On Notation in Multivariate Calculus

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There is a standard joke, which says that to distinguish a physicist from a mathematician, it is enough to define $f(x, y) = x^2 + y^2$ and ask for $f(r, \theta)$. The physicist, recognizing f as the function computing the squared distance from the origin in the plane, and (r, θ) as the standard parameters representing polar coordinates, will return $f(r, \theta) = r^2$. The mathematician, seeing f as a function from $\mathbb{R}^2$ to $\mathbb{R}$, and (r, θ) as mere symbols, will make the simple substitution $f(r, \theta) = r^2 + \theta^2$.

On a purely technical level, the mathematician is, of course, right. As any good introductory calculus course will tell you, writing $f(x, y) = x^2 + y^2$ is simply "shorthand" for defining $f : \mathbb{R}^2 \to \mathbb{R}$ and specifying that

$$f(x, y) = x^2 + y^2 \qquad \forall (x, y) \in \mathbb{R}^2.$$

In this interpretation, the symbols x and y are simply place-holders for arbitrary elements of $\mathbb{R}$, we could have just as well said

$$f(s, t) = s^2 + t^2 \qquad \forall (s, t) \in \mathbb{R}^2$$

and defined the same f.

After some amount of time, the same sort of calculus class might go on to define *partial differentiation.* Partial differentiation, we are told, is what you get when you differentiate a function "with respect to" one of the variables, while holding the other variables "fixed". For instance, if we differentiate our f with respect to the variable x, we obtain a new function $\frac{\partial f}{\partial x}$, whose value at every point is given by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad \forall (x, y) \in \mathbb{R}^2.$$

This definition, I think, should bother you. We just finished explaining that our variables x and y were mere place-holders. We can, by our reasoning, change x to s and y to t, and hence define the same object via the expression

$$\frac{\partial f}{\partial s}(s, t) = \lim_{h \to 0} \frac{f(s + h, t) - f(s, t)}{h} \qquad \forall (s, t) \in \mathbb{R}^2.$$

But this fails! We were attempting to define $\frac{\partial f}{\partial x}$. This new expression, whatever it is, appears to be defining some *other thing* $\frac{\partial f}{\partial s}$! Somehow, the object $\frac{\partial f}{\partial x}$ appears to be dependent on the "free variable" x – the thing that was supposed to be arbitrary.

When faced with this example, one is tempted to make an objection. For instance, is it not poor form to reuse the symbol "x" as an argument in the definition of $\frac{\partial f}{\partial x}$? Is it not more clear if I instead write

$$\frac{\partial f}{\partial x}(s,t) = \lim_{h\to 0} \frac{f(s+h,t) - f(s,t)}{h} \qquad \forall (s,t) \in \mathbb{R}^2,$$

and avoid the problem? Well, yes, but if one is using the symbol "s" to specify the first argument to $\frac{\partial f}{\partial x}$, then why does the notation for $\frac{\partial f}{\partial x}$ emphasize differentiation *with respect to* "x"? Would the second partial derivative with respect to the first argument be denoted $\frac{\partial^2 f}{\partial s \partial x}$ instead of $\frac{\partial^2 f}{\partial x^2}$? And if not, what significance does x play that warrants its position in the denominator?

If you press a calculus instructor on this subject, they might tell you that $\frac{\partial f}{\partial x}$ is simply a case of "bad notation". There's no contradiction here, rather what we *really mean* when we write $\frac{\partial f}{\partial x}$ is to differentiate f with *respect to the first argument*, the thing that we happen to call x, and that perhaps we should more properly write something like $\partial_1 f$ instead. Indeed, some books will do this; let's see how it works in practice.

In elementary calculus books, or ones aimed at students outside of mathematics, one often finds basic exercises on applying differentiation rules. For instance, we can easily imagine running into exercises such as:

Exercises:

1. Compute the following:
   (a) $\frac{\partial}{\partial x}(x^2 + y^2)$
   (b) $\frac{\partial}{\partial y}(x^2 + y^2)$
   (c) ...
2. ...

These exercises are already somewhat peculiar. If x and y are free variables, as is to be believed, what are they doing to the right of the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, which act on functions? Perhaps the question is to be interpreted as asking for a *formal* differentiation in the ring $\mathbb{R}[x,y]$? Or maybe the expression x^2+y^2 is to be understood as implicitly defining a function $\mathbb{R}^2 \to \mathbb{R}$, which is then differentiated? But although the precise mathematical meaning of the question is unclear, the intent is clear enough – we are being asked to differentiate the expression with respect to one of the variables.

