodesic subvariety different from S_Γ. Our first main result is the following.

Theorem 1.2.1. *If $\Gamma \subset G$ is non-arithmetic, then S_Γ contains only finitely many maximal totally geodesic subvarieties.*

The problem was originally proposed informally by Reid and, independently, by McMullen for real hyperbolic lattices [25, Question 7.6], [56, Question 8.2]. For $\mathrm{SO}(1,n)$ this was recently proven by Bader, Fisher, Miller and Stover [3] and, for closed hyperbolic 3-manifolds, by Margulis and Mohammadi [54]. See also [35] for the first result of this kind. Around the same time Bader, Fisher, Miller and Stover annouced also a proof of the finiteness of the maximal totally geodesic (both real and complex) subvarieties of a ball quotient. The proof has then appeared in [4]. Such approaches build on superrigidity theorems and use results on equidistribution from homogeneous dynamics. As we explain in section 1.3, we are able interpret the problem as a phenomenon of *unlikely intersections* inside a period domain for polarised $\mathbb{Z}$-variations of Hodge structures and deduce the finiteness from a strong Ax-Schanuel theorem established by Bakker and Tsimerman [7]. Indeed we will see that Theorem 1.2.1 is predicted by a conjecture of Klingler [48, Conjecture 1.9], which was our main motivation for studying such finiteness statements. The main novelty is to use $\mathbb{Z}$-Hodge theory, rather than $\mathbb{R}$-Hodge theory. In Section 6.4, we show that such approach gives a new proof of Margulis commensurability criterion for arithmeticity for complex hyperbolic lattices in $\mathrm{PU}(1,n)$, at least for $n>1$.

Our second main result looks at totally geodesic subvarieties from the functional transcendence point of view, in the sense of [50]. Whenever a complex algebraic variety S has a semi-algebraic universal cover $\pi : X \to S$, one can formulate Ax–Schanuel and Zilber–Pink type conjectures. For more details see [79, Section 2.2] and [50]. For example an *abstract Ax–Lindemann–Weierstrass* would assert the following. Let V be a semialgebraic subvariety of X, and let S' be the Zariski closure of $\pi(V)$. Then S' is *bi-algebraic*, that is the irreducible components of $\pi^{-1}(S')$ are semialgebraic in X.

As in the case of Shimura varieties, we prove that totally geodesic subvarieties are bi-algebraic and have a natural group theoretical description in terms of *real sub-Shimura datum*, see Definition 3.1.6. We prove the non-arithmetic Ax-Schanuel conjecture, generalising the Ax-Schanuel conjecture for Shimura varieties [49, 59] to non-arithmetic ball quotients.

Theorem 1.2.2. *Let $W \subset X \times S_\Gamma$ be an algebraic subvariety and $\Pi \subset X \times S_\Gamma$ be the graph of $\pi : X \to S_\Gamma$. Let U be an irreducible component of $W \cap \Pi$ such that*

$$\operatorname{codim} U < \operatorname{codim} W + \operatorname{codim} \Pi,$$

the codimension being in $X \times S_\Gamma$ or, equivalently,

$$\dim W < \dim U + \dim S_\Gamma.$$

If the projection of U to S_Γ is not zero dimensional, then it is contained in a strict totally geodesic subvariety of S_Γ.

Mok [58] has developed a different perspective on functional transcendence problems for ball quotients, and using methods of several complex variables, algebraic geometry and Kähler geometry recently managed to prove a form of Ax-Lindemann-Weierstrass, which follows from the above theorem. See also the earlier work of Chan and Mok [13] regarding compact ball quotients.

Finally notice that both statements depend only on the commensurability class of Γ. Therefore, we may and do replace Γ by a finite index subgroup to assume that all lattices we consider are torsion free.

1.3. **Strategy.** The starting point for both the main theorems is to embed S_Γ in a period domain for polarised integral variations of Hodge structures. We start with the key observation that all lattices Γ in $G = \mathrm{PU}(1, n)$ satisfy a form of *infinitesimal rigidity* (this obviously fails when $n = 1$). Namely Garland and Raghunathan [36], buling on the work initiated by Calabi-Vesentini and Weil [11, 84, 85], proved that the first Eilenberg–MacLane cohomology group of all lattices Γ in G with respect to the adjoint representation is trivial. This is enough to see that Γ determines a totally real number field K and a K-form of G, which we denote by $\mathbf{G}$. Since X is a Hermitian symmetric space, S_Γ supports a natural polarised variation of $\mathbb{R}$-Hodge structures which we denote by $\mathbb{V}$. Using a recent work of Esnault and Groechenig [33], and Simpson's theory [74], we prove.

Theorem 1.3.1. *Every element in Γ has trace in the ring of integers of a totally real number field K. As a consequence, up to conjugation by G, Γ lies in $\mathbf{G}(\mathcal{O}_K)$. Moreover $\mathbb{V}$ induces a $\mathbb{Z}$-variation of Hodge structures $\widehat{\mathbb{V}}$ on S_Γ.*

Let $\widehat{\mathbf{G}}/\mathbb{Q}$ be the generic Mumford–Tate group of $\widehat{\mathbb{V}}$. By the theory of period domains and period maps of Griffiths [38], $\widehat{\mathbb{V}}$ induces a commutative diagram in the complex analytic category

$$\begin{array}{ccc} X & \xrightarrow{\quad\tilde{\psi}\quad} & D = D_{\widehat{G}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ {S_\Gamma}^{\mathrm{an}} & \xrightarrow{\quad\psi\quad} & \widehat{\mathbf{G}}(\mathbb{Z})\backslash D \end{array},$$

where $\psi : {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is the period map associated to $\widehat{\mathbb{V}}$ and D. We prove that $\widehat{\mathbf{G}}$ is the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$. Such a map ψ generalises the theory of modular embeddings of triangle groups [89, 15] and Deligne–Mostow lattices [16].

From a functional transcendence point of view, there are two *algebraic structures* on S_Γ: the former coming from the fact that its universal covering X is semialgebraic, the latter coming from the fact that S_Γ supports a polarised integral VHS. Depending on the special structure we choose,

we have two, a priori different, definitions of *bi-algebraic subvarieties*. For example we say that an irreducible algebraic subvariety $W \subset S_\Gamma$ is Γ-*bi-algebraic* if some component of $\pi^{-1}(W)$ is algebraic in X and that W is $\mathbb{Z}$-*bi-algebraic* if some component of $\tilde{\psi}(\pi^{-1}(W))$ is an algebraic subvariety of $D_{\widehat{G}}$. We prove that the Γ-*bi-algebraic* subvarieties are exactly the totally geodesic ones. What is the relation between the two special structures? Since $\tilde{\psi}$ is only a holomorphic map, such question is far from being trivial. For example, if D' is a Mumford–Tate sub-domain of D, then any analytic irreducible component of $\psi^{-1}(\pi(D'))$ is an algebraic subvariety of S_Γ by a famous theorem of Cattani, Deligne and Kaplan [12]. For a proof of the fact that $\psi^{-1}(\pi(D'))$ is algebraic, that is it contains finitely many components, see the recent work of Bakker, Klingler and Tsimerman [6].

To prove Theorems 1.2.1 and 1.2.2 we need the following.

Theorem 1.3.2. *The totally geodesic subvarieties of S_Γ are precisely the $\mathbb{Z}$-bi-algebraic ones, equivalently the Γ-bi-algebraic ones.*

Let V be an algebraic subvariety of D, and Y an anlytical irreducible component of $\psi^{-1}(\pi(V))$. then it is Γ-bi-algebraic. Theorem 1.3.2 implies, among other things, that if Y is algebraic, then it Γ-bi-algebraic. This refines the algebraicity result of Cattani, Deligne and Kaplan. Moreover Theorem 1.3.2 reduces Theorem 1.2.1 to counting the $\mathbb{Z}$-bi-algebraic subvarieties in S_Γ and therefore to an unlikely intersection problem. A similar strategy appeared also in [89], where certain Riemann surfaces C, associated to non-arithmetic lattices, are embedded in Shimura varieties and the André–Oort conjecture is used to deduce that C contains only finitely many CM-points, see also [31], where Wolfart's programme is completed.

The difficulty in applying the techniques of [49, 59] to Theorem 1.2.2 lies in the fact that one cannot apply the André–Deligne monodromy theorem [1, 21] to compute the monodromy of subvarieties Y of S_Γ that are not contained in any strict totally geodesic subvariety. Thanks to Theorem 1.3.2 we are be able to prove that the monodromy of such Y's is indeed G, by applying André–Deligne to $\psi(Y)$.

Finally, in section 5.6, we prove some consequences of Theorem 1.3.2 which may be of independent interest.

Corollary 1.3.3. *Let H be a subgroup of G of hermitian type. If $\Gamma \cap H$ is Zariski dense in H, then $\Gamma \cap H$ is a lattice in H.*

The second application is abouth non-arithmetic monodromy groups, as first studied by Nori [65]. After Sarnak [71], a subgroup of a lattice Δ of a real algebraic group R is called *thin subgroup* if it has infinite-index in Δ and it is Zariski dense in R. A link between non-arithmetic lattices and thin subgroups was also noticed by Sarnak, see indeed [71, Section 3.2]. We present a geometric way of constructing many thin subgroups, as long as a non-arithmetic lattice Γ is given.

Corollary 1.3.4. *Let W be an irreducible subvariety of S_Γ which is not contained in any strict Γ, or equivalently $\mathbb{Z}$, bi-algebraic subvariety. Unless Γ is arithmetic, the image of*

$$\pi_1(W) \to \pi_1(S_\Gamma) \cong \Gamma$$

gives rise to a thin subgroup of $\widehat{\mathbf{G}}(\mathbb{Z})$.

1.4. **Outline of paper.** We first fix some notations and discuss preliminary results about lattices in semisimple Lie groups without compact factors. In Section 3, we extend the theory of Shimura varieties starting from a *real Shimura datum* (G, X, Γ), generalising many classical results and defining Γ-special (and weakly Γ-special) subvarieties. Section 4 is devoted to $\mathbb{Z}$-Hodge theory and period domains. Here it is explained how to realise S_Γ inside a period domain for $\mathbb{Z}$-VHS. In section 5, we compare the Mumford–Tate and the monodromy groups associated to the natural $\mathcal{O}_K$-VHS on S_Γ and the $\mathbb{Z}$-VHS we have constructed, proving a tight relation between the two worlds. Building on the results of the previous sections, we prove Theorem 1.2.1 in section 6. We prove the non-arithmetic Ax-Schanuel conjecture in Section 7. Finally, we discuss arithmetic aspects of the theory, and, in Section 8, we discuss various notions of special points and André–Oort type conjectures.

1.5. **Acknowledgements.** The second author thanks Bader for an inspiring lecture on his work with Fisher, Miller and Stover in Orsay, in October 2019. The authors are grateful to Klingler for several discussion on related topics and for his *cours de l'IHES* on Tame Geometry and Hodge Theory. It is a pleasure to thank Bader, Fisher, Miller and Stover for having explained us their approach and various conversations around superrigidity. Finally we are grateful to Esnault, Klingler and Labourie for comments on a previous draft of the paper.

The first author would like to thank the IHES for two research visits in March 2019 and 2020. His work was supported by the Engineering and Physical Sciences Research Council [EP/ L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

2. Notation, terminology and recollections

In this section we fix some notation, and discuss the first properties of lattices in semisimple groups without compact factors that we need. In particular we explain the arithmeticity criteria of Mostow-Vinberg and of Margulis.

2.1. **Notations.** We make free use of the following standard notations. We denote by G real algebraic groups, and by $\mathbf{G}$ algebraic groups defined over some number field, which usually is either $\mathbb{Q}$, or a totally real number field. Let K be a totally real field. The K-forms of $\mathrm{SU}(1,n)$ are known, by the work of Weil [83], to be obtained as $\mathrm{SU}(h)$ for some Hermitian form h on F^r, where F is a division algebra with involution over a quadratic imaginary extension L of K and $n+1 = r\deg(F)$.

Regarding algebraic groups we have

- Let G be a connected real algebraic group. By *rank of G* we always mean the real rank of the group G, i.e. the dimension of a maximal real split torus of G;
- If G is reductive, which for us requires also that G is connected, we write G^{ad} for the *adjoint* of G, i.e. the quotient of G by its centre (it is a semisimple group with trivial center);
- We write $\mathbb{S}$ for the *Deligne torus*. That is the real torus obtained as Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ of the group $\mathbb{C}^*$. The Weil restriction of $\mathbf{G}/K$ from K to $\mathbb{Q}$ is usually denoted by $\widehat{\mathbf{G}}$.

Regarding the subgroups of real algebraic groups, we have

- A discrete subgroup Γ of a locally compact group G is a *lattice* if $\Gamma\backslash G$ has a finite invariant measure.

- All lattices considered in this paper are also assumed to be torsion free. Selberg's Lemma asserts if G is semisimple, that Γ has a torsion-free subgroup of finite index, see for example [61, Theorem 4.8.2];
- Given H an algebraic subgroup of G, we write Γ_H for $\Gamma \cap H(\mathbb{R})$;
- A lattice $\Gamma \subset G$ in a connected semisimple group without compact factors is *reducible* if G admits infinite connected normal subgroups H, H' such that $HH' = G$, $H \cap H'$ is discrete and $\Gamma/(\Gamma_H \cdot \Gamma_{H'})$ is finite. A lattice is *irreducible* if it is not reducible;
- Two subgroups Γ_1, Γ_2 of G are said to be *commensurable* with each other if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2;
- The commensurator $\mathrm{Comm}(\Gamma)$ of Γ is a subgroup of G defined as follows:

$$\mathrm{Comm}(\Gamma) := \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable with each other}\};$$

- A subgroup $\Gamma \subset G$ is *arithmetic* if there exists a semisimple linear algebraic group $\mathbf{G}/\mathbb{Q}$ and a surjective morphism with compact kernel $p : \mathbf{G}(\mathbb{R}) \to G$ such that Γ lies in the commensurability class of $p(\mathbf{G}(\mathbb{Z}))$. Here we denote by $\mathbf{G}(\mathbb{Z})$ the group $\mathbf{G}(\mathbb{Q}) \cap v^{-1}(\mathrm{GL}(V_{\mathbb{Z}}))$ for some faithful representation $v : G \to \mathrm{GL}(V_{\mathbb{Q}})$, where $V_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space, and $V_{\mathbb{Z}}$ a lattice in $V_{\mathbb{Q}}$. Torsion free arithmetic subgroups are lattices;
- A closed subgroup Γ of G is *cocompact* if $\Gamma\backslash G$ is compact. Every cocompact, torsion-free, discrete subgroup of G is a lattice.

2.2. **Local rigidity.** Let G be a semisimple algebraic group without compact factors and Γ be a subgroup of G. Denote by

$$\mathrm{Ad} : \Gamma \subset G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

the adjoint representation in the automorphism of the Lie algebra $\mathfrak{g}$ of G.

Definition 2.2.1. We define the *trace field* of Γ as the field generated over $\mathbb{Q}$ by

$$\{\mathrm{tr}\,\mathrm{Ad}\,\gamma : \gamma \in \Gamma\}.$$

If Γ is a lattice, its trace field is a finitely generated field extension of $\mathbb{Q}$, which depends only on the commensurability class of Γ. Indeed it is a well known fact that lattices are finitely generated (or even finitely presented), see for example [61, Theorem 4.7.10] and references therein.

Definition 2.2.2. An irreducible lattice $\Gamma \subset G$ is *locally rigid* if there exists a neighbourhood U of the inclusion $i : \Gamma \hookrightarrow G$ in $\mathrm{Hom}(\Gamma, G)$, such that any elements of U is conjugated to i.

The trace field K of a locally rigid lattice is a number field. By Borel density theorem, lattices in G are Zariski dense. For details see for example [61, Corollary 4.5.6]. In particular Γ determines a K-form of G, which we denote by $\mathbf{G}$, such that, up to conjugation by an element in G, Γ lies in $\mathbf{G}(K)$ (and K is minimal with this property). For references see [82], [55, Chapter VIII, Proposition 3.22] and [24, Proposition 12.2.1].

Lattices in $G = \mathrm{PU}(1, n)$, necessarily for $n > 1$, are known to be locally rigid (and the trace field is a totally real number field, see Theorem 4.3.1). For completeness we recall a more general[1] result [36, Theorem 0.11], which builds on the study of lattices initiated by Selberg, Calabi and Weil [86].

[1]More general in the sense that it allows also to consider $G = \mathrm{SL}_2(\mathbb{C})$. Even if local rigidity for non-cocompact lattices in $\mathrm{SL}_2(\mathbb{C})$ can fail.

Theorem 2.2.3. *If G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, then, for every lattice Γ in G there exists $g \in G$ and a subfield $K \subset \mathbb{R}$ of finite degree over $\mathbb{Q}$, such that $g\Gamma g^{-1}$ is contained in the set of K-rational points of $\mathbf{G}$.*

Since lattices are finitely generated, we know that there exist a finite set of finite places Σ of K such that Γ lies in $\mathbf{G}(\mathcal{O}_{K,\Sigma})$ (once a representation is fixed, and up to conjugation by G). In the next subsection we discuss when lattices are contained in $\mathbf{G}(\mathcal{O}_K)$.

2.3. **Lattices and integral points.** Let $K \subset \mathbb{R}$ be a totally real number field and $\mathcal{O}_K$ its ring of integers. The following is [24, Corollary 12.2.8], see also [62, Lemma 4.1].

Theorem 2.3.1 (Mostow–Vinberg arithmeticity criterion)**.** *Let $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$ be a lattice in G. Γ is arithmetic if and only if for every embedding $\sigma : K \to \mathbb{R}$ different form the fixed embedding $K \subset \mathbb{R}$, the group G_σ is compact.*

Remark 2.3.2. If $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$ is a non-arithmetic lattice, then $\mathbf{G}(\mathcal{O}_K)$ is not discrete in G and Γ has infinite index in $\mathbf{G}(\mathcal{O}_K)$.

Let $\mathbf{G}$ be a semisimple algebraic group defined over a totally real number field $K \subset \mathbb{R}$, and write G for its real points. Denote by Ω_∞ the set of all archimedean places of K and write $\widehat{\mathbf{G}}$ for the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$, see [87, Section 1.3]. It has a natural structure of $\mathbb{Q}$-algebraic group and

$$\widehat{\mathbf{G}}(\mathbb{R}) = \prod_{\sigma \in \Omega_\infty} G_\sigma,$$

where G_σ is the real group $\mathbf{G} \times_{K,\sigma} \mathbb{R}$.

For details and proofs we refer to [61, Section 5.5].

Proposition 2.3.3. *The subgroup $\mathbf{G}(\mathcal{O}_K)$ of G embeds as an arithmetic subgroup of $\widehat{\mathbf{G}}$ via the natural embedding*

$$\mathbf{G}(\mathcal{O}_K) \hookrightarrow \widehat{\mathbf{G}}, \;\; g \mapsto (\sigma(g))_{\sigma \in \Omega_\infty}.$$

Indeed $\mathbf{G}(\mathcal{O}_K)$ becomes $\widehat{\mathbf{G}}(\mathbb{Z})$. We also have:

- If $\mathbf{G}$ is simple, then $\mathbf{G}(\mathcal{O}_K)$ gives rise to an irreducible lattice;
- If, for some $\sigma \in \Omega_\infty$, G_σ is compact, then $\mathbf{G}(\mathcal{O}_K)$ gives rise to a cocompact lattice.

2.4. **Arithmeticity criteria, after Margulis.** All the result discussed in the section can be found in [55] and are due to Margulis.

Theorem 2.4.1 (Arithmeticity)**.** *Let G be a semisimple group without compact factor. If G has rank strictly bigger than one and $\Gamma \subset G$ is an irreducible lattice, then Γ is arithmetic.*

The proof of Theorem 2.4.1 is based on applying the *supperrigidity theorem*, which is recalled below, to representations obtained from different embeddings of the trace field of Γ into local fields.

Theorem 2.4.2 (Superrigidity)**.** *Let G be a semisimple group of of rank at least 2, and $\Gamma \subset G$ an irreducible lattice. Let E be a local field of characteristic zero, and $\mathcal{G}'$ be an adjoint, absolutely simple E-group. Then every homomorphism $\Gamma \to \mathcal{G}'(E)$ with Zariski-dense and unbounded image extends uniquely to a continuous homomorphism $G \to \mathcal{G}'(E)$.*

When the local field E is $\mathbb{R}$, we refer to the above statement as *Archimedean superrigidity*.

The following is known as the *commensurability criterion for arithmeticity*, and it will be reproven in Section 6.4.