It's worth noting what happens if we now try to clarify matters by introducing our improved notation.

Exercises (better notation):

1. Compute the following:
   (a) $\partial_1(x^2 + y^2)$
   (b) $\partial_2(x^2 + y^2)$
   (c) ...
2. ...

The new version is decidedly worse. We are now not only supposed to interpret $x^2 + y^2$ as a function, but we are being asked to figure out which of x and y is the first argument! Perhaps it is obvious, by simple convention, that x "ought" to be before y. But if x and y are still mere symbols, we can choose different symbols to rewrite the exercises as,

Exercises (better notation):

1. Compute the following:
   (a) $\partial_1(\xi^2 + \eta^2)$
   (b) $\partial_2(\xi^2 + \eta^2)$
   (c) ...
2. ...

which leaves the question completely ambiguous.

It's tempting to dismiss this concern as a consequence of mere laziness. If these books wanted to do this properly, it might be argued, they would define a proper $f : \mathbb{R}^2 \to \mathbb{R}$, and then ask for $\partial_1 f$ or $\partial_2 f$. Or perhaps simple tasks like applying formal differentiation rules are not important enough to get a proper mathematical treatment, justifying the occasional use of the poor notation $\frac{\partial}{\partial x}$. Maybe when one wants to be careful, the correct approach is to use ∂_1 and ∂_2, using $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ only when the intended meaning is obvious or when there is no need to state things rigorously.

But such issues are not limited to trivial exercises. People who actually *use* calculus, rather than simply study it, are not content with merely differentiating with respect to the first or second argument as our notation ∂_1 and ∂_2 allows. Before long you'll run into a physicist, who upon seeing our favourite $f(x, y) = x^2 + y^2$, will ask for $\frac{\partial f}{\partial r}$!

It was one thing for the variables with which we defined f to appear in the notation $\frac{\partial f}{\partial x}$, since checking whether x was the first or second argument is routine enough that mentally translating the question to one asking for $\partial_1 f$ is not particularly demanding. But the symbol "r" bears no relation to f, as an argument in the definition or otherwise, and so $\frac{\partial f}{\partial r}$ is another matter entirely. What is meant by this is that somewhere off to the side there is an equation, something like $r = \sqrt{x^2 + y^2}$, which tells you what "r" is. The request for $\frac{\partial f}{\partial r}$ is then an instruction to "re-express" the function f "in terms of" r, that is to write $f(r) = r^2$, and then compute $\frac{\partial f}{\partial r}(r) = 2r$. We can then reverse the substitution, and find that $\frac{\partial f}{\partial r}(x, y) = 2\sqrt{x^2 + y^2}$.

This procedure is noteworthy for several reasons. Firstly, the equation $r = \sqrt{x^2 + y^2}$ is, as it stands, meaningless. The symbols x and y are arbitrary place-holders, as we have established, and so without any context for what they are supposed to hold the place of, there is no sense in putting them in an equation. Secondly, our function f was briefly transformed from a function of two arguments into a function of one, differentiated, and then the derivative re-expressed as a function of the original two arguments. Since functions, being objects in mathematics, do not undergo time evolution, and hence do not have varying numbers of arguments from moment to moment, we are once again in need of clarification.

Turning again to our new notation, we find (somehow) that the "correct" way to define our operator $\frac{\partial}{\partial r}$ is as follows:

$$(\partial_r f)(x, y) = \frac{x}{\sqrt{x^2 + y^2}}(\partial_1 f)(x, y) + \frac{y}{\sqrt{x^2 + y^2}}(\partial_2 f)(x, y)$$

$$\forall (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

How one arrives at such an expression is somewhat of a mystery, and it's not immediately clear what it has to do with the simple request of substituting the x and y for r, and performing the differentiation, and re-expressing the result in terms of x and y (or why in fact it should give the same answer for any function f where such a substitution is possible).

Physicists, and most mathematicians, are not typically bothered by such things. In day to day use, the symbols x, y and r are treated as having an existence of their own; one speaks of "functions of x" or "functions of y" as if the arguments were not *numbers* but were somehow intimately tied up with the variables themselves. Nowhere is this more apparent than in our original example, where our physicist re-expressed $f(x, y) = x^2 + y^2$ as $f(r, \theta) = r^2$. Somehow, the function f was more than just an association between elements of $\mathbb{R}^2$ and elements of $\mathbb{R}$, but is an object which takes different forms depending on the variables used to express it.

In physical terms, this statement is obvious. The function f computes the squared distance of a point from the origin in the plane, and how one computes such a distance obviously depends on what information one is given. If we are given the point's Cartesian coordinates, the result is a simple matter of applying the Pythagorean Theorem, whereas if we are instead given the radial distance and an angle, we can simply return the square of the radial distance, and the angle is of no use to us.

In this viewpoint, our f is not really a function mapping $\mathbb{R}^2 \to \mathbb{R}$, but rather an assignment of real numbers to points in a geometric object. We are given different presentations of what might be called *The Euclidean Plane*, one in terms of the Cartesian "co-ordinates" (x, y) and another in terms of the polar "co-ordinates" (r, θ), and we return a sort-of *model* of f with respect to the particular presentation of The Euclidean Plane we have been given. The model is, naturally, a function from $\mathbb{R}^2 \to \mathbb{R}$, but each is thought of as a mere version of some more abstract object f.

There is a formalism we can use to try to capture this notion, that of the calculus of manifolds. A manifold is an abstract geometric object; something like a sphere, the plane, a torus or some higher-dimensional surface. In physics, the most important manifold is something called *spacetime*, a four-dimensional combination of space and time which is thought to encode the interplay between space and time in our universe.

So suppose we wish to study our function f, and understand it like the physicist does. In physics, the objects of interest are not merely mathematical, but are thought of as having some existence in the "real world". For instance, consider a completely blank sheet of paper spread out on a table, but extending infinitely in all directions. The sheet of paper we will view as a plane in three dimensions; let us denote such a plane by P.

Now suppose we wish to define our function f as a function $f : P \to \mathbb{R}$, that is, we wish to associate a squared *distance from the origin* to every point $p \in P$. The first question is: well, what origin? The sheet of paper is completely blank; there is no sense in which any particular point is more "the origin" than any other. What's more, the paper is infinite, and so assigning each point $p \in P$ a real number one at a time is impossible.

Now imagine overlaying the paper with an infinite transparency on which is imprinted an infinite grid. The grid lines can be straight and orderly, as they are for a Cartesian grid, or they can be curved and distorted, as they would be if you took a Cartesian grid imprinted on a rubber sheet and twisted and stretched them. In each case, there is a marked point, something called $(0, 0)$, and two distinguished paths passing through every point. Following the first path rightward until the next marked intersection we arrive at a point labelled $(1, 0)$. The path we follow need not be straight, it may curve violently, travelling in all manner of directions on the underlying sheet of paper. With two different grids, we may very well arrive at two different points labelled $(1, 0)$ on the paper; in fact, the grids need not agree which point we call $(0, 0)$.

Travelling along the second sort of path from $(1, 0)$ downward we arrive at $(1, -1)$. This journey, like the other one, can take an equally wide range of forms. We notice that inter-spaced between the grid lines are other, fainter lines marked with pairs of numbers like $(1, -0.5)$ and $(1, -0.75)$. With these increasingly fainter lines, we can travel all over the page, and find a distinct pair of real numbers associated to each point by any particular grid.

Now suppose we want to describe our f. The grid lines have given us a way of labelling every point in our empty, infinite sheet of paper. We can now choose a point we like, call it the origin, find its location (x_0, y_0) on some grid, and given another location (x_1, y_1) on the same grid, compute a real number using x_0, y_0, x_1, and y_1.

In wishing to describe our function f we have been forced to make a choice – a choice of grid. We have had to artificially associate to each point $p \in P$ a pair of real numbers, and then perform the computation in terms of those real numbers, despite the fact that P is *purely geometric*, and has no such data associated to it. What we end up with is not a function $f : P \to \mathbb{R}$, as we wanted, but some function $f_G : \mathbb{R}^2 \to \mathbb{R}$, dependent on our choice of grid G. Our choice of grid can be thought of as a choice of two maps $x, y : P \to \mathbb{R}$ (or alternatively, $(x, y) : P \to \mathbb{R}^2$) which associate to each $p \in P$ an "x-coordinate" $x(p)$ and a "y-coordinate" $y(p)$.