Theorem 2.4.3. *Assume G is without compact factors and Γ is an irreducible lattice. Then Γ is arithmetic if and only if it has an infinite index in $\mathrm{Comm}(\Gamma)$.*

2.5. **Non-arithmetic complex hyperbolic lattices.** Regarding commensurability classes of non-arithmetic lattices in $\mathrm{PU}(1,n)$, we have:

$n=2$. By the work of Deligne, Mostow and Deraux, Parker, Paupert, there are 22 known commensurability classes in $\mathrm{PU}(1,2)$. See [28, 29] and references therein;

$n=3$. By the work of Deligne, Mostow and Deraux, there are 2 commensurability classes of non-arithmetic lattices in $\mathrm{PU}(1,3)$. In both cases the trace field is $\mathbb{Q}(\sqrt{3})$ and the lattices are not cocompact. See [27] and references therein.

For $n>3$ non-arithmetic lattices are currently not known to exist. One of the biggest challenge in the study of complex hyperbolic lattices is to understand for each n how many commensurability classes non-arithmetic lattices exist in $\mathrm{PU}(1,n)$.

3. Real Shimura data and general properties

We extend the (geometric) theory of Shimura varieties, more precisely as defined by Deligne in terms of Shimura datum [22, 23], to include quotients of Hermitian symmetric spaces by arbitrary lattices. We then discuss a generalisation of the theory of toroidal compactifications to non-arithmetic lattices and various definability results, proving in the most general form all the results we need in the rest of the paper.

3.1. **Definitions and recollections.** Let G be a reductive group and Γ a lattice in G.

Definition 3.1.1. A Γ-*factor* of G is either a one dimensional split torus in the centre of G, or a normal subgroup N of $G\sim G'\times N$ such that the image of Γ along the projection from G to N is a lattice. We call a Γ-factor *irreducible*, if the image of Γ in N is an irreducible lattice.

In the above definition of Γ-factor, N is not required to be irreducible, and may have compact semisimple factors. Moreover the centre of G is a product of irreducible tori which are Γ-factors.

Proposition 3.1.2. *Let H be a reductive subgroup of G such that Γ_H is a lattice. Then H can be written as a finite product of Γ-irreducible factors.*

Proof. Given an arbitrary lattice Γ in G there is a unique direct product decomposition $G=\prod_{i\in I}G_i$, where G_i is normal in G, such that Γ_{G_i} is an irreducible lattice in G_i and $\prod_i\Gamma_{G_i}$ has finite index in Γ. See for example [70, Theorem 5.22, page 86]. □

Definition 3.1.3. A *real Shimura datum* is a triplet (G,X,Γ) where G is a reductive algebraic group, $\Gamma\subset G$ is a lattice such that its image in G^{ad} is Zariski dense, and X is a G-orbit in the set of morphisms of algebraic groups $\mathrm{Hom}(\mathbb{S},G)$ such that for some (all) $x\in X$ the *real Shimura–Deligne axioms* are satisfied:

RSD0. The image of Γ in each GL_2-irreducible factor is an arithmetic lattice;

RSD1. The adjoint representation $\mathrm{Lie}(G)$ is of type $\{(-1,1),(0,0),(1,-1)\}$;

RSD2. The involution $x(\sqrt{-1})$ of G^{ad} is a Cartan involution;

RSD3. G has no simple compact Γ-factors.

Without the axiom RSD0. the above definition would include every Riemann surface.

Remark 3.1.4. Let X be a G-orbit in the set of morphisms of algebraic groups $\mathrm{Hom}(\mathbb{S}, G)$ satisfying RSD1, RSD2 and RSD3. Let $x \in X$ be such that $x : \mathbb{S} \to G$ factorises trough H, for some subgroup H of G. Then the H orbit of α in X satisfies RSD1 and RSD2. See for example [14, Proposition 3.2].

We have a notion of morphism of real Shimura data.

Definition 3.1.5. Let (G_1, X_1, Γ_1) and (G_2, X_2, Γ_2) be real Shimura data. A *morphism of real Shimura data*

$$(G_1, X_1, \Gamma_1) \to (G_2, X_2, \Gamma_2)$$

is a morphism of real algebraic groups $f : G_1 \to G_2$ such that for each $x \in X_1$, the composition $f \circ x : \mathbb{S} \to G_2$ is in X_2 and $f(\Gamma_1) \subset \Gamma_2$.

We need to work with a more general definition of real sub-Shimura datum, allowing the axiom RSD0 to fail. Indeed, as long as the Hermitian space X has dimension strictly bigger than one, we want to consider arbitrary sub-Shimura datum. For example, in Theorem 1.2.1, we want to consider every totally geodesic subvariety of dimension one.

Definition 3.1.6. Let (G, X, Γ) a real Shimura datum, H a subgroup of G and Γ_H a lattice in H. A *real sub-Shimura datum* is a triplet (H, X_H, Γ_H) where X_H is a H-orbit in the set $\mathrm{Hom}(\mathbb{S}, G)$ satisfying RSD1, RSD2 and RSD3 of Definition 3.1.3.

Definition 3.1.7. Let (G, X, Γ) be a real Shimura datum. We define its *adjoint real Shimura datum*, simply denoted by

$$(G, X, \Gamma)^{\mathrm{ad}},$$

as the triplet $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, \Gamma^{\mathrm{ad}})$ where X^{ad} is the G^{ad}-conjugacy class of morphisms $x : \mathbb{S} \to G \to G^{\mathrm{ad}}$ and Γ^{ad} is the image of Γ in G^{ad}. It is again a real Shimura datum.

Definition 3.1.8. Let (G, X, Γ) be a real Shimura datum. We denote by S_Γ the quotient $\Gamma\backslash X$ and we refer to it as a *Shimura variety*.

Every sub-Shimura datum $(H, X_H, \Gamma_H) \subset (G, X, \Gamma)$, as in Definition 3.1.6, induces a closed subvariety $S_{\Gamma_H} = \Gamma_H\backslash X_H$ of S_Γ.

Proposition 3.1.2 can be rephrased as follows.

Proposition 3.1.9. *Any real Shimura datum (G, X, Γ) can be uniquely written as a product of a finite number of real sub-Shimura datum (G_i, X_i, Γ_i) such that the Γ_i are irreducible lattices and*

$$\prod_i S_{\Gamma_i} \to S_\Gamma$$

is a finite covering (of complex manifolds).

The proof of the following appears as [22, Proposition 1.1.14 and Corollary 1.1.17].

Theorem 3.1.10 (Deligne)**.** *Let (G, X, Γ) be a real Shimura datum. Then X has a unique structure of a complex manifold such that, for every real representation $\rho : G \to \mathrm{GL}(V_{\mathbb{R}})$, $(V_{\mathbb{R}}, \rho \circ h)_{h \in X}$ is a holomorphic family of Hodge structures. For this complex structure, each family $(V_{\mathbb{R}}, \rho \circ h)_{h \in X}$ is a variation of Hodge structures, and X is a finite disjoint union of Hermitian symmetric domains.*

Choosing a connected component X^+ of X, we can also define *connected real Shimura data* and connected Shimura varieties. In what follows we often implicitly work over some connected component.

Remark 3.1.11. Let $(\mathbf{G}, \mathcal{X})$ be a Shimura datum in the sense of Deligne (i.e. $\mathbf{G}$ is assumed to be a $\mathbb{Q}$-group). For any faithful representation $\mathbf{G} \to \mathrm{GL}(V_{\mathbb{Z}})$, the triplet $(\mathbf{G}_{\mathbb{R}}, \mathcal{X}, \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}(V_{\mathbb{Z}}))$ defines a real Shimura datum. Given a Shimura datum $(\mathbf{G}, \mathcal{X})$ as before and $K_{\mathbb{A}_f}$ a compact open subgroup of the finite adelic points of $\mathbf{G}$, the triplet $(\mathbf{G}_{\mathbb{R}}, \mathcal{X}, \mathbf{G}(\mathbb{Q}) \cap K_{\mathbb{A}_f})$ is a real Shimura datum. We refer to the latter case as *congruence Shimura datum*.

Theorem 3.1.12 (Baily-Borel, Siu-Yau, Mok)**.** *Every Shimura variety S_Γ has a unique structure of a quasi-projective complex algebraic variety.*

Proof. If the lattice Γ is arithmetic, it was proven by Baily and Borel [5]. Using Proposition 3.1.9 we may assume that Γ is irreducible. Thanks to Theorem 2.4.1, if G has rank bigger or equal than two, then Γ is arithemtic. Regarding rank one groups we have that X is isomorphic to the complex unit ball $\mathbb{B}^n \subset \mathbb{C}^n$, for some $n \geq 1$. If $n = 1$ the axiom RSD0. enusres that Γ is arithmetic. If $n > 1$ Siu-Yau [75] and Mok [57] proved that S_Γ is biholomorphic to a quasi-projective variety. To see that S_Γ has a unique algebraic structure, one can apply [26] in place of the classical Borel's extension theorem and then argue as in the arithmetic case. □

Definition 3.1.13. Let $W \subset S_\Gamma$ be an irreducible algebraic subvariety. We say that W is Γ-*special* if it is induced by a real sub-Shimura datum (H, X_H, Γ_H) of (G, X, Γ). We say that W is *weakly-special* if there exists a real sub-Shimura datum (H, X_H, Γ_H) of (G, X, Γ) such that its adjoint splits as a product

$$(H, X_H, \Gamma_H)^{\mathrm{ad}} = (H_1, X_1, \Gamma_1) \times (H_2, X_2, \Gamma_2),$$

and W is the image of $X_1^+ \times \{x_2\}$ for some $x_2 \in X_2$.

Definition 3.1.14. Given a real Shimura datum (G, X, Γ), we say that S_Γ is an *arithmetic Shimura variety* if Γ is arithmetic. If Γ is irreducible and non-arithmetic, we say that S_Γ is a *non-arithmetic ball quotient*.

3.2. **Real sub-Shimura data in rank one.** Let $(G = \mathrm{PU}(1, n), X, \Gamma)$ be a real Shimura datum[2], for some $n > 1$. Let X_H be a Hermitian symmetric sub-domain of X with automorphism group H. Such spaces will be often referred to as *real sub-Shimura couples* of (G, X). In this section we explicitly describe the map $H \to G$ inducing the inclusion of X_H in G. We prove that every H appearing in this way can be assumed to be a semisimple group.

Let $V_{\mathbb{C}}$ be a $n+1$-dimensional complex vector space with $\varphi : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ a hermitian form of signature $(1, n)$. Fix a $\mathbb{C}$-basis $(e_1, \ldots, e_{n+1})$ such that the quadratic form associated to φ becomes

$$z_1 \overline{z}_1 - \sum z_i \overline{z}_i.$$

The group of $G = \mathrm{PU}(1, n)$ can be defined in $\mathrm{GL}(V_{\mathbb{C}})$ as equivalence classes of matrices M satisfying

$$\overline{M}^t g M = g,$$

[2]The main results of the paper remain true if one starts with Γ an irreducible lattice in $\mathrm{PU}(1, n) \times K$, where K is compact.

where $g = \mathrm{diag}(1, -1, \ldots, -1)$. Since $g = \overline{g}$ the group can be seen in $\mathrm{GL}(V_{\mathbb{R}})$. Let $W_{\mathbb{C}}$ be a sub-vector space of $V_{\mathbb{C}}$ such that φ restricts to a hermitian form $W_{\mathbb{C}} \times W_{\mathbb{C}} \to \mathbb{C}$ and write $V_{\mathbb{C}}$ as the direct sum of $W_{\mathbb{C}}$ and its orthogonal complement. We can arrange a basis of $W_{\mathbb{C}}$ in such a way that the quadratic form associated to $\varphi_{|W_{\mathbb{C}}}$ corresponds to a matrix of the form $\mathrm{diag}(1, -1, \ldots, -1)$.

Let $H \subset G \subset \mathrm{GL}(V_{\mathbb{R}})$ be the subgroup stabilizing $W_{\mathbb{C}}$. Since H has to stabilise also the orthogonal of $W_{\mathbb{C}}$, it is isomorphic to $\mathrm{PU}(1, m) \times C$, where m is the rank of $W_{\mathbb{C}}$, from some compact subgroup C in G. And every H associated to an X_H arises in this way. Up to conjugation by an element in G, any $\alpha : \mathbb{S} \to G$ can be written as

$$z \mapsto \mathrm{diag}(z, \overline{z}, 1, \ldots, 1).$$

Given a subgroup $H \subset G$ as before, we may therefore assume that $\alpha(\mathbb{S}) \subset H$, for some choice of a basis for $W_{\mathbb{C}}$.

Notice also that a similar description holds when $G = \mathrm{GU}(1, n)$.

3.3. **Toroidal compactification of non-arithmetic ball quotients.** For this secition assume that G has real rank one. A compactification of S_Γ as a complex spaces with isolated normal singularities was obtained by Siu and Yau [75]. Hummel and Schroeder [44] and then Mok [57, Theorem 1] showed that S_Γ admits a unique smooth toroidal compactification, which we denote by $\overline{S_\Gamma}$. If the parabolic subgroups of Γ are unipotent, the boundary $\overline{S_\Gamma} - S_\Gamma$ is a disjoint union of abelian varieties. The minimal compactification S_Γ^{BB}, which is proven to be projective-algebraic in [57], can be recovered by blowing down the boundary.

Given a parabolic subgroup $P \subset G$ we write:

- N_P for the unipotent radical of P;
- U_P for the centre of N_P. We may identify U_P with its Lie algebra $\mathrm{Lie}(U_P) \cong \mathbb{R}$.

Definition 3.3.1. A parabolic subgroup $P \subset G$ is called Γ-rational if its unipotent radical N_P intersects Γ as a lattice.

We denote by $\overline{S_\Gamma}$ *the* toroidal compactification of S_Γ, as in [57, Theorem 1]. Notice that, since G has rank one, there is only one toroidal compactification. Indeed $U_P \cong \mathbb{R}$ and there are no choices of fans to be made.

3.3.1. *First properties of the toroidal compactification.* From now, one speaking of toroidal compactifications, we replace Γ by a finite index subgroup and always assume that Γ is torsion free and that the parabolic subgroups of Γ are unipotent.

Theorem 3.3.2. *The toroidal compactification $\overline{S_\Gamma}$ of S_Γ is a smooth compactification with strict normal crossing divisor at infinity $B := \overline{S_\Gamma} - S_\Gamma$.*

Proof. The fact that S_Γ can be compactified as in the arithmetic case follows from [57]. From [57, Corollary 7.6, Chapter III], we have that $\overline{S_\Gamma}$ is indeed smooth and that the boundary is smooth. □

Denote by $\Delta \subset \mathbb{C}$ the complex disk and by Δ^* the punctured disk. As in the arithmetic case, there exists an open cover $\{U_\alpha\}_\alpha$ of $\overline{S_\Gamma}$ such that $U_\alpha = \Delta^n$ and $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$. We have, see also [57, Theorem 7.2, Chapter III].

Proposition 3.3.3. *For any Γ-rational parabolic P, the set $U_P \cap \Gamma$ is isomorphic to $\mathbb{Z}$. The image of the fundamental group of $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$ in the fundamental group of $\overline{S_\Gamma}$ lies in $\Gamma_{U_P} \cong \mathbb{Z}$.*

Proof. The first part can be found in [57, Section 1.3]. For the second part, we work with the local coordinates as in [64, pages 255-256]. Let $X^\vee$ be the compact dual of X and $\mathcal{X} \subset X^\vee$ be the Borel embedding. Assume that we are working with a Γ-rational boundary component $F = \{b\}$ corresponding to a Γ-rational parabolic P, and let V be the quotient of the unipotent radical of P by it centre. It is a real vector space of rank $n-1$ (where $n = \dim S_\Gamma$). For any boundary component $b \in X$, set

$$\mathcal{X}_b := \bigcup_{g \in U_P \otimes \mathbb{C}} g \cdot \mathcal{X} \subset X^\vee.$$

There exists a canonical holomorphic isomorphism

$$j : \mathcal{X} \cong \mathbb{C}^{n-1} \times \mathbb{C} \times F,$$

where $\mathbb{C}^{n-1} = V_{\mathbb{C}}$ and the latter copy of $\mathbb{C}$ is $U_P \otimes \mathbb{C}$. We can naturally identify the universal cover of $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$ with

$$\mathcal{D} \cong \{(z_1, \dots, z_{n-1}, z_n, b) \in \mathbb{C}^{n-1} \times \mathbb{C} \times F : \mathrm{Im}(z_n) \geq 0\}. \tag{3.3.1}$$

The group $U_P \otimes \mathbb{C}$ acts on $\mathcal{D}_F$, in these coordinates, by $(z_1, \dots, z_{n-1}, z_n, b) \mapsto (z_1, \dots, z_{n-1}, z_n + a, b)$. Observe that we can factorise the map $\pi : \mathcal{X} \to S_\Gamma$ as

$$\mathcal{X} \xrightarrow{\exp_F} \exp_F(\mathcal{X}) \to S,$$

where $\exp_F : \mathbb{C}^{n-1} \times \mathbb{C} \times F \to \mathbb{C}^{n-1} \times \mathbb{C}^* \times F$ is simply $\exp(2\pi i -)$ on the $\mathbb{C}$-component and the identity on $\mathbb{C}^{n-1}$. Moreover $\exp_F(\mathcal{X})$ is $\Gamma_{U_P} \backslash \mathcal{X}$. To conclude, notice that we have a commutative diagram:

$$\begin{array}{ccc} \exp_F(\mathcal{X}) & \hookrightarrow & \exp_F(\mathcal{X})^\vee \\ \downarrow & & \downarrow \\ S_\Gamma & \hookrightarrow & \overline{S_\Gamma} \end{array},$$

since, as in the arithmetic case, the boundary of $\exp_F(\mathcal{X})^\vee$ is mapped onto the boundary of $\overline{S_\Gamma}$. □

Finally we describe in a more explicit way what we discussed in the proof of Proposition 3.3.3. We follow [76, Section 1] and [57, Equation 8]. Identify X with the complex ball $\mathbb{B}^n \subset \mathbb{C}^n$. For any b in the boundary of $\mathbb{B}^n$, let $\Gamma_b \subset \Gamma$ be the set of elements fixing b. The Siegel domain D representation of X with b corresponding to infinity is

$$D := \{(z_1, \dots, z_n) : \mathbb{C}^n \in \mathrm{Im}(z_n) > \sum_{i=1}^{n-1} |z_i|^2\}.$$

For any $N > 0$ set

$$D^{(N)} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : \mathrm{Im}(z_n) > \sum_{i=1}^{n-1} |z_i|^2 + N\}. \tag{3.3.2}$$

Let W_b be the set of parabolic isometries fixing b. If N is large enough, two points $x, y \in D^{(N)}$ are equivalent mod Γ if and only if they are equivalent modulo the lattice Γ_{W_b}. Moreover, thanks to

the work of Margulis and Gromov (see [75, Section 2], and references therein), for N sufficiently large one can take $D^{(N)}$ to be in the set

$$A_b := \{x \in \mathbb{B}^n : \min_{\gamma \in \Gamma_b} d(x, \gamma x) < \epsilon\},$$

for some $\epsilon > 0$ (which depends on Γ). Consider the set

$$E := \{x \in \mathbb{B}^n : \min_{\gamma \in \Gamma} d(x, \gamma x) \geq \epsilon\}.$$

Notice that the image of E along $\pi : \mathbb{B}^n \to S_\Gamma$ is compact. Finally can write S_Γ as the union of a finite number of $\pi(A_b)$s and $\pi(E)$.

3.3.2. *Metric at infinity.* Let ω be the $(1,1)$ form on S_Γ inducing the natural hermitian metric. In this section we explain the behaviour of ω near the boundary. Similar computations in the non-arithmetic setting appear also in To's paper [76], see for example [76, Section 2], and take $f = \mathrm{Id} : X \to X$ in Proposition 2.1 in *op. cit.*.

Fix an open cover $\{U_\alpha\}_\alpha$ of $\overline{S_\Gamma}$ such that $U_\alpha = \Delta^n$ and $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$.

Definition 3.3.4. We say that ω is of *Poincaré-growth* with respect to the toroidal compactification $\overline{S_\Gamma}$ if, for any α, ω restricted to U_α is bounded by

$$C_\alpha \left(\sum_{k=1}^{n-1} |dz_k|^2 + \frac{|dz_n|^2}{|z_n|^2 |\log z_n|^2} \right).$$

The above definition does not depend on the choice of coordinates. Whenever we say that ω is of Poincaré-growth, it will be understood that we refer to the growth of ω with respect to the toroidal compactification S_Γ.

As in the arithmetic case, we have the following. See [57, Section 1.2] and [76, Section 2].

Theorem 3.3.5. *The Kähler form ω on S_Γ is of Poincaré-growth.*

Finally we notice that the holomorphic tangent bundle TS_Γ extends to a holomorphic vector bundle $\overline{TS_\Gamma}$ on $\overline{S_\Gamma}$ as follows: in an open neighbourhood $U_\alpha = \Delta^n$ of $\overline{S_\Gamma}$ where $U_\alpha \cap S_\Gamma = \Delta^{n-1} \times \Delta^*$, a local holomorphic basis of $\overline{TS_\Gamma}$ on U_α is given by

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}}, z_n \frac{\partial}{\partial z_n}.$$

3.4. **Some algebraicity results.** We first describe Siegel sets in G and X, for the action of Γ. This is needed to define semialgebraic fundamental sets for the action of Γ on X and for proving that every real sub-Shimura datum of (G, X, Γ) gives an algebraic subvariety of S_Γ. As usual, the only case we need to describe is when G has rank one.