In manifold theory, our "choice of grid" is called a "choice of chart". A collection of charts is an *atlas*, in analogy with a book of maps (e.g. of the Earth). When one wishes to describe a function f on a manifold and perform some computation, one is often forced to pick a chart. For instance, if we choose a Cartesian chart (a Cartesian grid), in which the point $(0, 0)$ labels our desired origin, we find that

$$f(x, y) = x^2 + y^2.$$

But what is this expression? The entities x and y are now *functions* $x, y : P \to \mathbb{R}$, so the expression on the right-hand-side is to be interpreted as *point-wise multiplication and addition of functions* rather than addition and multiplication of real numbers. The left-hand-side, therefore, is also a function $f(x, y) : P \to \mathbb{R}$, thought of as "$f$ expressed in the chart (x, y)". Note that in this interpretation, we are not *evaluating* f "at" (x, y), but merely emphasizing the chart we've used to define it. In particular, we have $f = f(x, y)$. If we ask for $f(x, y)(p)$ (f evaluated at p), the result is $x(p)^2 + y(p)^2 \in \mathbb{R}$. Since x and y also lets us label points $p \in P$ via the map $p \mapsto (x(p), y(p))$, we can also ask for something like "$f(3, 4)$", by which we really mean something like[1]

$$\begin{aligned} f(x, y)((x, y)^{-1}(3, 4)) &= x((x, y)^{-1}(3, 4))^2 + y((x, y)^{-1}(3, 4))^2 \\ &= 3^2 + 4^2 \\ &= 25 \end{aligned}.$$

In this formalism, we can again ask for a partial derivative $\frac{\partial f}{\partial x}$. Now "$x$" is an object in its own right, and $\frac{\partial f}{\partial x}$ can be justifiably thought of as some combination of the two. The expression $\frac{\partial f}{\partial x} = 2x$ is also a valid one, defining a function $\frac{\partial f}{\partial x} : P \to \mathbb{R}$. Even the naive calculus books, with their questions of the form $\frac{\partial}{\partial x}(x^2 + y^2)$, are vindicated, as $\frac{\partial}{\partial x}$ is now an operator mapping functions $(P \to \mathbb{R})$ to functions $(P \to \mathbb{R})$, and $x^2 + y^2$ is such a function.

But what of $\frac{\partial}{\partial r}$? In the physicist's viewpoint, "r" was "defined" by the equation $r = \sqrt{x^2 + y^2}$. When x and y were free variables, such an equation had no meaning, but now it serves to define a function $r : P \to \mathbb{R}$. Likewise, we

[1]Note that we are assuming that (x, y), when thought of as a function $P \to \mathbb{R}^2$, is invertible.

can define $\theta = \arctan(\frac{y}{x})$, again as a function on P. The operator $\frac{\partial}{\partial r}$ can now be defined via

$$\frac{\partial f}{\partial r} = (\partial_1(f \circ (r,\theta)^{-1})) \circ (r,\theta).$$

What does this say? Exactly what we want! The expression $f \circ (r,\theta)^{-1}$ is an instruction to express our f *in terms of* the co-ordinate chart (r,θ). Our notation ∂_1 now finds a use in telling us to take the partial derivative of $f \circ (r,\theta)^{-1}$ (now *genuinely* a function $\mathbb{R}^2 \to \mathbb{R}$) with respect to the first argument, and afterwards we pre-compose it with (r,θ) to arrive at a map $\frac{\partial f}{\partial r} : P \to \mathbb{R}$. The definition mirrors our original intent: we now have a formal way of saying express the function "in terms of r and θ" and take the derivative.

But, how to compute $f \circ (r,\theta)^{-1}$? Up until now we have avoided even defining f directly, with good reason. P, after all, is supposed to be an infinite sheet of paper; isn't this a problem?

The solution comes in the form of a clever trick. We can observe that

$$f \circ (r,\theta)^{-1} = (f \circ (x,y)^{-1}) \circ ((x,y) \circ (r,\theta)^{-1}) \ ,$$

by the associativity of function composition. The right-hand-side is now the composition of two functions. The first is $f \circ (x,y)^{-1}$, which is merely the function $\mathbb{R}^2 \to \mathbb{R}$ we evaluated earlier at $(3,4) \in \mathbb{R}^2$. The second is a *chart transition function*, $(x,y) \circ (r,\theta)^{-1}$, which can be computed directly from the relations $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$. To do this, we can rearrange these expressions to arrive at the familiar $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Now suppose we have a point (r_0,θ_0) in the image of (r,θ). We can then consider a point $p = (r,\theta)^{-1}(r_0,\theta_0) \in P$. Using our new relations, we now evaluate that