3.4.1. *Siegel sets.* Garland and Raghunathan [36] extended Borel's reduction theory [10] to arbitrary lattices in rank one groups. Notice that Borel follows a different convention, in this paper G always acts on X on the left. Since we use the reverse order of multiplication, compared to [10] and [36], the inequalities defining the set A_t are reversed.

Write $X \cong G/K_0$ for some fixed compact maximal $K_0 \subset G$ and denote by x_0 the base point $\mathrm{Id}\, K_0$ of X. Let P be a minimal Γ-rational parabolic subgroup and let L be the unique Levi subgroup L of P which is stable under the Cartan involution associated to K_0. Consider the Langlands

decomposition

$$P = N_P AM,$$

where A is the split component of P with respect to the basepoint x_0. That is $L = AM$, A is a split torus, $L \cap K$ is a maximal compact in L, and M is the maximal anisotropic subgroup of the connected centralizer $Z(A)$ of A in P. Since $Z(A)$ is compact, M lies in K. For any real number $t > 0$ let

$$A_t := \{a \in A : a^\alpha \geq t\},$$

where α is the unique positive simple root of G with respect to A and P.

A Siegel set for G for the data (K_0, P, A) is a product

$$\Sigma'_{t,\Omega} = \Omega \cdot A_t \cdot K_0 \subset G,$$

where Ω is a compact neighbourhood of the identity in N_P. Finally we consider Siegel sets in X

$$\Sigma_{t,\Omega} = \Omega \cdot A_t \cdot x_0 \subset X,$$

where $\Sigma_{t,\Omega}$ denotes the image of $\Sigma'_{t,\Omega}$ in X.

It is interesting to compare Siegel sets with the $D^{(N)}$s described in equation 3.3.2 in the previous section. Arguing as in [49, Proposition 3.2], we have.

Proposition 3.4.1. *Let $\Sigma = \Sigma_{t,\Omega}$ be a Siegel set in X for the action of Γ. Then Σ is covered by a finite union of open subsets Θ with the following properties. For each Θ there exists Γ-rational boundary component $F = \{b\}$ corresponding to a Γ-rational parabolic P, a positive integer N large enough, relatively compact subsets $U' \subset U(P)$, $Y' \subset \mathbb{C}^{n-1}$ such that Θ, up to a bi-isometry, can be described as*

$$\Theta \cong \{(z_1, \ldots, z_n) \in \mathbb{C}^{n-1} \times U(P)_{\mathbb{C}}, \mathrm{Re}(z_n) \in U', (z_1, \ldots z_n) \in Y' : \mathrm{Im}(z_n) > \sum_{i=1}^{n-1} |z_i|^2 + N\} \subset D^{(N)}.$$

Definition 3.4.2. A *fundamental set* for the action of Γ on X is a connected open subset $\mathcal{F}$ of X such that $\Gamma \cdot \overline{\mathcal{F}} = X$ and the set

$$\{\gamma \in \Gamma : \gamma \mathcal{F} \cap \mathcal{F} \neq \emptyset\}$$

is finite.

Theorem 3.4.3 (Garland-Raghunathan)**.** *For any Siegel set $\Sigma_{t,\Omega}$ the set*

$$\{\gamma \in \Gamma : \gamma \Sigma_{t,\Omega} \cap \Sigma_{t,\Omega} \neq \emptyset\}$$

is finite. There exists a Siegel set $\Sigma_{t_0,\Omega}$ and a finite set Ξ of G such that:

- *For all $b \in \Xi$, $\Gamma \cap bN_P b^{-1}$ is a lattice in $bN_P b^{-1}$;*
- *The set $\Xi \cdot \Sigma_{t_0,\Omega}$ is a fundamental set for the action of Γ on X.*

Furthermore, when Ω is chosen to be semialgebraic, the associated Siegel sets $\Sigma_{t,\Omega}$ are semialgebraic. In particular the fundamental set $\mathcal{F} := \Xi \cdot \Sigma_{t_0,\Omega}$ is semialgebraic.

Proof. The first part of the statement is [36, Theorem 4.10]. It is formulated for *admissible* discrete subgroups of G (see [36, Definition 0.4]), a class that includes any lattice $\Gamma \subset G$, as proved in [36, Theorem 0.7].

When Ω is chosen to be semialgebraic, the Siegel set for G

$$\Omega \cdot A_t \cdot K_0,$$

is semialgebraic since it is defined as product in G of semialgebraic sets and therefore its image $\Sigma_{t,\Omega}$ in X is again semialgebraic. Finally, since Ξ is finite $\Xi \cdot \Sigma_{t_0,\Omega}$ is semialgebraic. □

Finally we discuss some functoriality properties of Siegel sets. Let H be a semisimple group of G intersecting Γ as a lattice. As described in section 3.2, there is a real sub-Shimura datum (H, X_H, Γ_H) associated to H. The next proposition describes the intersection of Siegel sets in X with X_H (assuming that the dimension of X_H is non-zero).

Proposition 3.4.4. *Let (H, X_H, Γ_H) be a real sub-Shimura datum of $(G = \mathrm{PU}(1,n), X, \Gamma)$. Any Siegel sets of H is contained in a Siegel set for G.*

Proof. Since we are interested only in Siegel sets in X_H and X we may assume that $H = \mathrm{PU}(1,m)$ for some $m < n$. Let P be a minimal Γ-rational parabolic subgroup of P and P_H be $P \cap H$. As above, consider the decompositions relative to K_0 and $K_0 \cap H$

$$P = N_P A M, \;\; P_H = N_H A_H M_H.$$

As explained in section 3.2 we have that $N_P \cap P_H = N_H$, $A \cap P_H = A_H$ and $M \cap P_H = M_H$. The result then follows by the construction of Siegel sets explained above. □

3.4.2. *Fundamental sets and o-minimality.* We have all the ingredients to prove that the uniformising map, opportunely restricted, is definable, extending [49, Theorem 4.1] to cover arbitrary lattices. For a general introduction to o-minimality, we refer to [81]. Thanks to Theorem 3.1.12, S_Γ has a canonical structure of a complex algebraic manifold, and therefore of an $\mathbb{R}_{\mathrm{an,exp}}$ manifold, see also [49, Appendix A]. The following can be proven as [49, Theorem 4.1] replacing all references to [2] with the arguments of section 3.3.

Theorem 3.4.5. *Let (G, X, Γ) be a real Shimura datum and let $\mathcal{F}$ be a fundamental set for the action of Γ in X. The restriction $\pi : \mathcal{F} \to S_\Gamma$ of the uniformising map $\pi : X \to S_\Gamma$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

Remark 3.4.6. More generally one could extend [6, Theorem 1.1] to cover the case of arbitrary lattices. Since we have already described toroidal compactifications, we prefer to follow the approach of [49], which applies here verbatim. Arguing as in [6] one can even prove that any complex analytic map from an algebraic variety to ${S_\Gamma}^{\mathrm{an}}$ is indeed algebraic. In an even grater generality, such result has been recently proven by Deng in [26].

Finally the following will be implicitly used whenever speaking of Γ-special subvarieties.

Corollary 3.4.7. *Let (H, X_H, Γ_H) be a real sub-Shimura datum of $(G = \mathrm{PU}(1,n), X, \Gamma)$. The induced map $S_{\Gamma_H} \to S_\Gamma$ is algebraic.*

Proof. By definition $(H, X_H, \Gamma_H) \to (G, X, \Gamma)$ induces a commutative diagram

$$\begin{array}{ccc} X_H & \xrightarrow{\tilde{i}} & X \\ \downarrow & & \downarrow \\ S_{\Gamma_H} & \xrightarrow{i} & S_\Gamma \end{array}.$$

For some $x \in X_H$, we can write the graph of $\tilde{i}$ as

$$\{h.(x, \tilde{i}(x)),\ h \in H\}.$$

Let $\mathcal{F}_H$ be a fundamental set for the action of Γ_H in X_H. By Proposition 3.4.4, we have that the graph of $\tilde{i}$ restricted to $\mathcal{F}_H$ is a definable set. Let $\mathcal{F}$ be a fundamental set for Γ in X, containing $\mathcal{F}_H$, and consider the restriction of the uniformising map $X_H \times X \to S_{\Gamma_H} \times S_\Gamma$ to $\mathcal{F}_H \times \mathcal{F}$:

$$\pi_{|\mathcal{F}_H \times \mathcal{F}} : \mathcal{F}_H \times \mathcal{F} \to S_{\Gamma_H} \times S_\Gamma.$$

By Theorem 3.4.5, it is a definable map, and it follows that $\pi_{|\mathcal{F}_H \times \mathcal{F}}(\mathrm{graph}(\tilde{i}_{|\mathcal{F}_H}))$ is definable. We have eventually proven that $i : S_{\Gamma_H} \hookrightarrow S_\Gamma$ is definable. The result then follows from Peterzil-Starchenko's o-minimal GAGA, which is explained below. □

The following is [67, Theorem 4.4 and Corollary 4.5].

Theorem 3.4.8 (Peterzil-Starchenko)**.** *Let S be a smooth complex algebraic manifold. Let $Y \subset S$ be a closed complex analytic subset which is also a definable subset (for some o-minimal structure expanding $\mathbb{R}_{\mathrm{an}}$). Then Y is an algebraic subset of S.*

4. $\mathbb{Z}$-Hodge theory on non-arithmetic quotients and rigidity

Thanks to Theorem 3.1.10, faithful linear real representations of G, induce real variation of Hodge structures on both X and S_Γ. In this section we investigate how to construct a *natural* $\mathbb{Z}$-VHS on S_Γ. The first step is proving that the traces of the image of any $\gamma \in \Gamma$ along the adjoint representation lies in the ring of integers of a totally real number field.

4.1. **Motivation.** Given a complex algebraic variety S, we denote by S^{an} the complex points $S(\mathbb{C})$ with its natural structure of a complex analytic variety. We denote by $\pi_1(S)$ the topological fundamental group of S^{an}. Unless it is necessary, we omit the base point in the notation. Regarding VHS we consider only polarised and pure VHS. More precisely we consider only $\mathbb{Q}$, K, $\mathbb{Z}$ and $\mathcal{O}_K$-VHS, as defined in [38], where K is a totally real field number field.

Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum. Recall that there is an isogeny from $\lambda : \mathrm{SU}(1,n) \to \mathrm{PU}(1,n)$, and we may assume that, up to replacing Γ by a finite index subgroup, Γ is the image along λ of a lattice in $\mathrm{SU}(1,n)$. For this section we assume this is indeed the case and we let $V_{\mathbb{C}}$ be a complex vector space of dimension $n+1$, and $\mathrm{SU}(1,n) \to \mathrm{GL}(V_{\mathbb{C}})$ be the standard representation of $\mathrm{SU}(1,n)$. Let $\mathbb{V}$ be the local system associated to

$$\pi_1(S_\Gamma) \to \mathrm{GL}(V_{\mathbb{C}}). \tag{4.1.1}$$

If Γ is an arithmetic subgroup of G, it is a well know fact that $\mathbb{V}$ is induced by a $\mathbb{Z}$-VHS corresponding to a family of principally polarised abelian varieties.

Simpson [73, 74] has conjectures predicting that certain $\mathbb{C}$-VHS come from geometry, in particular implying that they admit an integral structure (see indeed [73, Conjecture 5]). The motivating question (which we do not answer here) is the following.

Conjecture 4.1.1 (Simpson)**.** *Let $S_\Gamma = \Gamma\backslash X$ be a smooth quasi-projective ball quotient. Then the standard representation $\pi_1(S_\Gamma) = \Gamma \to \mathrm{GL}(V_{\mathbb{C}})$ is of geometric origin (in the sense of* [53, Section 1]*).*

As we will see in a moment, the above conjecture, at least when Γ is cocompact, is a special case of [73, Conjecture 4] because of the rigidity of Γ, as established by Weil [86]. See also [53, Corollary 7.11] for a related discussion.

We use recent progress in such field of Esnault and Groechenig to prove that lattices in G, not necessarily cocompact, have integral traces. As notices in [53, Remark 7.12], if Γ is cocompact, such integrality follows directly from the work of Esnault and Groechenig [33]. This will be used to construct a $\mathbb{Z}$-VHS on S_Γ closely related to the original one. Eventually we construct *generalised modular embeddings* of S_Γ in some period domain, see Theorem 4.3.1.

4.2. **Cohomological rigidity and integral traces.** The main result of the section is the following.

Theorem 4.2.1. *Let Γ be a lattice in $G = \mathrm{PU}(1,n)$ for some $n > 1$, K its trace field and $\mathbf{G}$ the K-form of G determined by Γ. There exists a finite index subgroup $\Gamma' \subset \Gamma$ with integral traces. Equivalently, up to conjugation by G, Γ' lies in $\mathbf{G}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers K.*

In section Section 2.2 we discussed locally rigid lattices. To construct a $\mathbb{Z}$-VHS we need a stronger rigidity: namely *cohomological rigidity*. Buliding on a study initiated by Weil [86] in the cocompact case, Garland-Raghunathan [36, Theorem 1.10], proved the following.

Theorem 4.2.2 (Garland-Raghunathan)**.** *Let G be a semisimple Lie group, not locally isomorphic to SL_2, nor to $\mathrm{SL}_2(\mathbb{C})$. For any lattice Γ in G, the first Eilenberg–MacLane cohomology group of Γ with respect to the adjoint representation is zero. In symbols:*

$$H^1(\Gamma, \mathrm{Ad}) = 0.$$

To see why such vanishing is related to a rigidity result, observe that the the space of first-order deformations of $\rho : \Gamma \hookrightarrow G$ is naturally identified with $H^1(\Gamma, \mathrm{Ad})$, where

$$\mathrm{Ad} : \Gamma \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}) \tag{4.2.1}$$

is the adjoint representation.

The following is [33, Theorem 1.1], whose argument relies on Drinfeld's theorem on the existence of ℓ-adic companions over a finite field. Theorem 4.2.1 will be a simple of application of such result.

Theorem 4.2.3 (Esnault–Groechenig)**.** *Let S be a smooth connected quasi-projective complex variety. Then a complex local system $\mathbb{V}$ on S is integral, i.e. it comes as extension of scalars from a local system of projective $\mathcal{O}_L$-modules of finite type (for some number field $L \subset \mathbb{C}$), whenever it is:*

1. *Irreducible;*
2. *Quasi-unipotent local monodromies around the components at infinity of a compactification with normal crossings divisor $i : S \hookrightarrow \overline{S}$;*
3. *Cohomologically rigid, that is $\mathbb{H}^1(\overline{S}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ vanishes;*
4. *Of finite determinant.*

Here $i_{!*}\,\mathrm{End}^0(\mathbb{V})$ denotes the intermediate extension seen as a perverse sheaf as in [9]. See [33, Remark 2.4] for more details. Moreover $\mathbb{H}^1(\overline{S}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ is the Zariski tangent space at the moduli point of $\mathbb{V}$ of the Betti moduli stack of complex local systems of given rank with prescribed determinant and prescribed local monodromies along the components of the normal crossing divisor $\overline{S} - i(S)$.

Proof of Theorem 4.2.1. In the proof we are free to replace Γ with a finite index subgroup, and so may and do assume that, as in section 3.3.1, Γ is torsion free and that the parabolic subgroups of Γ are unipotent. Moreover, as explained in section 4.1, we assume also that Γ lifts to $\mathrm{SU}(1,n)$.

Let S_Γ be the ball quotient $\Gamma\backslash X$. Let $\mathbb{V}$ be the local sytem on S_Γ associated to the standard representation of $\mathrm{SU}(1,n)$, as in (4.1.1). Thanks to Bass-Serre theory [8], $\mathbb{V}$ is integral in the sense of Theorem 4.2.3 if and only if the image of every $\gamma \in \Gamma$ in $\mathrm{GL}(V_{\mathbb{C}})$ has traces in the ring of integers of some number field $E \subset \mathbb{C}$ (see also [19, Lemma 7.1]). This implies that the traces of $\mathrm{Ad}(\gamma)$ are in $\mathcal{O}_K$, where K is the trace field of Γ (which will be proven to be a totally real number field in the next section).

As recalled in Section 3.3, S_Γ is a smooth quasi-projective variety which admits a smooth toroidal compactification

$$i : S_\Gamma \hookrightarrow \overline{S_\Gamma},$$

with smooth boundary[3]. By construction the local system $\mathrm{End}^0(\mathbb{V})$ corresponds to the adjoint representation described in equation (4.2.1).

Notice that, since Γ is irreducible and Zariski dense in G, also $\mathbb{V}$ is irreducible (this of course depends on our choice of the faithful representation of $\mathrm{SU}(1,n)$ in $\mathrm{GL}(V_{\mathbb{C}})$). To prove the integrality of $\mathbb{V}$ we are left to check conditions (2), (3) and (4) of Theorem 4.2.3. The fourth condition is satisfied, since every element of $\mathrm{SU}(1,n)$ has finite determinant.

Proof of (2). Let B be $\overline{S_\Gamma} - i(S_\Gamma)$. Write B as union of its N disjoint irreducible components: $B = \bigcup_i B_i$. We notice here that the singular locus of B is empty, and therefore $\overline{S_\Gamma}$ is equal to what is denoted by U in [33, Section 2]. Let P_i be Γ-rational parabolic corresponding to B_i. As proven in Proposition 3.3.3 the local monodromy at every B_i corresponds to an element $T_i \in \Gamma \cap U_{P_i}$, which is certainly unipotent. □

Proof of (3). Here we show that $\mathbb{V}$ is cohomologically rigid (without boundary conditions). Recall that

$$H^1(S_\Gamma, \mathrm{End}^0(\mathbb{V})) = H^1(\Gamma, \mathrm{Ad}) = 0,$$

where the last equality follows from Theorem 4.2.2. To show that $\mathbb{H}^1(\overline{S_\Gamma}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ vanishes, it is enough that it injects in $H^1(S_\Gamma, \mathrm{End}^0(\mathbb{V}))$. This follows from the description of $\mathbb{H}^1(\overline{S_\Gamma}, i_{!*}\,\mathrm{End}^0(\mathbb{V}))$ appearing in [33, Proposition 2.3], more precisely see page 4284 line 8 and Remark 2.4 in *op. cit.*. □

Eventually we have checked all the conditions of Theorem 4.2.3 and therefore concluded that the traces of $\mathrm{Ad}(\gamma)$ are in $\mathcal{O}_K$. Again by Bass-Serre theory we conclude that, up to G-conjugation, Γ lies in $\mathbf{G}(\mathcal{O}_K)$, concluding the proof of Theorem 4.2.1. □

4.3. **Weil restriction, after Deligne-Simpson.** Let (G, X, Γ) be an arbitrary real Shimura datum and write it at as a product of (G_i, X_i, Γ_i) such that each Γ_i is an irreducible lattice in G_i (as in Proposition 3.1.9). As recalled in Theorem 2.4.1, if G_i has rank strictly bigger than one, then Γ_i is an arithmetic lattice, which implies that Γ_i has integral traces. If G_i is of rank one and isomorphic to GL_2, RSD0 of Definition 3.1.3 ensures that Γ_i is also an arithmetic lattice. Therefore we may

[3]As recalled in the first paragraph of Section 3.3, the boundary is actually a disjoint union of N abelian varieties. Moreover, since G has rank one, the toroidal compactification of S_Γ does not depend on any choices.

apply Theorem 4.2.1 to the remaining G_is of rank one, and conclude that each Γ_i has integral traces. It follows that we can write, for some real number field K, every real Shimura datum as follows:

$$(G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K)).$$

The following is the main result of the section.

Theorem 4.3.1. *Let $(G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$ be a real Shimura datum. The field K is totally real, and for each natural embedding $\sigma : K \to \mathbb{R}$ the representation*

$$\pi_1(S_\Gamma) \to G_\sigma$$

is induced by a $\mathcal{O}_K$-VHS, denoted by $\mathbb{V}^\sigma$. Moreover the $\mathcal{O}_K$-VHS

$$\widehat{\mathbb{V}} := \bigoplus_{\sigma : K \to \mathbb{R}} \mathbb{V}^\sigma \tag{4.3.1}$$

has a natural structure of $\mathbb{Z}$-VHS of weight zero.

The proof is inspired by the arguments of [74, Theorem 5] and [22]. For completeness and related discussions we refer also to [19, Section 10] and [53, Proposition 7.1].

Proposition 4.3.2 (Simpson)**.** *If for each embedding $\sigma : K \to \mathbb{C}$, the local system $\mathbb{V}^\sigma$ associated to the representation*

$$\pi_1(S_\Gamma) \to G_\sigma(\mathbb{C}) \to \mathrm{GL}(V_\mathbb{C})$$

underlies a polarised complex VHS of weight zero, then the direct sum of

$$\widehat{\mathbb{V}} = \bigoplus_{\sigma : K \to \mathbb{C}} \mathbb{V}^\sigma$$

has a natural structure of $\mathbb{Z}$-VHS.

Since the groups G_σ are all isomorphic over $\mathbb{C}$, if $G = \mathrm{PU}(1, n)$, the proof of Theorem 4.3.1 shows that we can write G_σ as $\mathrm{PU}(p_\sigma, q_\sigma)$ for some positive non zero integers p_σ, q_σ such that $p_\sigma + q_\sigma = n + 1$. To be more precise, in the proof of Proposition 4.3.2 one has to use all the embeddings of E into $\mathbb{C}$, where, as in the proof of Theorem 4.2.1, we let $E \subset \mathbb{C}$ be the field generated by the traces of Γ in $\mathrm{GL}(V_\mathbb{C})$. In our exposition a choice of CM type of E is implicitly made. For more details we refer to the proof of [53, Proposition 7.1].

Proof of Theorem 4.3.1. The fact that $\widehat{\mathbb{V}}$ has a natrual structure of $\mathbb{Z}$-VHS, rather than of $\mathbb{Q}$-VHS, is the content of Bass-Serre theory [8]. For each embedding $\sigma : K \to \mathbb{C}$, let $G_\sigma(\mathbb{C})$ be complex group $\mathbf{G} \times_{K,\sigma} \mathrm{Spec}(\mathbb{C})$. Consider the representation

$$\rho_\sigma : \pi_1(S_\Gamma) \to \mathbf{G}(\mathcal{O}_K) \to G_\sigma(\mathbb{C}).$$

Here we prove that ρ_σ is induced by an $\mathcal{O}_K$-VHS of weight zero $\mathbb{V}^\sigma$. Since

$$H^1(\Gamma, \mathrm{Ad} \circ \rho_\sigma) = H^1(\Gamma, \mathrm{Ad}) \otimes_{K,\sigma} \mathbb{C} = 0,$$

the twisted representation ρ_σ is again cohomologically rigid (see also [19, Lemma 6.6]). Moreover $\mathbb{V}^\sigma$ has also quasi-unipotent monodromy at infinity by construction. Indeed, let P_i be a Γ-rational parabolic corresponding to an irreducible divisor B_i of the boundary, and denote by T_i the local monodromy for $\mathbb{V}$ and by ${T_i}^\sigma$ the local monodromy for $\mathbb{V}^\sigma$. By construction of $\mathbb{V}^\sigma$, the element ${T_i}^\sigma$ is obtained by $T_i \in \Gamma_{U_{P_i}} \subset \Gamma \subset \mathbf{G}(\mathcal{O}_K)$ by applying σ to its entries. Since being unipotent is a

geometric condition, it is enough to check that $T_i^\sigma \in G_\sigma \otimes \mathbb{C} \cong G \otimes \mathbb{C}$ is unipotent, which holds true because T_i is unipotent.

Eventually we can apply Theorem 4.3.3, which is recalled below, to conclude that each $\mathbb{V}^\sigma$ is a $\mathcal{O}_K$-VHS. To compute the weight of $\mathbb{V}^\sigma$, notice that it corresponds to a map

$$\mathbb{S} \to G_\sigma,$$

and each of the G_σ is of adjoint type (since $G = \mathrm{PU}(1,n)$ is of adjoint type, and they are geometrically isomorphic). Therefore each $\mathbb{V}^\sigma$ has weight zero. The reason being that the weight homomorphism is a map from $\mathbb{G}_m$ to the centre of G_σ, see [23, 1.1.13.]. □

Simpson proved that every representation of the fundamental group of a smooth projective variety can be deformed to a $\mathbb{C}$-VHS. In particular the rigid ones are immediately $\mathbb{C}$-VHS. The following is [19, Theorem 8.1], see also references therein.

Theorem 4.3.3 (Corlette)**.** *Suppose $\mathbb{V}$ is local system with quasi-unipotent monodromy at infinity. If $\mathbb{V}$ is rigid, then it underlines a $\mathbb{C}$-VHS.*

4.4. **Integral variations of Hodge structures and period maps.** Starting from a real Shimura datum (G, X, Γ), we first showed the existence of a minimal totally real field such that $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$, where $\mathbf{G}$ is the K-form of G determined by Γ. If the trace field plays a role in the argument, we write the real Shimura datum as $(G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$.

4.4.1. *Recap of period domains.* Everything can be found in the book [38] (see also [48]). Let $(H_\mathbb{Z}, q_\mathbb{Z})$ be a polarised $\mathbb{Z}$-Hodge structure. Let $\mathbf{R}$ be the $\mathbb{Q}$-algebraic group $\mathrm{Aut}(H_\mathbb{Q}, q_\mathbb{Q})$. We have

- The space D of $q_\mathbb{Z}$-polarised Hodge structures on $H_\mathbb{Z}$ with specified Hodge numbers is an homogeneous for R;
- Choosing a reference Hodge structure we have $D = R/M$ where M is a subgroup of the compact unitary subgroup $R \cap U(h)$ with respect to the Hodge form h of the reference Hodge structure.
- D is naturally an open sub-manifold of the flag variety $D^\vee$.

Let S be a smooth quasi-projective complex variety. By *period map*

$$S^{\mathrm{an}} \to \mathbf{R}(\mathbb{Z})\backslash D$$

we mean a holomorphic locally liftable Griffiths transverse map. It is equivalent to the data of a $\mathbb{Z}$-VHS on S with generic Mumford–Tate group $\mathbf{R}$[4]. See also [48, Section 3].

4.4.2. *Generalised modular embeddings.* From the real Shimura datum $(G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$, we produced a $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ on S_Γ. As recalled in the previous section, we have a Mumford–Tate domain associated to $(\widehat{\mathbf{G}}, M)$, were $\widehat{\mathbf{G}}$, as in section 2.3, is the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{G}$[5]. and M a compact in $\widehat{G}$

As in section 2.3, we write:

- r for the degree of K over $\mathbb{Q}$;

[4]We always assume that the image of the monodromy representation associated to such VHS is contained in $\mathbf{R}(\mathbb{Z})$, implicitly assuming that, if this is not satisfied, we replace $\mathbf{R}(\mathbb{Z})$ by a finite index subgroup.

[5]To be more precise here we should take the generic Mumford-Tate group of $\widehat{\mathbb{V}}$, which, a priori, could be a strict subgroup of $\widehat{\mathbf{G}}$,

- $\sigma_i : K \to \mathbb{R}$ the real embeddings of K, ordered in such a way that σ_1 is simply the identity on K;
- $\mathrm{pr}_i : \widehat{\mathbf{G}} \to \mathbf{G}$ for the map of K-group schemes obtained by projecting onto the ith factor of $\widehat{\mathbf{G}} \otimes K$.

Let $\psi : {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ be the period map associated to the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ we constructed in 4.3.1. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\psi}} & D = D_{\widehat{G}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ {S_\Gamma}^{\mathrm{an}} & \xrightarrow{\psi} & \widehat{\mathbf{G}}(\mathbb{Z})\backslash D \end{array}.$$

By construction $D_{\widehat{G}}$ is the product of X and a homogeneous space under the group $\prod_{i=2,\dots,r} G_{\sigma_i}$. We notice here that, by Theorem 2.3.1, the group $\prod_{i=2,\dots,r} G_{\sigma_i}$ is compact if and only if Γ is an arithmetic lattice. Given such decomposition we can write:

$$\tilde{\psi}(x) = (x, x_{\sigma_2}, \dots, x_{\sigma_r}),$$

where x_{σ_i} is the Hodge structure obtained by the fibre of $\mathbb{V}^{\sigma_i}$, or more precisely of its lift to X, at x. Finally it is important to observe that $\tilde{\psi}$ is holomorphic and Γ-equivariant, in the sense that for each $\gamma \in \Gamma$

$$\tilde{\psi}(\gamma x) = (\gamma x, \rho_{\sigma_2}(\gamma) x_{\sigma_2}, \dots, \rho_{\sigma_r}(\gamma) x_{\sigma_r}),$$

where $\rho_{\sigma_i} : \Gamma \to G_{\sigma_i}$ is obtained by applying $\sigma_i : K \to \mathbb{R}$ to the coefficients of $\Gamma \subset \mathbf{G}(\mathcal{O}_K)$.

Remark 4.4.1. It is interesting to compare the above diagram with a famous result of Corlette. Let S be a closed connected Kähler manifold, with universal covering $\tilde{S}$ and fundamental group Γ. Let

$$\rho : \Gamma \to \widehat{G}$$

be a group homomorphism form Γ into a real reductive group. If the Zariski closure of Γ in $\widehat{G}$ is reductive, then Corlette [17] proved that there exists a ρ-equivariant harmonic map from $\tilde{S}$ to the quotient of $\widehat{G}$ by a compact maximal subgroup $\widehat{K_0}$. If S is a ball quotient S_Γ, we have constructed a holomorphic map from X to $\widehat{G}/M$, where M is a compact. We can compose the map $\tilde{\psi}$ with $\widehat{G}/M \to \widehat{G}/\widehat{K_0}$, but the result wont be holomorphic.

4.4.3. *Definition of $\mathbb{Z}$-special subvarieties (after Klingler).* We specialise the definitions of Klingler [48] for the period map $\psi : {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ associated to the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ (see also [6]) when $G = \mathrm{PU}(1, n)$ for some $n > 1$. In [7, Section 1.2] the following spaces are called *weak Mumford–Tate domain*. We omit the word *weak* because we are ultimately interested in subvarieties of S_Γ, where there are no possibilities of moving $\mathbb{Z}$-special subvarieties in families.

Definition 4.4.2. Let D be a (polarised) period domain. A *Mumford–Tate sub-domain* D' of D is an orbit $M.x$ where $x \in D$ and M is the real group associated to $\mathbf{M}$ a normal algebraic $\mathbb{Q}$-subgroup of $\mathbf{MT}(x)$. In fact D' is a complex sub-manifold of D and $\pi(D') = \mathbf{M}(\mathbb{Z})\backslash M.x$ is a complex analytic subvariety of $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$, which we call *Mumford–Tate subvariety*.

The following is [12, Theorem 1.1].

Theorem 4.4.3 (Cattani–Deligne–Kaplan)**.** *Let D' be a Mumford–Tate sub-domain of D. Then $\psi^{-1}(\pi(D'))$ is an algebraic subvariety of S_Γ.*

Definition 4.4.4. Subvarieties of S_Γ of the form $\psi^{-1}(\pi(D'))$, for some Mumford–Tate sub-domain $D' \subset D$ are called $\mathbb{Z}$-*special*. An irreducible subvariety W of S_Γ is $\mathbb{Z}$-*Hodge generic* if it is not contained in any strict $\mathbb{Z}$-special subvariety.

Since the intersection of two Mumford–Tate sub-domains is again a Mumford–Tate sub-domain, we have the following.

Proposition 4.4.5. *Irreducible components of intersections of $\mathbb{Z}$-special subvarieties are again $\mathbb{Z}$-special.*

Proof. For $i = 1, 2$, let D_{R_i} be a Mumford–Tate sub-domain of D associated to a $\mathbb{Q}$-subgroup $\mathbf{R}_i \subset \widehat{\mathbf{G}}$. Let

$$x \in D_{1,2} := D_{R_1} \cap D_{R_2} : \mathbb{S} \to \widehat{G}.$$

The $\mathbb{Q}$-Zariski closure of $x(\mathbb{S})$ is contained in both $\mathbf{R}_1$ and $\mathbf{R}_2$. Thanks to Remark 3.1.4 $R_1 \cap R_2$ satisfies RSD1. and RSD2. It follows that $D_{1,2}$ is the orbit under $R_1 \cap R_2$ of some Hodge generic point $x \in D_{1,2}$. Throwing away the compact $\mathbb{Q}$-factors of the adjoint of $\mathbf{R}_1 \cap \mathbf{R}_2$ on which the projection of x is trivial, we may assume that $\mathbf{R}_1 \cap \mathbf{R}_2$ describes a Mumford–Tate sub-domain $D_{R_1 \cap R_2}$ of $D_{\widehat{G}}$. □

By definition S_Γ is always $\mathbb{Z}$-special. By functoriality of the above construction, we can already see that Γ-special subvarieties give examples of $\mathbb{Z}$-special subvarieties. Proving the converse requires much more work and we will be able to do that only when G has rank one.

Proposition 4.4.6. *Let $(H, X_H, \Gamma_H \subset \mathbf{H}(\mathcal{O}_K))$ be a real sub-Shimura datum of $(G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$. Denote by $\widehat{\mathbf{H}}$ the Weil restriction from K to $\mathbb{Q}$ of $\mathbf{H}$. The subvariety S_{Γ_H} of S_Γ can be written as the preimage along the period map*

$$\psi : {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$$

of the sub-domain $\widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}} \subset \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$.

Proof. As explained in Theorem 3.1.12 and Corollary 3.4.7, S_{Γ_H} is a smooth algebraic subvariety of S_Γ. We can restrict the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$ from S_Γ to S_{Γ_H}, to obtain a commutative diagram

$$\begin{array}{ccc} X_H & \xrightarrow{\tilde{\psi}_{|X_H}} & D_{\widehat{H}} \\ \downarrow & & \downarrow \\ {S_{\Gamma_H}}^{\mathrm{an}} & \xrightarrow{\psi_{|{S_{\Gamma_H}}^{\mathrm{an}}}} & \widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}} \end{array},$$

which, by functoriality, naturally injects in the fundamental diagram for S_Γ. This is enough to conclude. □

Remark 4.4.7. It could happen that $\mathbf{H}$ is defined over a smaller field. Indeed the trace field of Γ_H, which we denote by K_H, may be strictly contained in K. In this case we can consider the Weil restriction $\mathrm{Res}_{\mathbb{Q}}^{K_H}\mathbf{H}$ and the associated period domain

$$D' := D_{\mathrm{Res}_{\mathbb{Q}}^{K_H}\mathbf{H}} \subset D_{\widehat{G}}.$$

We remark here that S_{Γ_H} can also be obtain as $\psi^{-1}\pi(D')$. Indeed D' is contained in $D_{\widehat{H}}$ and $\dim \psi(S_{\Gamma_H}) \cap \pi(D') = \dim \psi(S_{\Gamma_H}) \cap \widehat{\mathbf{H}}(\mathbb{Z})\backslash D_{\widehat{H}}$.

5. Comparison between the Γ-special and $\mathbb{Z}$-special structures

Let $(G = \mathrm{PU}(1,n), X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$ be a real Shimura datum. In this section we introduce and compare notions of *Mumford–Tate* and *monodromy* groups for subvarieties of S_Γ with respect to both the $\mathcal{O}_K$ and the $\mathbb{Z}$-variation of Hodge structures we introduced in Theorem 4.3.1. Eventually proving Theorem 1.3.2.

5.1. **Computing the monodromy of S_Γ for the $\mathbb{Z}$-Hodge structure.** Consider the real representation of $\pi_1(S_\Gamma)$ associated to $\widehat{\mathbb{V}}$, which, by construction, is given by

$$\rho_{\widehat{\mathbb{V}}} : \Gamma \to \widehat{G} = \prod_{\sigma\in\Omega_\infty} G_\sigma, \ \ \gamma \mapsto (\sigma_1(\gamma) = \gamma, \dots, \sigma_r(\gamma)). \tag{5.1.1}$$

In this section we compute the algebraic monodromy of subvarieties of S_Γ, for the $\mathbb{Z}$-VHS $\widehat{\mathbb{V}}$. We will prove, among other things, that $\rho_{\widehat{\mathbb{V}}}$ has Zariski dense image in $\widehat{G}$.

Let W be an irreducible algebraic subvariety of S_Γ, and let $\widetilde{W}$ be an analytic component of $\pi^{-1}(W)$. For the length of this section we omit base points in the notation of the fundamental group and we replace W by its smooth locus.

We denote as follows the Γ-monodromy of W, i.e. the monodromy of W with respect to the VHS $\mathbb{V}$:

$${}^{\Gamma}\mathbf{M}_W := \overline{\mathrm{Im}(\pi_1(W) \to \pi_1(S_\Gamma) = \Gamma)}^{\mathrm{Zar},0}.$$

Lemma 5.1.1. *The K-algebraic group ${}^{\Gamma}\mathbf{M}_W$ is non-trivial and its real points are not compact.*

Proof. Heading for a contradiction suppose that the intersection between Γ and ${}^{\Gamma}M_W$ is finite. By Riemann existence theorem, this implies that $\widetilde{W}$ is algebraic (not only semialgebraic) in X. But X admits a bounded realisation and therefore can not contain any positive dimensional algebraic subvarieties. Finally, since $\Gamma \cap {}^{\Gamma}M_W$ is a discrete and infinite subgroup of ${}^{\Gamma}M_W$, ${}^{\Gamma}M_W$ can not be compact. □

The group ${}^{\Gamma}\mathbf{M}_W$ is defined over K, but it could happen that it is defined over a smaller field. In the next definition we look at the field of definition of the adjoint of ${}^{\Gamma}\mathbf{M}_W$, in the sense of Section 2.2.

Definition 5.1.2. The field generated by the traces of the elements in $\Gamma \cap {}^{\Gamma}M_W$ is called the *trace field of* W and denoted by K_W.

We denote as follows the $\mathbb{Z}$-monodromy of W, i.e. the monodromy of W with respect to the VHS $\widehat{\mathbb{V}}$:

$${}^{\mathbb{Z}}\mathbf{M}_W := \overline{\mathrm{Im}(\pi_1(W) \to \widehat{\mathbf{G}}(\mathbb{Z}))}^{\mathrm{Zar},0}.$$

It is a connected $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$.

Some simple observations relating the two special structures, relying only on functoriality:

- If W is contained in a Γ-special subvariety associated to $(H, X_H, \Gamma_H \subset \mathbf{H}(\mathcal{O}_K))$, then the Γ-monodromy of W is contained in $\mathbf{H}$;
- If W is contained in a $\mathbb{Z}$-special subvariety associated $(R, D_R, \mathbf{R}(\mathbb{Z}))$ (as defined in 4.4.3), then the $\mathbb{Z}$-monodromy of W is contained in $\mathbf{R}$.

We define $\widehat{{}^{\Gamma}\mathbf{M}_W}$ as the Weil restriction from K_W to $\mathbb{Q}$ of the adjoint of ${}^{\Gamma}\mathbf{M}_W$. It is a $\mathbb{Q}$-algebraic group whose real points can be identified as the product of the $\left({}^{\Gamma}\mathbf{M}_W^{\mathrm{ad}}\right)_\sigma$, varying $\sigma : K_W \to \mathbb{R}$. Each factor has a structure of K_W-algebraic group. Since K_W is contained in K, the factors have also a structure of K-group scheme.

Theorem 5.1.3. *The $\mathbb{Z}$-monodromy of W is the Weil restriction from K_W to $\mathbb{Q}$ of the Γ-monodromy of W. In symbols we write:*

$$\widehat{{}^{\Gamma}\mathbf{M}_W} = {}^{\mathbb{Z}}\mathbf{M}_W^{\mathrm{ad}}.$$

Given the representation

$$\pi_1(W) \to {}^{\Gamma}\mathbf{M}_W^{\mathrm{ad}},$$

the representation associated to the restriction of $\widehat{\mathbb{V}}$ to W is obtained, by equation 5.1.1, using each embedding $\sigma : K \to \mathbb{R}$:

$$\pi_1(W) \to {}^{\Gamma}\mathbf{M}_W^{\mathrm{ad}} \to \widehat{{}^{\Gamma}\mathbf{M}_W}.$$

In particular the adjoint of ${}^{\mathbb{Z}}\mathbf{M}_W$ is contained in $\widehat{{}^{\Gamma}\mathbf{M}_W}$.

Proof. If $K_W = \mathbb{Q}$, there is nothing to be proven, so we may assume that $[K_W : \mathbb{Q}] > 1$. For each $\sigma : K_W \to \mathbb{R}$, denote by $\mathbf{M}_\sigma$ the image of the adjoint of ${}^{\mathbb{Z}}\mathbf{M}_W$ along the projection

$$\mathrm{pr}_\sigma : \widehat{\mathbf{G}} \to \mathbf{G}_\sigma.$$

We first prove that $\mathbf{M}_\sigma$ is exactly the adjoint of $\left({}^{\Gamma}\mathbf{M}_W\right)_\sigma$. By equation 5.1.1, we have that $\mathbf{M}_\sigma$ contains the image of the monodromy representation associated to the VHS $\mathbb{V}^\sigma_{|W}$, which is

$$g_W : \pi_1(W) \to \left({}^{\Gamma}\mathbf{M}_W\right)_\sigma.$$

Since the image of $\pi_1(W)$ is Zariski dense in ${}^{\Gamma}\mathbf{M}_W$, we have that the image of g_W is contained in $\mathbf{M}_\sigma$ and it is Zariski dense in $\left({}^{\Gamma}\mathbf{M}_W\right)_\sigma$. It follows that

$$\left({}^{\Gamma}\mathbf{M}_W\right)_\sigma^{\mathrm{ad}} = \mathbf{M}_\sigma. \tag{5.1.2}$$

For any two of the r_W archimedean places σ_i, σ_j of K_W we prove that ${}^{\mathbb{Z}}\mathbf{M}_W$ surjects onto

$$\mathbf{M}_{\sigma_i} \times \mathbf{M}_{\sigma_j}.$$

This is enough to conclude when $r_W = 2$, and then we argue by induction. Let $\mathbf{H}$ be the intersection of ${}^{\mathbb{Z}}\mathbf{M}_W$ with $\mathbf{M}_{\sigma_i} \times \mathbf{M}_{\sigma_j}$. It is an algebraic group whose minimal field of definition is K_W.

For $\ell \in \{i, j\}$, we have a short exact sequences (in the category of algebraic K_W-groups)

$$1 \to K_\ell \to \mathbf{H} \to \mathbf{M}_{\sigma_\ell} \to 1.$$

Since we are working with semisimple group, the only possibilities for K_ℓ is to be either finite or equal to $\mathbf{M}_{\sigma_{\ell'}}$, for ℓ' such that $\{\ell, \ell'\} = \{i, j\}$. If $K_\ell = \mathbf{M}_{\sigma_{\ell'}}$ then

$$\mathbf{H} = \mathbf{M}_{\sigma_i} \times \mathbf{M}_{\sigma_j},$$

proving the claim. If both kernels are finite we argue using Goursat's Lemma.

Lemma 5.1.4 (Goursat's Lemma)**.** *Let G_1, G_2 be algebraic groups and H be a closed subgroup of $G_1 \times G_2$ such that the two projections from H to G_1 and G_2 are surjective. For $i = 1, 2$, let K_i be the kernel of $H \to G_i$. The image of H in $G_1/K_2 \times G_2/K_1$ is the graph of an isomorphism $G_1/K_2 \to G_2/K_1$.*

Assume that both K_i and K_j are finite groups. By applying Lemma 5.1.4 we can write $\mathbf{H}$ as the graph of an isomorphism of K_W group schemes

$$f : \mathbf{M}_{\sigma_i}/K_j \to \mathbf{M}_{\sigma_j}/K_i.$$

In particular, up to a quotient by a finite group, we obtain a commutative square

$$\begin{array}{ccc} \mathbf{M}_{\sigma_i}(K_W) & \xrightarrow{f_{K_W}} & \mathbf{M}_{\sigma_j}(K_W) \\ \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle \sigma_j} \\ \mathbf{M}_{\sigma_i}(\mathbb{R}) & \xrightarrow{f_{\mathbb{R}}} & \mathbf{M}_{\sigma_j}(\mathbb{R}) \end{array}.$$

This diagram induces an equivalence between the two places σ_i, σ_j. Indeed, since K_W is the field generated by the traces of the $\gamma \in \Gamma \cap {}^{\Gamma}M_W$, and

$$\mathbb{Q}\{\mathrm{tr}(\mathrm{ad}(\gamma)) : \gamma \in \sigma_i(\Gamma \cap {}^{\Gamma}M_W)\} = \mathbb{Q}\{\mathrm{tr}(\mathrm{ad}(\gamma)) : \gamma \in \sigma_j(\Gamma \cap {}^{\Gamma}M_W)\}$$

we obtain that $\sigma_i(K_W) = \sigma_j(K_W)$. This is impossible because σ_i, σ_j are places of K_W, the smallest field of definition of ${}^{\Gamma}\mathbf{M}_W$.

To finish the proof we argue by induction. Assume that ${}^{\mathbb{Z}}\mathbf{M}_W$ surjects onto any products of m factors, for some $m < r_W$, and we prove that it surjects onto any products of $m + 1$ factors. Let S be a set of $m + 1$ places of K_W and write $S = S' \cup \{\sigma_\ell\}$ for some $\sigma_\ell \in S$. Set

$$\mathbf{M}_{S'} := \prod_{\sigma \in S'} \mathbf{M}_\sigma, \quad \mathbf{M}_S := \prod_{\sigma \in S} \mathbf{M}_\sigma,$$

and consider the exact sequence

$$1 \to K_{S'} \to \mathbf{H} \to \mathbf{M}_{S'} \to 1.$$

The kernel $K_{S'}$ is either finite, or equal to $\mathbf{M}_{\sigma_\ell}$. In the latter case, we conclude that $\mathbf{H} = \mathbf{M}_S$. In the former case, consider the short exact sequence

$$1 \to K_\ell \to \mathbf{H} \to \mathbf{M}_{\sigma_\ell} \to 1.$$

By applying Lemma 5.1.4, we can have an isomorphism of K-group schemes

$$\mathbf{M}_{\sigma_\ell} \cong \mathbf{M}_{S'}/K_\ell.$$

Since $M_{S'}$ is a product of simple groups, the right hand side has to be isomorphic to (a finite quotient of) $\mathbf{M}_{\sigma_{\ell'}}$, for some $\ell' \in S'$. We therefore obtain the same commutative square as in the case where S had cardinality two, concluding the proof. □

Since the Γ-monodromy of S_Γ is $\mathbf{G}$, and the field of definition of G is K, we have proven the following.

Corollary 5.1.5. *The $\mathbb{Z}$-monodromy group of S_Γ is $\widehat{\mathbf{G}}$, i.e. Γ has Zariski dense image in $\widehat{G}$.*

5.2. **Mumford–Tate level.** Let $\alpha \in X$ and write $\hat{\alpha}$ for $\tilde{\psi}(\alpha) \in D$. Recall that α corresponds to a morphism of real groups

$$\alpha : \mathbb{S} \to G.$$

As explained in Section 4.4.1 also elements in D corresponds to maps from the Deligne torus to a real group. By construction of $\widehat{\mathbb{V}}$, it means that $\hat{\alpha}$ corresponds to

$$\hat{\alpha} = (\alpha_1 = \alpha, \alpha_2, \dots, \alpha_r) : \mathbb{S} \to \widehat{\mathbf{G}}_{\mathbb{R}} = \prod_{i=1}^{r} G_{\sigma_i}. \tag{5.2.1}$$

We can consider the Mumford–Tate group of $\hat{\alpha}$, i.e. the $\mathbb{Q}$-Zariski closure of $\mathrm{Im}(\hat{\alpha})$ in $\widehat{\mathbf{G}}$ which we denote by

$$\mathbf{MT}(\hat{\alpha}) = \mathbf{MT}(\widehat{\mathbb{V}}_\alpha) \subset \widehat{\mathbf{G}}.$$

As in section 4.4, we denote by $\mathrm{pr}_1 : \widehat{\mathbf{G}}_K \to \mathbf{G}$ the map of K-groups obtained by projecting onto the first factor. Write $\mathbf{H}_\alpha$ for $\mathrm{pr}_1(\mathbf{MT}(\hat{\alpha}))$.

Definition 5.2.1. We define the *K-Mumford–Tate group* of α, denoted by $\mathbf{MT}(\alpha)$, as the K-Zariski closure of $\alpha(\mathbb{S}) \subset G$ in $\mathbf{G}$.

Lemma 5.2.2. *We have a factorisation of the map α:*

$$\alpha : \mathbb{S} \to H_\alpha \subset G.$$

Moreover $\mathbf{MT}(\alpha) \subset \mathbf{H}_\alpha$.

Proof. Applying pr_1 to $\hat{\alpha}(\mathbb{S}) \subset \mathbf{MT}(\hat{\alpha})$, we have $\mathrm{pr}_1(\hat{\alpha}(\mathbb{S})) \subset \mathbf{H}_\alpha = \mathrm{pr}_1(\mathbf{MT}(\hat{\alpha}))$. By equation 5.2.1, $\mathrm{pr}_1(\hat{\alpha}(\mathbb{S})) = \alpha(\mathbb{S})$. Finally, since H_α is a K-group, it contains the K-Zariski closure of $\alpha(\mathbb{S})$, which, by definition, is $\mathbf{MT}(\alpha)$. □

5.3. **Γ-generic Mumford–Tate.** In this section we defined the Γ-generic Mumford–Tate group of W and prove that it contains the Γ-monodromy.

Given $\alpha \in \widetilde{W}$, let $\mathrm{MT}(\alpha)$ the real points of K-group $\mathbf{MT}(\alpha)$ and notice that $\widetilde{W}$ is contained in the union of the $\mathrm{MT}(\alpha).\alpha$, running α in $\widetilde{W}$. Since all $\mathbf{MT}(\alpha)$ are defined over some number field, there are only countably many possible groups appearing in such union. In particular $\widetilde{W}$ is contained in a countable union of $\mathrm{MT}(\alpha).\alpha$. Since $\widetilde{W}$ is irreducible and algebraic, we have proven the existence of an $\alpha \in \widetilde{W}$ such that $\widetilde{W} \subset \mathrm{MT}(\alpha).\alpha$. Moreover, for any $\beta \in \widetilde{W}$, $\mathbf{MT}(\beta)$ is a subgroup of $\mathbf{MT}(\alpha)$.

Definition 5.3.1. The Γ-*Mumford–Tate group of* W, denoted by $\mathbf{MT}(W)$, is $\mathbf{MT}(\alpha)$ for some $\alpha \in \widetilde{W}$ such that $\widetilde{W} \subset \mathrm{MT}(\alpha).\alpha$.

It follows that W is contained in $\pi(\mathrm{MT}(W).\alpha) = \Gamma_{\mathrm{MT}(W)} \backslash \mathrm{MT}(W).\alpha$. By the functoriality of the topological[6] fundamental group π_1, we obtain

$$\pi_1(W) \to \Gamma_{\mathrm{MT}(W)} \to \Gamma,$$

[6]Here $\pi(\mathrm{MT}(W).\alpha)$ is not, a priori, an algebraic subvariety of S_Γ.

where the latter arrow is the inclusion of $\Gamma \cap \mathrm{MT}(W)$ in Γ. We proved

Proposition 5.3.2. *The Γ-Mumford–Tate group of W contains the Γ-monodromy. In symbols:*

$$^{\Gamma}\mathbf{M}_W \subset \mathbf{MT}(W).$$

5.4. **Monodromy and Mumford–Tate groups.** The following can be found in [1, Theorem 1] and [21] (see also [50, Theorem 4.10.]).

Theorem 5.4.1 (Deligne, André)**.** *Let W a smooth quasi-projective variety supporting a $\mathbb{Z}$-VHS and $s \in W$ a Hodge generic point. Let M_s be the algebraic monodromy of W (at s). Then M_s is a normal subgroup of the derived group $\mathbf{MT}(s)^{\mathrm{der}}$ of the Mumford–Tate group at s.*

Let W be a subvariety of S_Γ of positive dimension. Applying Theorem 5.4.1 to W, we see that

$$^{\mathbb{Z}}\mathbf{M}_W \lhd \mathbf{MT}(\hat{\alpha})^{\mathrm{der}}/\mathbb{Q}.$$

By looking at them as K_W-groups and applying pr_1, thanks to equation 5.1.2, we have

$$^{\Gamma}\mathbf{M}_W \lhd \mathbf{H}_\alpha^{\mathrm{der}}. \tag{5.4.1}$$

Theorem 5.4.2. *Let W be an irreducible algebraic subvariety of S_Γ, of positive dimension. Let $\alpha \in \widetilde{W}$ be such that $\mathbf{MT}(\alpha) = \mathbf{MT}(W)$. The derived subgroup of the Mumford–Tate group of $\widehat{\mathbb{V}}$ at α is the Weil restriction from K_W to $\mathbb{Q}$ of $\mathbf{MT}(\alpha)^{\mathrm{der}}$ (as in Definition 5.2.1). In symbols we write:*

$$\widehat{\mathbf{MT}(\alpha)} = \mathbf{MT}(\hat{\alpha})^{\mathrm{der}}.$$

Proof. Putting Proposition 5.3.2, Lemma 5.2.2 and equation 5.4.1 together, we have

$$^{\Gamma}\mathbf{M}_W \subset \mathbf{MT}(W) \subset \mathbf{H}_\alpha,$$

where the first group is a normal subgroup of $\mathbf{H}_\alpha^{\mathrm{der}}$. By Lemma 5.1.1, $^{\Gamma}M_W$ is not compact. We can write each of the $^{\Gamma}M_W, MT(W)^{\mathrm{der}}, H_\alpha^{\mathrm{der}}$ as a product of a simple real group of rank one and a compact subgroup. Thanks to the description of the embeddings of semisimple subgroups of $\mathrm{PU}(1,n)$ of Section 3.2 we have that the compact factor of $^{\Gamma}M_W$ is contained in the compact factor of $MT(W)^{\mathrm{der}}$ which is contained in the compact factor of H_α^{der}. Since such compact Γ-factors do not play a role in the description of real sub-Shimura data, we conclude that $\widehat{\mathbf{MT}(\alpha)} = \mathbf{MT}(\hat{\alpha})^{\mathrm{der}}$. □

5.5. **Proof of Theorem 1.3.2.** We are ready to complete the proof of Theorem 1.3.2 and to characterise totally geodesic subvariety of S_Γ in four equivalent ways. The following is the converse of Proposition 4.4.6.

Theorem 5.5.1. *Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum and let S_Γ be the associated Shimura variety. Any $\mathbb{Z}$-special subvariety W of S_Γ is Γ-special.*

Proof. Let $\alpha \in \widetilde{W}$ be such that $\pi(\alpha)$ is smooth, and $\mathbf{MT}(\alpha) = \mathbf{MT}(W)$. To keep the notation more compact, we write $\mathbf{M}$ for the K-Mumford–Tate of α. By Theorem 5.4.2, we can assume that

$$W = \psi^{-1}(\widehat{\mathbf{M}}(\mathbb{Z})\backslash D_{\widehat{M}}),$$

where $\widehat{\mathbf{M}}$ denotes the Weil restriction from K_W to $\mathbb{Q}$ of $\mathbf{M}$ and $\widehat{M}$ denotes the associated real group.

Since $\mathbf{M}$ contains the K-Mumford–Tate group of any x lying in $\tilde{\psi}^{-1}(D_{\widehat{F}})$, we have

$$\pi(M.\alpha) \subset \psi^{-1}(\widehat{\mathbf{M}}(\mathbb{Z})\backslash D_{\widehat{M}}).$$

Notice that $M.\alpha$ is the image of $D_{\widehat{M}}$ along the projection onto the first factor

$$D_{\widehat{G}} \to X.$$

In particular it is a Hermitian subspace of X. Let Y be an analytic irreducible component of $\tilde{\psi}^{-1}(D_{\widehat{F}})$, we can write

$$M.\alpha \subset Y \subset \bigcup_{\beta \in Y} \mathrm{MT}(\beta).\beta.$$

By looking at dimensions, we must have $Y = M.\alpha$, and so $W = \pi(F_\alpha)$. Being W algebraic, by Theorem 4.4.3, we conclude that $(M, M.\alpha, \Gamma_M \subset \mathbf{M}(K))$ is a real sub-Shimura datum of $(G, X, \Gamma \subset \mathbf{G}(K))$. As Lemma 5.5.2 below shows. □

Lemma 5.5.2. *Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum. Let X_H be a Hermitian symmetric subspace of X, associated to a subgroup $H \subset G$, such that $\pi(X_H)$ is an algebraic subvariety of S_Γ. Then (H, X_H, Γ_H) is a real sub-Shimura datum of (G, X, Γ). That is Γ_H is a lattice in H.*

Proof. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain such that $\Gamma'\backslash\Omega$ is a quasi-projective variety form some discrete subgroup Γ' of the automorphisms of Ω acting freely on Ω. Griffiths [40, Proposition 8.12] (see also discussion after [40, Question 8.13]) proved that $\Gamma'\backslash\Omega$ has finite Kobayashi-Eisenman volume [32]. Since X_H is a quotient of H by a compact subgroup, Γ_H is a lattice if and only if $\Gamma_H\backslash X_H$ has finite volume with respect to an Haar measure μ_H on H. Notice that here $X_H = \mathbb{B}^m$ for some $m < n$. Standard arguments in hyperbolic complex analysis [51] and Griffiths' result applied to $\Omega = X_H$ imply that Γ_H is indeed a lattice. See also [32, Proposition 2.4 (page 57)] for the fact that Kobayashi-Eisenman volume is equal to the volume associated to Kähler form on $\mathbb{B}^m$. In the special case of ball quotients, the result follows also from Mumford–Hirzebruch's proportionality theorem [64, Theorem 3.2], at least once it is extended to cover the case of non-arithmetic quotients (see discussions in Section 3.3). □

5.6. **Some corollaries.** The first corollary we discuss is a generalisation of the André–Deligne monodromy theorem.

Corollary 5.6.1. *Let $W \subset S_\Gamma$ be an irreducible algebraic subvariety, which is not contained in any strict Γ-special subvariety, then the image of the monodromy representation*

$$\pi_1(W) \to \Gamma$$

is Zariski dense in G.

Proof. Theorem 5.5.1 shows that W is not contained in any strict $\mathbb{Z}$-special subvariety. Therefore there exists $\alpha \in \widetilde{W}$ such that $\mathbf{MT}(\hat{\alpha})$ is $\widehat{\mathbf{G}}$. By the André–Deligne monodromy theorem, which was recalled as Theorem 5.4.1, the $\mathbb{Z}$-monodromy of W is indeed $\widehat{\mathbf{G}}$. Eventually Theorem 5.4.2 implies that the Γ-monodromy of W is $\mathbf{G}$. That is, the image of $\pi_1(W) \to \Gamma$ is Zariski dense in $\mathbf{G}$. □

Corollary 5.6.2. *Let $W \subset S_\Gamma$ be an irreducible algebraic subvariety. The following are equivalent:*

(1) W is totally geodesic;
(2) W is bi-algebraic, i.e. some (equivalently any) analytic component of the preimage of W along $\pi : X \to S_\Gamma$ is algebraic;
(3) W is a Γ-special subvariety;
(4) $W = \psi^{-1}(\pi(Y))$ for some algebraic subvariety Y of $D_{\widehat{G}}$.

If G was a group of arbitrary rank, one should distinguish between Γ-special subvarieties and weakly-Γ-special subvarieties, as in the arithmetic case.

Proof. By definition (3) implies (1) and (1) implies (2). Condition (4) is equivalent to W being $\mathbb{Z}$-special by [48, Proposition 7.4.]. The equivalence between (3) and (4) then follows from Proposition 4.4.6 and Theorem 5.5.1. To show that (2) implies (3) we can argue as in [80, Section 4]. We may and do assume that W is not contained in any strict Γ-special nor $\mathbb{Z}$-special subvarieties. The same proof of [80] applies, replacing André–Deligne with Corollary 5.6.1 which implies that the Γ-monodromy of W is $\mathbf{G}$. Since G has rank one, we do not have to consider product situations, as in [80, Equation (6)], and we can conclude directly that W is S_Γ. □

Remark 5.6.3. Mok [58], by completely different techniques, recently proved that (2) implies (1). See also [13, Theorem 1.3] for the case where Γ is cocompact.

Corollary 5.6.2 implies that the irreducible components of the intersection of Γ-special subvarieties are either points or Γ-special subvarieties. In particular we can consider the smallest Γ-special subvariety containing an irreducible $W \subset S_\Gamma$. More precisely we can prove the following.

Corollary 5.6.4. *Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum. For $i = 1, 2$, let H_i be a semisimple subgroup of G of hermitian type such that $\Gamma_i := \Gamma_{H_i}$ is a lattice in H_i. Then $\Gamma_{1,2} := \Gamma_1 \cap \Gamma_2$ is a lattice in $H_{1,2} := H_1 \cap H_2$.*

If one of the two Γ_i is cocompact, then the result follows also from [70, Lemma 1.19 (page 26)].

Proof. The set $\Gamma_{1,2}$ is a discrete subgroup of $H_{1,2}$. If $H_{1,2}$ is compact, then $\Gamma_{1,2}$ has to be finite, and there is nothing to prove. We may and do assume that $H_{1,2}$ is a rank one subgroup of G and it is without compact Γ-factors. For $i = 1, 2$, as explained in section 3.2, there are real sub-Shimura data associated to the H_is:

$$(H_i, X_i, \Gamma_i \subset \mathbf{H}_i(\mathcal{O}_K)) \subset (G, X, \Gamma \subset \mathbf{G}(\mathcal{O}_K)).$$

Consider an irreducible component, of positive dimension, S' of $S_{\Gamma_1} \cap S_{\Gamma_2}$. Thanks to Corollary 5.6.2, S' is Γ-special subvariety. Remark 3.1.4 shows that $H_{1,2}$ satisfies RSD1. and RSD2. Arguing as in Proposition 4.4.5, the $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$ given by $\widehat{\mathbf{H_1}} \cap \widehat{\mathbf{H_2}}$ describes S' as a $\mathbb{Z}$-special subvariety. It follows that the real sub-Shimura datum associated to S' is $(H_{1,2}, X_{1,2}, \Gamma_{1,2} \subset \mathbf{H}_{1,2}(K))$. This implies that $\Gamma_{1,2}$ is a lattice in $H_{1,2}$. □

Definition 5.6.5. A smooth irreducible algebraic subvariety $W \subset S_\Gamma$ is *special* if it satisfies one of the equivalent conditions of Corollary 5.6.2. We define the *special closure of* W as the smallest special subvariety of S_Γ containing W. We say that W is *Hodge generic* if it is not contained in any strict special subvariety.

The fact that a non-arithmetic Γ gives rise to a thin subgroup of $\widehat{\mathbf{G}}(\mathbb{Z})$ follows already from Corollary 5.1.5. As announced in the introduction, we can prove Corollary 1.3.4.

Corollary 5.6.6. *Let $(G = \mathrm{PU}(1,n), X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$ be a real Shimura datum. Let W be a Hodge generic subvariety of S_Γ. If Γ is non-arithmetic, the image of*

$$\pi_1(W) \to \Gamma$$

gives rise to a thin subgroup of $\widehat{\mathbf{G}}(\mathbb{Z})$.

Proof. As in Remark 2.3.2, since Γ is non-arithmetic, Γ has infinite index in $\mathbf{G}(\mathcal{O}_K)$ and so the same holds for its image in $\widehat{\mathbf{G}}(\mathbb{Z})$. Because of the Hodge genericity assumption, as explained in the proof of Corollary 5.6.1, the image of $\pi_1(W) \to \Gamma \to \widehat{\mathbf{G}}(\mathbb{Z})$ is Zariski dense and therefore a thin subgroup of $\widehat{\mathbf{G}}(\mathbb{Z})$. □

Finally the following was announced in the introduction as Corollary 1.3.3.

Corollary 5.6.7. *Let (F, X_F) be a real sub-Shimura couple of $(G = \mathrm{PU}(1,n), X)$. If Γ_F is Zariski dense in F, then Γ_F is a lattice in F.*

Proof. Since $\Gamma_F \subset F$ is Zariski dense in F, it determines a the K-form of F, which we denote by $\mathbf{F}$. In the proof of Theorem 5.5.1 we started with $\mathbf{F}$ such that the exists some $\alpha \in X$ whose K-Mumford–Tate group is $\mathbf{F}$, and we concluded that Γ_F is a lattice by Lemma 5.5.2. As in the case of arithmetic Shimura varieties the set of $\alpha \in X_F$ such that $\mathbf{MT}(\alpha)$ is strictly contained in $\mathbf{F}$ is countable. A counting argument shows the existence of an $\alpha \in X_F$ such that $\mathbf{MT}(\alpha) = \mathbf{F}$. □

6. Finiteness of special subvarieties

Let $(G = \mathrm{PU}(1,n), X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$ be a non-arithmetic real Shimura datum and S_Γ the associated Shimura variety. In this section we prove Theorem 1.2.1, which, thanks to Corollary 5.6.2, becomes the following.

Theorem 6.0.1. *If S_Γ is non-arithmetic, then it contains only finitely many maximal special subvarieties.*

A special subvariety $S' \subset S_\Gamma$, in the sense of Definition 5.6.5, is called maximal if the only special subvariety of S_Γ strictly containing S' is S_Γ itself. The maximality condition can not be dropped in the statement. For example S_Γ could contain a special subvariety $S' \subsetneq S_\Gamma$ associated to a real Shimura datum (H, X_H, Γ_H) where Γ_H is arithmetic and S' could contain countably many special subvarieties, as the next remark explains.

The strategy of the proof, which is inspired by the arguments of [78, 43, 20], is to show that maximal special subvarieties are parametrised by a countable and definable set. This is enough to conclude that they form a finite set. A key input to show the countability is the Ax-Schanuel for the pair $(S_\Gamma, \widehat{\mathbb{V}})$, as recently proven by Bakker and Tsimerman [7]. We recall its statement in section 6.2 and then prove the result. Finally we explain how a similar argument can be used to reprove Margulis' commensurability criterion for arithmeticity for lattices in $\mathrm{PU}(1,n)$ and $n > 1$.

6.1. Motivation: Klingler's Zilber-Pink conjecture.

Recently Klingler [48] has proposed a generalisation of the Zilber–Pink Conjecture for arbitrary irreducible smooth quasi-projective complex varieties supporting a (mixed) $\mathbb{Z}$-VHS. See [48, Section 5.1.] and references therein. In this section we explain his conjecture for the pair $(S_\Gamma, \widehat{\mathbb{V}})$ and how it predicts Theorem 1.2.1.

We start with some definitions from [48].

Definition 6.1.1. Let $W \subset S_\Gamma$ be an irreducible subvariety. The *Hodge codimension* of W is the codimension of the tangent space at a Hodge-generic smooth point of $\psi(W)$ in the corresponding horizontal tangent space of $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$:

$$\text{H-cd}(W) := \operatorname{rk} T_h(\widehat{\mathbf{G}}(\mathbb{Z})\backslash D) - \dim W.$$

Let $\hat{\mathfrak{g}}$ be the Lie algebra of $\widehat{G}$. Notice that the rank of $T_h(\widehat{\mathbf{G}}(\mathbb{Z})\backslash D)$ is the dimension, over $\mathbb{C}$, of the -1 part of $\hat{\mathfrak{g}}$, with respect to the Hodge-filtration induced from $\widehat{\mathbb{V}}$. Notice that

$$T_h(\widehat{\mathbf{G}}(\mathbb{Z})\backslash D) = T(\widehat{\mathbf{G}}(\mathbb{Z})\backslash D)$$

holds if and only if $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D$ is a Shimura variety. If this is the case, the Hodge codimension of W is just the codimension of W in its special closure.

Definition 6.1.2. An irreducible subvariety $W \subset S_\Gamma$ is said to be *Hodge-optimal* if for any irreducible subvariety $W \subsetneq Y \subset S_\Gamma$ we have

$$\text{H-cd}(W) < \text{H-cd}(Y).$$

The following is [48, Conjecture 1.9].

Conjecture 6.1.3 (Klingler)**.** *There are only finitely many Hodge optimal subvarieties of S_Γ.*

Notice that the conjecture is meaningful only if Γ is non-arithmetic. Otherwise the period map $\psi : {S_\Gamma}^{\text{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is an isomorphism, the Hodge codimension of S_Γ is zero, and therefore there are no Hodge-optimal subvarieties.

Proposition 6.1.4. *If Γ is non-arithmetic, maximal totally geodesic subvarieties of S_Γ are Hodge optimal. Therefore Conjecture 6.1.3 implies Theorem 1.2.1.*

Proof. Thanks to Corollary 5.6.2, totally geodesic subvarieties of S_Γ are $\mathbb{Z}$-special. Let W be a maximal $\mathbb{Z}$-special subvariety and $W \subsetneq Y \subset S_\Gamma$. Since Y has to be Hodge generic in S_Γ, we have

$$\text{H-cd}(Y) = \text{H-cd}(S_\Gamma) + \operatorname{codim}_{S_\Gamma}(Y).$$

Since W is $\mathbb{Z}$-special its Hodge codimension is strictly smaller than the Hodge codimension of S_Γ. Therefore

$$\text{H-cd}(W) < \text{H-cd}(S_\Gamma) \leq \text{H-cd}(Y),$$

proving that W is Hodge-optimal. □

Finally it is also interesting to notice that the Hodge codimension can detect arithmeticity.

Proposition 6.1.5. *Let $W \subset S_\Gamma$ be irreducible smooth subvariety. W has Hodge codimension 0 if and only if it is associated to a real sub-Shimura datum*

$$(H, X_H, \Gamma_H) \subset (G, X, \Gamma),$$

and Γ_H is arithmetic. Equivalently the period map $\psi : {S_\Gamma}^{\text{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$, restricted to W is dominant.

Remark 6.1.6. Let (H, X_H, Γ_H) be a sub-Shimura datum of a non-arithmetic real Shimura datum (G, X, Γ) . It may and does happen that Γ_H is arithmetic. Some examples of non-arithmetic lattices $\Gamma \subset G = \text{PU}(1, 2)$ such that Γ_H is arithmetic are discussed in [88].

Another interesting conjecture is [48, Conjecture 5.7]. In general, it predicts that the closure of the positive dimensional subvarieties of Shimura type of a smooth quasi-projective variety supporting a $\mathbb{Z}$-VHS, is of Shimura type. Such conjecture for the pair $(S_\Gamma, \widehat{\mathbb{V}})$ is indeed implied by Conjecture 6.1.3 and Theorem 1.2.1.

Corollary 6.1.7. *Suppose that the union of positive dimensional special subvarieties of S_Γ, which are of Shimura type with dominant period map, is Zariski dense in S_Γ, then $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is a Shimura variety and the period map*

$$\psi : {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$$

is an isomorphism.

6.2. **Bakker–Tsimerman's Ax-Schanuel.** The following follows from [7, Theorem 1.1]. We refer to it as the $\mathbb{Z}$-Ax–Schanuel (or simply $\mathbb{Z}$-AS).

Theorem 6.2.1 (Bakker–Tsimerman)**.** *Let $\widehat{W} \subset {D_{\widehat{G}}}^\vee \times S_\Gamma$ be an algebraic subvariety. Let $\widehat{U}$ be an irreducible component of $\widehat{W} \cap S_\Gamma \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} D_{\widehat{G}}$ such that*

$$\operatorname{codim} \widehat{U} < \operatorname{codim} \widehat{W} + \operatorname{codim} S_\Gamma \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} D_{\widehat{G}},$$

the codimension being in $S_\Gamma \times D_{\widehat{G}}$. Then the projection of $\widehat{U}$ to S_Γ is contained in a strict weak Mumford–Tate subvariety of S_Γ.

Thanks to Corollary 5.6.2, and the fact that G has rank one, we conclude that the projection of $\widehat{U}$ to S_Γ is contained in a strict special subvariety of S_Γ, unless it is zero dimensional.

6.3. **Proof of Theorem 1.2.1.** We first need a lemma.

Lemma 6.3.1. *Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum. There exists a compact subset C of S_Γ such that every special subvariety $S' \subset S_\Gamma$ has non empty intersection with C.*

Proof. If Γ is arithmetic this appears as [14, Lemma 4.5], for the general case see [52, Proposition 3.2.]. Here it is important that G has rank one, and therefore that there is no difference between totally geodesic subvarieties and what is called in [14] *strongly special* subvarieties. □

Let $(H, X_H, \Gamma_H \subset \mathbf{H}(\mathcal{O}_K))$ be a real sub-Shimura datum of (G, X, Γ), where $\mathbf{H}$ is a strict semisimple K-subgroup of $\mathbf{G}$. Denote by $\widehat{\mathbf{H}}$ the $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$ obtained as Weil restriction, from K to $\mathbb{Q}$, of $\mathbf{H}$, and by $\widehat{H}$ the associated real group[7]. Notice that

$$\tilde{\psi}(X_H) \subset D_{\widehat{H}} \cap \tilde{\psi}(X). \tag{6.3.1}$$

Let $\mathcal{F} \subset X$ be a fundamental set for the action of Γ as in Section 3.4.1, and set

$$\mathcal{C} := \pi^{-1}(C) \cap \mathcal{F}.$$

Lemma 6.3.2. *The set*

$$\Pi_0(\mathbf{H}) := \{(x, \hat{g}) \in \mathcal{C} \times \widehat{G} : \mathrm{Im}(\tilde{\psi}(x) : \mathbb{S} \to \widehat{G}) \subset \hat{g}\widehat{H}\hat{g}^{-1}\}, \tag{6.3.2}$$

is definable.

[7]Here we could consider the Weil restriction from the trace field K_H of Γ_H, rather than from K, of $\mathbf{H}$, but, as explained in Remark 4.4.7, both groups describe the same Γ-special subvariety.

Proof. Thanks to the description of Siegel sets of Section 3.4.3, we can assume that $\mathcal{C}$ is the closure of an open semialgebraic subset of $\mathcal{F}$. Note that the restriction of $\tilde{\psi} : X \to D_{\widehat{G}}$ to $\mathcal{C}$ is holomorphic and therefore definable in $\mathbb{R}_{\mathrm{an}}$. Eventually the condition on the image of $\tilde{\psi}(x)$ is a definable condition. □

Remark 6.3.3. The reason why we work with the compact $\mathcal{C}$, rather than with the fundamental set $\mathcal{F}$, is to avoid discussing the definability of (opportune restrictions of) $\tilde{\psi}$. However it should be possible to apply the arguments of [6, Theorem 1.5.] to prove that $\tilde{\psi}_{|\mathcal{F}}$ is definable. It would be a consequence of the fact that $\tilde{\psi}$ restricted to $D^{(N)}$, where $D^{(N)} \subset X$ was defined in 3.3.2, is definable, at least for N large enough, and that the complement of a compact subset of $\mathcal{F}$ is covered by a finite union of $D^{(N)}$s.

The condition on the image of $\tilde{\psi}(x)$ ensures that we are only considering conjugated of $\widehat{\mathbf{H}}$ by elements in $\hat{g}$ such that

$$\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)$$

has a complex structure. In particular $\hat{g}\widehat{H}\hat{g}^{-1}(\mathbb{C}).\tilde{\psi}(x)$ is an algebraic subvariety of the flag variety of $D_{\widehat{G}}$ whose intersection with $D_{\widehat{G}}$ is $\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)$. That is $\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)$ is an algebraic subvariety of $D_{\widehat{G}}$ in the sense of Section 6.2 and [49, Appendix B]. Associated to $(x, \hat{g}) \in \Pi_0(\mathbf{H})$ there is

$$S_{x,\hat{g}} := \psi^{-1}\pi(\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x)) \subset {S_\Gamma}^{\mathrm{an}},$$

that could happen to be a special subvariety of S_Γ, or not.

In what follows, by dimension we always mean the complex dimension. For $(x, \hat{g}) \in \Pi_0(\mathbf{H})$ consider the function computing the local dimension

$$d_X(x, \hat{g}) := \dim_{\tilde{\psi}(x)} \left(\hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x) \cap \tilde{\psi}(X) \right).$$

Given $j < n = \dim S_\Gamma$, consider the sets

$$\Pi_1^j(\mathbf{H}) := \{(x, \hat{g}) \in \Pi_0(\mathbf{H}) : d_X(x, \hat{g}) \geq j\},$$

$$\Sigma^j = \Sigma(\mathbf{H})^j := \{\hat{g}\widehat{H}\hat{g}^{-1} : (x, \widehat{g}) \in \Pi_1^j(\mathbf{H}) \text{ for some } x \in \mathcal{C}\}. \tag{6.3.3}$$

Proposition 6.3.4. *Let m be the maximum of the dimensions of the strict special subvarieties of S_Γ. The set $\Sigma^m = \Sigma(\mathbf{H})^m$ is finite.*

In the proof we use the $\mathbb{Z}$-AS recalled in Section 6.2.

Proof. It is enough to prove that Σ^m is definable and countable. First observe that, since the local dimension of a definable set at a point is a definable function, $\Pi_1^m(\mathbf{H})$ is a definable set. Consider the map

$$\Pi_1^m(\mathbf{H}) \to \widehat{G}/N_{\widehat{G}}(\widehat{H}), \quad (x, \widehat{g}) \mapsto \widehat{g}N_{\widehat{G}}(\widehat{H}),$$

where $N_{\widehat{G}}(\widehat{H})$ denotes the normaliser of $\widehat{H}$ in $\widehat{G}$. Such map is definable, and therefore its image, which is in bijection with Σ^m, is definable.

To prove that Σ is countable it is enough to show that each $\hat{g}\widehat{H}\hat{g}^{-1}$ is defined over a number field. We show that such $(x, \hat{g}) \in \Sigma^m$ gives rise to a special subvariety of S_Γ. That is

$$S_{x,\hat{g}} = \hat{g}\widehat{H}\hat{g}^{-1}.\tilde{\psi}(x) \cap \tilde{\psi}(X)$$

is a special subvariety of S_Γ. For simplicity simply write $D = D_{\widehat{G}}$. Consider the algebraic subset of $S_\Gamma \times D^\vee$, where $D^\vee$ denotes the flag variety associated to D

$$\widehat{W} := S_\Gamma \times \left(\hat{g}\widehat{H}\hat{g}^{-1}(\mathbb{C}).\tilde{\psi}(x)\right).$$

Let $\widehat{U}$ be a component at $\tilde{\psi}(x)$ of the intersection

$$\widehat{W} \cap S_\Gamma \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} D,$$

such that the projection of $\widehat{U}$ to S_Γ contains $S_{x,\hat{g}}$. We claim that, if Γ is non-arithmetic, $\widehat{U}$ is an atypical intersection. That is

$$\operatorname{codim}_{S_\Gamma \times D} \widehat{U} < \operatorname{codim}_{S_\Gamma \times D} \widehat{W} + \operatorname{codim}_{S_\Gamma \times D} \left(S_\Gamma \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D} D\right). \tag{6.3.4}$$

Assuming the claim, the $\mathbb{Z}$-AS asserts that the projection of $\widehat{U}$ to S_Γ is contained in a strict special subvariety S' of S_Γ. We have

$$S_{x,\hat{g}} \subset S' \subsetneq S_\Gamma.$$

By definition of Σ^m, $\dim S_{x,\hat{g}} \geq m$ and, since m is the maximum of the dimensions of the strict special subvarieties of S_Γ, $\dim S' \leq m$. It follows that $S_{x,\hat{g}} = S'$. That is $S_{x,\hat{g}}$ is a special subvariety of S_Γ and so $\hat{g}\widehat{H}\hat{g}^{-1}$ is defined over $\mathbb{Q}$. This concludes the proof, assuming equation (6.3.4).

To prove the equation (6.3.4), for each embedding $\sigma_1, \sigma_2, \ldots, \sigma_r : K \to \mathbb{R}$, ordered in such a way that σ_1 is just the identity on K, we set

$$d_i := \dim D_{G_{\sigma_i}},$$

$$d_{H_i} := \dim D_{H_{\sigma_i}}.$$

Notice that d_{H_i} is smaller or equal than d_i. We have

$$\dim \widehat{W} = d_1 + \sum_{i=1}^{r} d_{H_i},$$

$$\dim \widehat{U} = d_X(x, \hat{g}) \geq d_{H_1},$$

by definition of the set $\Pi_1(\mathbf{H})$. Putting everything together, we see that we have an equality in 6.3.4 if and only if

$$\operatorname{codim} \widehat{U} = \dim D - \dim D_{\widehat{H}} + \dim D \leq d_1 + \dim D - d_{H_1}.$$

Or simply if and only if, for all $i \neq 1$, $d_i = d_{H_i} = 0$. This happens if and only if each H_{σ_i} and G_{σ_i} are compact groups. By Theorem 2.3.1, it is equivalent to Γ being an arithmetic lattice in G. This concludes the proof. □

We have all the elements to finally prove Theorem 1.2.1, arguing by induction as in the proof of [78, Theorem 4.1].

Proof of Theorem 1.2.1. Let m be the maximum of the dimensions of the strict special subvarieties of S_Γ. For any $j \leq m$ consider the set

$$\mathcal{E}_j := \{\text{maximal special subvarieties of } S_\Gamma \text{ of dimension } j\}.$$

Up to conjugation by elements of $\widehat{G}$, there are only a finite number of semisimple subgroups of $\widehat{G}$. Therefore, Proposition 6.3.4 implies that there are only a finite number of special subvarieties of S_Γ of maximal dimension m. That is $\mathcal{E}_m$ is finite. We argue by induction (downward) on the js for which $\mathcal{E}_j \neq \emptyset$. Let $\mathbf{F}$ be a semisimple subgroup of $\mathbf{G}$ for which there exists a Shimura subvariety associated to $(F, X_F, \Gamma_F \subset \mathbf{F}(K))$ who lies in $\mathcal{E}_j$. Consider the set

$$\Sigma_2(\mathbf{F})^j := \{\hat{g}\widehat{F}\hat{g}^{-1} : (x, \widehat{g}) \in \Sigma(\mathbf{F})^j, S_{x,\widehat{g}} \notin \mathcal{E}_{j'}, \forall j' > j\},$$

where $\Sigma(\mathbf{F})^j$ is defined as in 6.3.3. The inductive assumption implies that the second condition is definable. The proof of Proposition 6.3.4 shows that $\Sigma_2(\mathbf{F})^j$ is countable. Since, up to conjugation by elements of $\widehat{G}$, there are only a finite number of possibilities for $\mathbf{F}$, we proved that $\mathcal{E}_j$ if finite. This concludes the proof of Theorem 1.2.1. □

6.4. **Commensurability criterion for arithmeticity of complex hyperbolic lattice.** We conclude this chapter interpreting the commensurability criterion for arithmeticity of Margulis as an unlikely intersection statement. We offer a new proof Theorem 2.4.3, in the complex hyperbolic case, following the strategy used in the proof of Theorem 1.2.1.

Let $(G = \mathrm{PU}(1, n), X, \Gamma)$ be a real Shimura datum and S_Γ be the associated Shimura variety.

Definition 6.4.1. A Γ-*special correspondence* for S_Γ is a Γ-special subvariety of $S_\Gamma \times S_\Gamma$ whose projections $\pi_i : S_\Gamma \times S_\Gamma \to S_\Gamma$ are finite and surjective.

Theorem 6.4.2. *Let $(G = \mathrm{PU}(1, n), X, \Gamma)$ be a real Shimura datum. The lattice Γ is arithmetic if and only if S_Γ admits infinitely many Γ-special correspondences.*

Remark 6.4.3. If $n = 1$, the axiom RSD0. of real Shimura datum implies that Γ is arithmetic and therefore that S_Γ admits infinitely many Γ-special correspondences. The proof of Theorem 6.4.2 carries on assuming only that Γ is (cohomologically[8]) rigid. However we are not able to recover Theorem 2.4.3 for *all* lattices in $\mathrm{PU}(1, 1)$.

The diagonal embedding $\Delta : S_\Gamma \to S_\Gamma \times S_\Gamma$ realises S_Γ as a Γ-special correspondence associated to the real sub-Shimura datum

$$(\Delta(G), \Delta(X), \Delta(\Gamma)) \subset (G \times G, X \times X, \Gamma \times \Gamma).$$

Notice also that $\Delta(S_\Gamma)$ is a strict maximal Γ-special subvariety of $S_\Gamma \times S_\Gamma$. Given $g \in G$, we consider the maps

$$i_g : G \to G \times G, \;\; f \mapsto (f, gfg^{-1}),$$
$$i_g : X \to X \times X, \;\; x \mapsto (x, g.x).$$

The subvariety $\pi \times \pi(i_g(X)) \subset S_\Gamma \times S_\Gamma$ is a Γ-special subvariety, in the sense of Definition 3.1.6, if and only if g lies in $\mathrm{Comm}(\Gamma)$. In this case $\pi \times \pi(i_g(X))$ is indeed the image of

$$i_g : S_{\Gamma \cap i_g(\Gamma)} \to S_\Gamma \times S_\Gamma.$$

If Γ is arithmetic then it has infinite index in its commensurator, and therefore, in the proof of Theorem 6.4.2 we may assume that Γ is non-arithmetic.

Using the map

$$\psi \times \psi : {S_\Gamma}^{\mathrm{an}} \times {S_\Gamma}^{\mathrm{an}} \to \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}} \times \widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}},$$

[8]Over a one dimensional base, all rigid systems are known to be cohomologically rigid [46, Theorem 1.1.2, Corollary 1.2.4].

we have also a definition of $\mathbb{Z}$-special correspondences and a similar description.

Definition 6.4.4. A $\mathbb{Z}$-*special correspondence* for S_Γ is a $\mathbb{Z}$-special subvariety of $S_\Gamma \times S_\Gamma$ whose projections $\pi_i : S_\Gamma \times S_\Gamma \to S_\Gamma$ are finite and surjective.

The arguments appearing in the proof of Theorem 1.3.2 show also the following.

Proposition 6.4.5. *Every Γ-special correspondence is $\mathbb{Z}$-special, and vice versa.*

From now on we just speak of *special* correspondences for S_Γ. We are ready to prove Theorem 6.4.2.

Proof of Theorem 6.4.2. As in section 6.3, let $\mathcal{C} \subset \mathcal{F} \subset X$ be the definable compact set such that we can assume that $X_H \cap \mathcal{C} \neq \emptyset$, for any real sub-Shimura datum $(H, X_H, \Gamma_H) \subset (G, X, \Gamma)$. Given $\hat{g} \in \widehat{G}$, we write

$$\hat{g} = (g_1, \dots, g_r) \in \prod_{i=1,\dots,r} G_{\sigma_i},$$

where $r = [K : \mathbb{Q}]$ and $\sigma_i : K \to \mathbb{R}$ are ordered in such a way that σ_1 induces the identity on K. Consider the map

$$i_{\hat{g}} : \widehat{G} \to \widehat{G} \times \widehat{G}, \quad \hat{f} \mapsto (\hat{f}, \hat{g}\hat{f}\hat{g}^{-1}),$$

and write $\widehat{\Delta}_{\hat{g}}$ for its image. Consider the set

$$\Pi_0 := \{(x, \hat{g}) \in \mathcal{C} \times \widehat{G} : g_1.x \in \mathcal{C}, \mathrm{Im}((\tilde{\psi} \times \tilde{\psi})(x, g_1.x) : \mathbb{S} \to \widehat{G} \times \widehat{G}) \subset \widehat{\Delta}_{\hat{g}}\}. \tag{6.4.1}$$

By Lemma 6.3.2, Π_0 is a definable set. Associated to $(x, \hat{g}) \in \Pi_0$ there is

$$C_{x,\hat{g}} := \psi^{-1} \times \psi^{-1} \left(\pi \times \pi(\widehat{\Delta}_{\hat{g}}.(\tilde{\psi}(x), \tilde{\psi}(g_1.x))) \right) \subset {S_\Gamma}^{\mathrm{an}} \times {S_\Gamma}^{\mathrm{an}},$$

that could happen to be a special correspondence for S_Γ, or not. Moreover any special correspondence for S_Γ arises as a $C_{x,\hat{g}}$, for some $(x, \hat{g}) \in \Pi_0$. Define

$$c(x, \hat{g}) := \dim_{(\tilde{\psi}(x), \tilde{\psi}(g_1.x))} \left(\widehat{\Delta}_{\hat{g}}.(\tilde{\psi}(x), \tilde{\psi}(g_1.x)) \cap \tilde{\psi} \times \tilde{\psi}(X \times X) \right).$$

The set

$$\Sigma := \{\widehat{\Delta}_{\hat{g}} : (x, \widehat{g}) \in \Pi_0 \text{ for some } x \in \mathcal{C}, c(x, \hat{g}) = n\}$$

is definable. To show that Σ is countable, we argue as in the proof of Proposition 6.3.4. More precisely we use $\mathbb{Z}$-AS to prove that $\widehat{\Delta}_{\hat{g}}$ comes from a $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}} \times \widehat{\mathbf{G}}$.

Assume that Γ is non-arithmetic and consider the algebraic subset of $S_\Gamma \times S_\Gamma \times D^\vee_{\widehat{G}} \times D^\vee_{\widehat{G}}$

$$\widehat{W} := S_\Gamma \times S_\Gamma \times \left(\widehat{\Delta}_{\hat{g}}(\mathbb{C}).(\tilde{\psi}(x), \tilde{\psi}(g_1.x)) \right).$$

Let $\widehat{U}$ be a component at $(\tilde{\psi}(x), \tilde{\psi}(g_1.x))$ of the intersection

$$\widehat{W} \cap S_\Gamma \times S_\Gamma \times_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D \times \widehat{\mathbf{G}}(\mathbb{Z})\backslash D} D_{\widehat{G}} \times D_{\widehat{G}},$$

such that the projection of $\widehat{U}$ to $S_\Gamma \times S_\Gamma$ contains $C_{x,\hat{g}}$. A direct computation shows that, if Γ is non-arithmetic, such intersection is atypical and the $\mathbb{Z}$-AS implies that $C_{x,\hat{g}}$, for $(x, \hat{g}) \in \Sigma$, is a special correspondence of S_Γ. It follows that $\widehat{\Delta}_{\hat{g}}$ is defined over a number field. We proved that Σ is definable and countable, therefore finite. That is S_Γ contains only finitely many special correspondences. $\square$

7. Proof of the non-arithmetic Ax-Schanuel conjecture

In this section we prove Theorem 1.2.2, which, thanks to Corollary 5.6.2 becomes the following, which will be usually called the Γ-AS.

Theorem 7.0.1 (Non-arithmetic Ax-Schanuel Conjecture)**.** *Let (G, X, Γ) be an irreducible Shimura datum. Let $W \subset X^\vee \times S_\Gamma$ be an algebraic subvariety and $\Pi \subset X \times S_\Gamma$ be the graph of $\pi : X \to S_\Gamma$. Let U be an irreducible component of $W \cap \Pi$ such that*

$$\operatorname{codim} U < \operatorname{codim} W + \operatorname{codim} \Pi,$$

the codimension being in $X \times S_\Gamma$ or, equivalently,

$$\dim W < \dim U + \dim S_\Gamma.$$

If the projection of U to S_Γ is not zero dimensional, then it is contained in a strict special subvariety of S_Γ.

Once we have a description of the toroidal compactification of S_Γ, as in section 3.3 and 3.3.1, using Corollary 5.6.2 and Corollary 5.6.1, the proof follows closely the strategy of [59]. For completeness and because there are some details to be changed (mainly about heights and the use of the Pila–Wilkie theorem), and some simplifications due to the fact that G ha rank one, we give a complete proof for the reader's convenience.

Let $Y \subset X$ be a positive dimensional algebraic subvariety. Applying the non-arithmetic Ax-Schanuel to $W = Y \times \overline{\pi(Y)}^{\mathrm{Zar}}$ we obtain the following, which recovers Mok's result [58].

Corollary 7.0.2 (Non-arithmetic Ax-Lindemann-Weierstrass)**.** *Let $(G = \mathrm{PU}(1, n), X, \Gamma)$ be a real Shimura datum and $Y \subset X$ be an algebraic subvariety. Then any irreducible component of the Zariski closure of $\pi(Y)$ in S_Γ is special.*

Remark 7.0.3. The non-arithmetic Ax-Lindemann-Weierstrass actually follows from the $\mathbb{Z}$-AS of Bakker-Tsimerman and Corollary 5.6.2. Indeed $\tilde{\psi}(Y)$ can be written as the intersection of $\tilde{\psi}(X)$ with $Y \times \prod_{i=2}^r D_i \subset D_{\widehat{G}} = X \times \prod_{i=2}^r D_i$. The $\mathbb{Z}$-AS applied to $Y \times \prod_{i=2}^r D_i \times \overline{\pi(Y)}^{\mathrm{Zar}}$, implies that the irreducible components of $\overline{\pi(Y)}^{\mathrm{Zar}}$ are $\mathbb{Z}$-special, which, by Corollary 5.6.2, implies the non-arithmetic Ax-Lindemann-Weierstrass as stated above. The same remarks applies to the Γ-AS for subvarieties of $X \times S_\Gamma$ of the form $A \times B$, where A is an algebraic subvariety of X, and B of S_Γ. In the non-product situation, the relation between the two AS is less clear.

7.1. **Inductive assumptions.** First notice that if $n = \dim X = 1$, there is nothing to prove, so we may assume that $\dim X > 1$. Heading for a contradiction let us suppose that the projection of U to S_Γ is *not* contained in any strict special subvariety and

$$\dim U > \dim W - \dim S_\Gamma.$$

Moreover, suppose that $\dim X$ is minimal among all counterexamples and, subject to that assumption, $\operatorname{codim} W + \operatorname{codim} \Pi - \operatorname{codim} U$ is as large a possible, and, subject to that assumption, $\dim U$ is maximal (and > 0). From now on let k be the $\dim U$.

7.2. **Hilbert scheme and definable Remmert-Stein and Chow.** We use the same notation as in [59, Section 3.1]. Let M be the Hilbert scheme of all subvarieties of $X^\vee \times {S_\Gamma}^{BB}$ with Hilbert polynomial given by the one of (a compactification of) W. Such M is an algebraic variety, and to a point $y \in M$ we denote by $W(y)$ the corresponding algebraic subvariety of $X \times S_\Gamma$. Consider

$$B := \{(x,s,y) \in X \times S_\Gamma \times M : (x,s) \in W(y)\},$$
$$A := \{(x,s,y) \in X \times S_\Gamma \times M : (x,s) \in W(y) \cap \Pi\}.$$

The latter is a closed complex analytic subset of $X \times S_\Gamma \times M$. Denote by $\theta : A \to M$ the projection $(x,s,y) \mapsto y$ and consider

$$A(k) := \{(x,s,y) \in A : \dim_{(x,s)} \theta^{-1}\theta((x,s,y)) \geq k\},$$

where $\dim_{(x,s)}$ denotes the dimension at (x,s) of the fibre of the projection in A (which is closed and complex analytic). Denote by Z the image of $A(k)$ along the natural projection

$$\varphi : X \times S_\Gamma \times M \to X \times S_\Gamma, (x,s,m) \mapsto (x,s).$$

Lemma 7.2.1. *Let $T := (\pi \times \mathrm{Id})(Z)$. It is a closed algebraic subvariety of $S_\Gamma \times S_\Gamma$.*

Proof. Let $\mathcal{F}$ be the fundamental set of Theorem 3.4.3. Thanks the Definable Chow [67], as in [59, Theorem 3.1], the result follows by the fact Z is Γ-invariant and $Z \cap (\mathcal{F} \times S_\Gamma)$ is definable (thanks to Theorem 3.4.5). □

Let $A(k)' \subset A(k)$ be an irreducible analytical component of $A(k)$ containing $U \times [W]$, where $[W] \in M$ is the point corresponding to $W \subset X \times S_\Gamma$. Let also

$$Z' := \varphi(A(k)') \subset Z, \quad V := (\pi \times \mathrm{Id})(Z').$$

Thanks to Lemma 7.2.1, V is an algebraic subvariety of $S_\Gamma \times S_\Gamma$. By construction V contains $(\pi \times \mathrm{Id})(U)$, therefore, by assumption, it is not contained in any strict special subvariety of the image of the diagonal embedding of $S_\Gamma \hookrightarrow S_\Gamma \times S_\Gamma$.

Consider the family F_0 of algebraic subvarieties of $X \times S_\Gamma$ corresponding to $A(k)'$ and let $\Gamma_0 \subset \Gamma$ be the subgroup of elements $\gamma \in \Gamma$ such that every member of F_0 is invariant by γ. For any $\mu \in \Gamma$ let $E_\mu \subset F_0$ be the subset corresponding to algebraic subvarieties invariant under μ. For all $\mu \in \Gamma - \Gamma_0$, E_μ is a strict algebraic subvariety of F_0. Hence every element W' of F_0 outside the countable union $\bigcup_{\mu \in \Gamma - \Gamma_0} E_\mu$ is invariant exactly by the subgroup Γ_0.

Lemma 7.2.2. *W is not invariant by any infinite subgroup of Γ.*

Proof. Suppose this is not true. It is enough to prove that Γ_0 is Zariski dense in G (which would imply that W is $X \times S_\Gamma$, which is a contradiction). Γ acts on $X \times S_\Gamma \times M$ component-wise and such action restricts to $A(k)$. For $\gamma \in \Gamma$ and $(x,s,[W]) \in A(k)$ we write:

$$\gamma \cdot (x,s,[W]) = (\gamma x, s, [\gamma W]).$$

The map $\varphi_{|A(k)} : A(k) \to Z$ is equivariant with respect to this Γ-action and, since $A(k) \to Z$ is proper and the action is discrete, it induces a proper map

$$\Gamma \backslash A(k) \to \Gamma \backslash Z \cong T$$

of analytical varieties. Moreover we may assume that it is surjective. Let $\Gamma \backslash A(k)'$ be the image of $A(k)'$ in $\Gamma \backslash A(k)$. Let Γ_1 be the image of the induced map

$$\pi_1(\Gamma \backslash A(k)') \to \pi_1(V) \to \Gamma.$$

We have just proven that Γ_1 is a finite index subgroup of the monodromy group of V (since the map $\Gamma\backslash A(k) \to V$ is a proper map of analytic varieties.). By construction F_0 is invariant under Γ_1. Let W' be a very general element of F_0 and $\gamma \in \Gamma_1$. We have

$$\mathrm{Stab}(\gamma \cdot W') = \gamma \cdot \mathrm{Stab}(W') \cdot \gamma^{-1},$$

so Γ_0, and hence its Zariski closure, are invariant under conjugation by Γ_1. Since V is Hodge generic, Corollary 5.6.1 implies that the Γ-monodromy of V is exactly G. □

To conclude the proof of the Γ-AS, it is enough to prove the following.

Proposition 7.2.3. *W is invariant by an infinite subgroup of Γ.*

7.3. **Pila-Wilkie and Hwang-To.** First we need some lemmas about the set

$$\Sigma(W) := \{g \in G : \dim_{\mathbb{R}} ((g \cdot W) \cap \Pi \cap (\mathcal{F} \times S_\Gamma)) = \dim_{\mathbb{R}} U\}, \tag{7.3.1}$$

were $\mathcal{F} \subset X$ is a fundamental set for the action of Γ, as in Theorem 3.4.5. Notice that $\Sigma(W)$ contains a $\gamma \in \Gamma$ whenever U intersects $\gamma^{-1}\mathcal{F} \times S_\Gamma$. From the definability of $\pi_{|\mathcal{F}} : \mathcal{F} \to S_\Gamma$, which was established in Theorem 3.4.5, we have the following.

Lemma 7.3.1. *The set $\Sigma(W)$ is definable.*

From now on we may and do assume that the identity element belongs to the set $\Sigma(W)$.

Lemma 7.3.2. *The volume of $((g \cdot W) \cap \Pi \cap (\mathcal{F} \times S_\Gamma))$ is uniformly bounded as g varies in $\Sigma(W)$.*

We argue using the results on toroidal compactifications and the metric at infinity of section 3.3. See also [49, Proposition 3.2 and Lemma 5.8].

Proof of Lemma 7.3.2. We may assume that $\mathcal{F}$ is a Siegel set. Thanks to the description of Siegel sets of Proposition 3.4.1, $\mathcal{F}$ can be covered by a finite union of sets Q that can be embedded in a product of one-dimensional sets $\prod_{i=1}^m J_i$ where each J_i comes with a $(1,1)$-volume form ω_i such that $\int_{J_i} \omega_i < \infty$ and $\sum_{i=1}^m \omega_i$ dominates the Kähler form of an invariant hyperbolic metric (as explained in Section 3.3.2). Let I be a subset of $\{1, \dots, m\}$ containing $\dim U$ elements. It is enough to show that the projection of

$$((g \cdot W) \cap \Pi \cap (\mathcal{F} \times S_\Gamma)) \cap Q \times S_\Gamma$$

onto $\prod_{i \in I} J_i$ has finite fibres of uniformly bounded cardinality. This follows since, as proven in the previous lemma, $\Sigma(W)$ is definable. □

Given $\gamma \in \Gamma$, we define the height of γ, denoted by $H(\gamma)$, as the maximum of the norm from K to $\mathbb{Q}$ of all the entries of γ seen as a matrix in $G = \mathrm{PU}(1,n)$. Thanks to Theorem 1.3.1, we can fix a free $\mathcal{O}_K$-module of finite rank, denoted by $V_{\mathcal{O}_K}$, and interpret elements in Γ as elements of $\mathrm{GL}(V_{\mathcal{O}_K})$.

Proposition 7.3.3. *For any positive real number R, let $B(x_0, R)$ denote the geodesic ball of X with centre x_0 and radius R. Every element γ of*

$$\{\gamma \in \Gamma : B(x_0, R) \cap \gamma^{-1}\mathcal{F} \neq \emptyset\}$$

has height $H(\gamma) = e^{O(R)}$.

Proof. Since $\tilde{\psi}$ does not increase the hyperbolic distance (see indeed [39, Corollary 9.3], and [32, Proposition 2.4 (page 57)]), we have that

$$\tilde{\psi}\left(B(x_0,R)\right) \subset B_D(\psi(x_0),R),$$

where $B_D(\psi(x_0),R)$ denotes a geodesic ball of radius R in D centred in $\psi(x_0)$. Let $\rho(\gamma)$ be the image of Γ in $\widehat{G}(\mathbb{Z})$. To apply [7, Theorem 4.2], we need to notice that the fundamental set $\mathcal{F} \subset X$ we are considering here, is essentially the same as the fundamental set described in [7, Section 3]. Indeed, as described in Proposition 3.4.1, $\mathcal{F}$ can covered by a finite union of open sets Θ, and each Θ is contained in a fundamental set as described in *op. cit.*. Eventually [7, Theorem 4.2] asserts that:

$$H(\rho(\gamma)) = e^{O(R)}.$$

where here $H(\rho(\gamma))$ denoted the height function of an elements in $\widehat{\mathbf{G}}(\mathbb{Z})$ seen in the Weil restriction from K to $\mathbb{Q}$ of $\mathrm{GL}(V_K)$. Let $\gamma_{i,j}$ be an entry of γ with maximal norm, we have

$$H(\rho(\gamma)) = \prod_{\sigma\in\Omega_\infty} (\sigma(\gamma_{i,j})) = H(\gamma),$$

and therefore $H(\gamma) = e^{O(R)}$. □

The Pila-Wilkie theorem [69] (and its later refinements) can be generalised to count K-points, when K is a number field $\mathbb{Q} \subset K \subset \mathbb{R}$. That is, let $\Theta \subset \mathbb{R}^n$ be a definable subset, $K \subset \mathbb{R}$ a number field of degree r. For a real number $T \geq 1$ set

$$\Theta(K,T) := \{x \in S \cap K^n : H(x) \leq T\},$$

$$N_K(\Theta,T) := |\Theta(K,T)|,$$

where $|\Theta(K,T)|$ denotes the cardinality of the set $\Theta(K,T)$. Let Θ^{alg} be the union of all positive dimensional semi-algebraic subsets of Θ. The following is [68, Theorem 1.5].

Theorem 7.3.4 (Pila)**.** *For $\epsilon > 0$, we have*

$$N_K(\Theta - \Theta^{\mathrm{alg}}, T) = O_{S,r,\epsilon}(T^\epsilon).$$

Thanks to Hwang–To [45, Corollary 3, page 1227], as in the proof of [59, Lemma 4.3], we finally have the following.

Proposition 7.3.5. *The set $\Sigma(W)$ contains a real semialgebraic curve $C_{\Sigma(W)}$ which contains infinitely many $\gamma \in \Gamma$.*

7.4. **End of the proof.** We are ready to prove Proposition 7.2.3. The argument is the same as the one appearing at end of the proof of [59, Lemma 4.3].

Proof of Proposition 7.2.3. For $c \in C_{\Sigma(W)}$, write $W_c := c \cdot W$. There are two possibilities:

- W_c is constant in c. In this case W is stable under $C_{\Sigma(W)} \cdot C_{\Sigma(W)}^{-1}$ and since $C_{\Sigma(W)}$ contains at least one element $\gamma' \neq \mathrm{Id}$ of Γ, we have that W is stable under the subgroup of Γ generated by γ'. Since Γ, since the beginning, is assumed to be torsion free, we conclude, proving the lemma.

- W_c is not constant in c. Notice that for $c' \in C_{\Sigma(W)} \cap \Gamma \neq \emptyset$ the projection of $W_c \cap \Pi$ is not contained in a strict special subvariety (by the assumption on U). It follows that $(\pi \times \mathrm{Id})(W_c \cap \Pi)$ is not contained in a strict special subvariety for all but countably many $c \in C$ (since each family of special subvarieties is associated to an algebraic subgroup of G defined over a number field). There are again two possibilities:
  - If $U \subset W_c$ for all $c \in C_{\Sigma(W)}$, then consider $W_c \cap W_{c'}$. We have $U \subset W_c \cap W_{c'}$ and
  $$\dim (W_c \cap W_{c'}) < \dim W$$
  contradicting the induction hypothesis on $\dim W - \dim U$.
  - If $U \nsubseteq W_c$ for some $c \in C_{\Sigma(W)}$, then $W_c \cap \Pi$ varies with $c \in C_{\Sigma(W)}$. Let $\widetilde{W}$ be the Zariski closure of $C \cdot W$. We have that $\dim \widetilde{W} = \dim W + 1$, and so
  $$\dim \left(\widetilde{W} \cap \Pi\right) = \dim U + 1$$
  contradicting the assumptions on $\dim U$.

The proposition is proven since the second case can not happen. □

Eventually we have proven Theorem 1.2.2.

8. Special points and their Zariski closure

In this section we present several notions of *special point* of Shimura varieties. We describe all of them and we formulate André–Oort like conjectures relating special points to special subvarieties. We also compare such conjectures with the ones appearing in [48, Section 5]. For simplicity we discuss only the irreducible case. So we fix $(G = \mathrm{PU}(1,n), X, \Gamma \subset \mathbf{G}(\mathcal{O}_K))$ a real Shimura datum and S_Γ the associated Shimura variety.

8.1. Γ-special points.

We first consider zero dimensional Γ-special subvarieties of S_Γ.

Definition 8.1.1. A point $x \in X$ is *pre-Γ-special* if the K-Zariski closure of the image of $x : \mathbb{S} \to G$ in $\mathbf{G}$ is commutative. A point $s \in S_\Gamma$ is *Γ-special* if it is the image along $\pi : X \to S_\Gamma$ of a pre-Γ-special point $x \in X$.

The following can be proven exactly as in the classical case of arithmetic Shimura varieties. For completeness we give some details of the proof. From now on, we fix an algebraic closure of the field of rational numbers, denoted by $\overline{\mathbb{Q}}$.

Proposition 8.1.2. *Every Γ-special subvariety contains an analytical dense set of Γ-special points. Pre-Γ-special point are defined over $\overline{\mathbb{Q}}$, with respect to the $\overline{\mathbb{Q}}$-structure given by looking at $X \subset X^\vee$ inside its associated flag variety.*

Proof. Let S_{Γ_H} be a Γ-special subvariety associated to (H, X_H, Γ_H). First notice that if one Γ-special point exists in S_{Γ_H}, then there is a dense set of Γ-special points. Indeed each point in $\mathbf{H}(K).x$ is again Γ-special and the set $\mathbf{H}(K).x$ is dense by the usual approximation property.

For the existence of a Γ-special point we argue as in [22, Section 5]. Let $x \in X_H$ and $T \subset G$ be a maximal torus containing the image of
$$x : \mathbb{S} \to H.$$
Write T as the centraliser of some regular element λ of the Lie algebra of T. Choose $\lambda' \in \mathbf{H}(K)$ sufficiently close to λ. It is still a regular element and its centralizer $\mathbf{T}'$ in $\mathbf{G}$ is a maximal torus in

$\mathbf{G}$. Since there are only finitely many conjugacy classes of maximal real tori in G, we may assume that T' and T are conjugated by some $h \in H$. The point $\pi(hx)$ is a Γ-special point in $\pi(X_H)$.

Finally if $x \in X$ is pre-Γ-special, we can factorise its associated Hodge cocharacter as

$$h_x : \mathbb{C}^* \to (\mathbf{T}_x)_{\mathbb{C}},$$

where $\mathbf{T}_x$ is a maximal K-torus. The above factorisation has to hold even over $\overline{\mathbb{Q}}$. Therefore the point associated to x in the flag variety is fixed by $\mathbf{T}_x(\overline{\mathbb{Q}})$. This is enough to conclude that x is defined over $\overline{\mathbb{Q}}$. For details we refer to [80, Proposition 3.7]. □

As in the case of Shimura varieties the Γ-AS, or more precisely the Γ-ALW, could be helpful in proving the following.

Conjecture 8.1.3 (Γ-André–Oort)**.** *An irreducible subvariety $W \subset S_\Gamma$ is Γ-special if it contains a Zariski dense set of Γ-special points.*

For an introduction to the André–Oort conjecture for Shimura varieties, we refer to [50].

8.2. **$\mathbb{Z}$-special points.** Another natural possibility is to look at zero dimensional intersections between $\psi({S_\Gamma}^{\mathrm{an}})$ and Mumford–Tate sub-domains of $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$.

Definition 8.2.1. A point $s \in S_\Gamma$ is *$\mathbb{Z}$-special* if it is a zero dimensional $\mathbb{Z}$-special subvariety.

A first relation between $\mathbb{Z}$-special point and Γ-special is given by the following.

Proposition 8.2.2. *Let $\pi(x) \in S_\Gamma$ be a $\mathbb{Z}$-special point, then $\pi(x)$ is Γ-special.*

Proof. Let $\mathbf{R}$ be a $\mathbb{Q}$-subgroup of $\widehat{\mathbf{G}}$ inducing a Mumford–Tate sub-domain D_R of $D_{\widehat{G}}$. If $\mathbf{R}(\mathbb{Z})\backslash D_R$ intersects $\psi({S_\Gamma}^{\mathrm{an}})$ in a finite number of points, then $R \cap G_{\sigma_1}$ is a compact subgroup of $\widehat{\mathbf{G}}(\mathbb{R})$. Let $x \in \tilde{\psi}^{-1}(D_R) \subset X$. It corresponds to a map

$$x : \mathbb{S} \to G,$$

and, by construction $x(\mathbb{S})$ is contained in $R \cap G_{\sigma_1}$. Since G is of adjoint type, $x(\mathbb{S})$ is contained in the centre of $R \cap G_{\sigma_1}$. However $R \cap G_{\sigma_1}$ has a natural structure of K-subgroup of $\mathbf{G}$, which we denote by $\mathbf{R}_\sigma$. It follows that $x(\mathbb{S})$ is contained in a commutative K-subgroup of $\mathbf{G}$. That is $\mathbf{MT}(x)$ is commutative and x is pre-Γ-special. □

In rank one, the Γ-AO Conjecture, thanks to Proposition 8.2.2, predicts that the irreducible components of the Zariski closure of $\mathbb{Z}$-special points are $\mathbb{Z}$-special subvariety.

Proposition 8.2.3. *If Γ is non-arithmetic, $\mathbb{Z}$-special points of S_Γ are atypical intersections in $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ between $\psi({S_\Gamma}^{\mathrm{an}})$ and Mumford–Tate sub-domains of $D_{\widehat{G}}$.*

Proof. Let $\mathbf{R} \subset \widehat{\mathbf{G}}$ be as in the proof of Proposition 8.2.2. The intersection between $\mathbf{R}(\mathbb{Z})\backslash D_R$ and $\psi({S_\Gamma}^{\mathrm{an}})$ is typical when

$$\mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}}\{s\} = \mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}} S_\Gamma + \mathrm{codim}_{\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}} \mathbf{R}(\mathbb{Z})\backslash D_R,$$

where s is one of the finite points lying in $\mathbf{R}(\mathbb{Z})\backslash D_R \cap \psi({S_\Gamma}^{\mathrm{an}})$. Let $r = [K : \mathbb{Q}]$. Equivalently

$$\sum_{i=1}^{r} d_i = \sum_{i=2}^{r} d_i + \left(d_1 + \sum_{i=2}^{r} d_i - d_{R_i} \right),$$

where d_i and d_{R_i} were defined in the proof of Proposition 6.3.4. That is s is typical if and only if $d_i = d_{R_i}$ for all $i \geq 2$, which is impossible unless $\mathbf{R} = \widehat{\mathbf{G}}$. □

As usual we expect that a subvariety of S_Γ having a dense set of atypical points is atypical.

Conjecture 8.2.4. *Let (G, X, Γ) be a real Shimura datum and $W \subset S_\Gamma$ an algebraic subvariety. W contains a Zariski dense set of $\mathbb{Z}$-special points if and only if is special and arithmetic.*

We can also observe that the intersection of Γ-special subvarieties gives rise to $\mathbb{Z}$-special, and therefore Γ-special, points.

Proposition 8.2.5. *Let S_{Γ_1}, S_{Γ_2} be Γ-special subvarieties of positive dimension in S_Γ intersecting in a finite number of points. If $s \in S_{\Gamma_1} \cap S_{\Gamma_2}$, then s is $\mathbb{Z}$-special.*

Proof. Since S_{Γ_1} and S_{Γ_2} are also $\mathbb{Z}$-special subvariety, every point in the intersection can be written as the the preimage along the period map of the intersection of two Mumford–Tate sub-domains. See also Proposition 4.4.5. The point s is therefore $\mathbb{Z}$-special. □

8.3. **Complex multiplication points.** Recall that, as in Theorem 1.3.1, we denote by $\widehat{\mathbb{V}}$ for the $\mathbb{Z}$-VHS on S_Γ induced by $\mathbb{V}$. Another interesting class of points is given by the following.

Definition 8.3.1. A point $s \in S_\Gamma$ is called a *CM-point* if the Mumford–Tate group of $\widehat{\mathbb{V}}$ at s is commutative.

Such points are both $\mathbb{Z}$ and Γ-special, but they are even more special. The following is a special case of [48, Conjecture 5.6] and it is indeed predicted by, the more difficult, Conjecture 8.2.4.

Conjecture 8.3.2. *Let (G, X, Γ) be a real Shimura datum and $W \subset S_\Gamma$ an algebraic subvariety. W contains a Zariski dense set of CM-points if and only if is special and arithmetic.*

If Γ is an arithmetic lattice, this is the classical André-Oort conjecture. If Γ is non-arithmetic, but $\widehat{\mathbf{G}}(\mathbb{Z})\backslash D_{\widehat{G}}$ is a Shimura variety, then the above conjecture follows from André–Oort conjecture for Shimura varieties of abelian type, which is now a theorem [77].

8.4. **Bi-arithmetic points.** In the classical case of Shimura varieties, another option is to look at *bi-arithmetic points* and *bi-arithmetic subvarieties*. That is points $x \in X(\overline{\mathbb{Q}})$ such that $\pi(x) \in S_\Gamma(\overline{\mathbb{Q}})$ and subvarieties $W \subset S_\Gamma$ which are defined over $\overline{\mathbb{Q}}$ and such that some component of $\pi^{-1}(W)$ is algebraic and defined over some number field. See also [50, section 4.2]. Let $(G = \mathrm{PU}(1, n), X, \Gamma)$ be a real Shimura datum[9]. We briefly show how to prove that S_Γ, and its special subvarieties S' of dimension > 1, have a model over a number field. This follows again from the rigidity of lattices.

8.4.1. *Models.* Let S be a smooth complex quasi-projective variety. We say that S admits a $\overline{\mathbb{Q}}$-*model* if there exists $Y/\overline{\mathbb{Q}}$ such that $Y \times_{\overline{\mathbb{Q}}} \mathbb{C} \cong S$, with respect to some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Two $\overline{\mathbb{Q}}$-models Y, Y' are certainly isomorphic, both over $\mathbb{C}$ and $\overline{\mathbb{Q}}$. We say that the S has a *unique* $\overline{\mathbb{Q}}$-structure is S has a $\overline{\mathbb{Q}}$-model and any two $\overline{\mathbb{Q}}$-models of S are uniquely isomorphic.

For example if Γ is cocompact, [72, Theorem 1] and [11, Theorem 1] imply the following[10].

[9] From axiom RSD0. a real Shimura datum $(G = \mathrm{PU}(1, 1), X, \Gamma)$ corresponds to a Shimura curve.

[10] Actually Shimura does not discuss the uniqueness of the $\overline{\mathbb{Q}}$-structure.

Theorem 8.4.1 (Shimura, Calabi, Visentini)**.** *Let $(G = \mathrm{PU}(1,n), X, \Gamma)$ be a real Shimura datum such that S_Γ is projective. Let Θ be the sheaf of germs of holomorphic sections of the tangent bundle of S_Γ. If the dimension of S_Γ is greater or equal to two, then, for $i = 0, 1$*

$$H^i(S_\Gamma, \Theta) = 0.$$

It follows that S_Γ admits a unique $\overline{\mathbb{Q}}$-structure.

If Γ is arithmetic, the above proof can be generalised to cover the case when S_Γ is not compact, by using Mumford's theory [64]. See indeed [34, 66]. With the tools described in Section 3.3 it should be possible to generalise such arguments to the case of non-arithmetic lattices. For length reasons, we prefer to give a different argument which uses only the fact that all lattices in $\mathrm{PU}(1,n)$, for $n > 1$, have entries in some number field, as explained in section 2.2. Even though such point of view will not say anything about the uniqueness of the $\overline{\mathbb{Q}}$-structure.

8.4.2. *The action of an automorphism of $\mathbb{C}$ on a Shimura variety.*

Theorem 8.4.2. *Let (G, X, Γ) be a real Shimura datum and $\sigma \in \mathrm{Aut}(\mathbb{C})$. There exists a real Shimura datum $({}^\sigma G, {}^\sigma X, {}^\sigma \Gamma)$ such that*

$${}^\sigma S_\Gamma \cong S_{{}^\sigma \Gamma}.$$

Moreover:

- *Γ is arithmetic if and only if ${}^\sigma\Gamma$ is arithmetic;*
- *G has rank one if and only if ${}^\sigma G$ has rank one.*

Proof. In the arithmetic case it follows from the work of Borovoi and Kazdhan [47]. In the non-arithmetic case, thanks to the work of Margulis, the theorem reduces to the case of ball quotients. The result can now be seen using the fact that the Chern numbers of S_Γ satisfy the equality from Yau's solution of the Calabi conjecture (in the cocompact case, and by Tsuji in the general case), which characterise ball quotients by their Chern numbers, and the fact that the Chern numbers are preserved under Galois conjugation. See [60, Proof of the reduction, page 257] and references therein.

Finally, since elements in the commensurator give special algebraic correspondences, and special correspondences are preserved by the action of $\mathrm{Aut}(\mathbb{C})$, we have that σ^{-1} induces an isomorphism

$$\mathrm{Comm}(\Gamma)/\Gamma \cong \mathrm{Comm}({}^\sigma\Gamma)/{}^\sigma\Gamma.$$

By Margulis (cf. sections 2.4.3 and 6.4), Γ is arithmetic if and only if ${}^\sigma\Gamma$ is arithmetic. See also the proof of [30, Proposition 1.2]. □

To complete our discussion of the action of $\mathrm{Aut}(\mathbb{C})$, we should explain what happens to special subvarieties. Indeed it is a well known fact that, in the theory of arithmetic Shimura varieties, $\mathrm{Aut}(\mathbb{C})$ maps special subvarieties to special subvarieties. In the next proposition, we briefly explain why this holds true in case of special curves in two-dimensional ball quotients. The general case should follow, using the numerical characterisation of totally geodesic subvarieties of (arbitrary) ball quotients that appears in [63, Addendum 2.4].

Proposition 8.4.3. *Let $(G = \mathrm{PU}(1,2), X, \Gamma)$ be a non-arithmetic ball quotient of dimension, and let (H, X_H, Γ_H) be any real sub-Shimura datum (possibly not satisfying RSD0.). Let $\sigma \in \mathrm{Aut}(\mathbb{C})$, then ${}^\sigma S_{\Gamma_H}$ is a ${}^\sigma\Gamma$-special subvariety of ${}^\sigma S_\Gamma$.*

Proof. We first give the details in the easier case where S_Γ is a smooth projective surface. Recall that a curve $C \subset S_\Gamma$ is totally geodesic if and only if

$$3C.C = -K_{S_\Gamma}.C, \tag{8.4.1}$$

where K_{S_Γ} denotes the canonical bundle on S_Γ. It is easy, if C satisfies (8.4.1), then ${}^\sigma C \subset {}^\sigma S_\Gamma$ is such that

$$3{}^\sigma C.{}^\sigma C = -K^\sigma.{}^\sigma C,$$

where K^σ is the canonical bundle of ${}^\sigma S_\Gamma$. That is ${}^\sigma C$ is totally geodesic. Thanks to Corollary 5.6.2, this proves the result in the special case of curves in projective surfaces. The quasi-projective case can be dealt in a similar way. □

Theorem 8.4.4. *Let (G, X, Γ) be a real Shimura datum. Then S_Γ admits a model over a number field.*

Proof. If Γ is arithmetic, the result follows for example from Faltings [34]. Therefore we may and do assume that $G = \mathrm{PU}(1, n)$, for some $n > 1$. Let ${S_\Gamma}^{BB}$ be Baily-Borel compactification of S_Γ, as discussed in Section 3.3. From [37, Criterion 1 (page 3)], the following are known to be equivalent:

a) ${S_\Gamma}^{BB}$ can be defined over $\overline{\mathbb{Q}}$;
b) The set $\{{}^\sigma {S_\Gamma}^{BB} : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ contains only finitely many isomorphism classes of complex projective varieties;
c) The set $\{{}^\sigma {S_\Gamma}^{BB} : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ contains only countably many isomorphism classes of complex projective varieties.

Since the action of $\mathrm{Aut}(\mathbb{C})$ on the Baily-Borel compactification of S_Γ preserves the smooth locus, it is enough to check that $\{{}^\sigma S_\Gamma : \sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})\}$ contains countably many isomorphism classes. By Theorem 8.4.2, it is enough to show that there are most countably many lattices in $G = \mathrm{PU}(1, n)$ (for some $n > 1$ fixed), up to conjugation by elements in G. But this follows from local rigidity, as discussed in Section 2.2. □

Finally, Proposition 8.4.3 shows that also the special subvarieties of S_Γ can be defined over some number field.

8.4.3. *Zariski closure of bi-arithmetic points.* Eventually we have:

Conjecture 8.4.5. *An irreducible subvariety $W \subset S_\Gamma$ contains a dense set of bi-arithmetic points if and only if it is bi-arithmetic.*

This is known to be equivalent to the André–Oort conjecture only for arithmetic Shimura varieties of abelian type (thanks to Wüstholz' analytic subgroup theorem). See [50, Example 4.17] and references therein. Finally it would be interesting to show that CM-points in S_Γ, in the sense of Definition 8.3.1, are bi-arithmetic.

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