$$\begin{aligned} x_0 &:= x(p) = r(p)\cos(\theta(p)) = r_0\cos(\theta_0) \\ y_0 &:= y(p) = r(p)\sin(\theta(p)) = r_0\sin(\theta_0), \end{aligned}$$

and so we see that $(x,y) \circ (r,\theta)^{-1}$ is the function mapping $(r_0,\theta_0) \mapsto (r_0\cos(\theta_0), r_0\sin(\theta_0))$. Applying $f \circ (x,y)^{-1}$ (the same sort of evaluation we did earlier) we now get a value of

$$x_0^2 + y_0^2 = r_0^2(\cos(\theta_0)^2 + \sin(\theta_0)^2) = r_0^2,$$

and applying ∂_1 to the resulting composition (now mapping $(r_0,\theta_0) \mapsto r_0^2$) we get a function mapping $(r_0,\theta_0) \mapsto 2r_0$ as we expect. The final pre-composition with (r,θ) merely serves to ensure that $\frac{\partial f}{\partial r} : P \to \mathbb{R}$ has the right domain.

Consider the miracle that has occurred here! We have managed to differentiate a function f whose *domain is an imaginary, infinite sheet of paper*. We have not even given a precise mathematical meaning to the object P, yet we can work, without issue, with the function and its partial derivatives simply by considering such things *relative to particular charts*, and understanding the transition maps $(x,y) \circ (r,\theta)^{-1}$ and $(r,\theta) \circ (x,y)^{-1}$ between them. Physicists use this sort of reasoning (although at a much more intuitive level) to discuss the manifold known as *spacetime* in General Relativity. Being an object in the "real world", spacetime does not have numeric labels at each point like $\mathbb{R}^n$ does, but one can nevertheless have functions on it, differentiate and integrate them, and use the whole arsenal of ordinary calculus by choosing grid lines for spacetime (choosing charts) and understanding their transitions.

But is this still mathematics? Are we not told, that in mathematics, everything is a set? If so, then what is P?

In practice, when manifold theory is taught, P is simply taken to be $\mathbb{R}^2$. The above analysis is then considerably simpler; the map $(x, y) : P \to \mathbb{R}^2$ is now simply $\mathrm{id}_{\mathbb{R}^2}$, the identity map on $\mathbb{R}^2$, and expressions like $f \circ (r, \theta)^{-1}$ can be computed directly. Somewhat awkwardly, if we wish to take our original definition

$$f(x, y) = x^2 + y^2 \qquad\qquad \forall (x, y) \in \mathbb{R}^2,$$

and "upgrade it" to one at the manifold level, the way to do it is to include a rather meaningless composition with $\mathrm{id}_{\mathbb{R}}^{-1}$ (i.e., what we called $(x, y)^{-1}$). The end result looks kind of silly, and in practice we may as well not distinguish between the two copies of $\mathbb{R}^2$ on either side of the function $\mathrm{id}_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$, the effect of which is to demote x and y back to their status as variables, and abuse notation by writing $\frac{\partial f}{\partial x}$. Moreover, trying to justify the notation $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial r}$ has taken a rather strenuous effort, much more than the effort it took rewrite what we meant in terms of "∂_1" and "∂_r".

Is this, then, the end? Is calculus forever doomed to notational inconsistencies?

> **Socrates**: *Very good; and now tell me what is the power which discerns, not only in sensible objects, but in all things, universal notions, such as those which are called being and not-being, and those others about which we were just asking – what organs will you assign for the perception of these notions?*
>
> **Theaetetus**: *You are thinking of being and not being, likeness and unlikeness, sameness and difference, and also of unity and other numbers which are applied to objects of sense; and you mean to ask, through what bodily organ the soul perceives odd and even numbers and other arithmetical conceptions.*
>
> **Socrates**: *You follow me excellently, Theaetetus; that is precisely what I am asking.*
>
> **Theaetetus**: *Indeed, Socrates, I cannot answer; my only notion is, that these, unlike objects of sense, have no separate organ, but that the mind, by a power of her own, contemplates the universals in all things.*
>
> — *Plato's Theaetetus*

Consider a (seemingly) unrelated question: what are the natural numbers $\mathbb{N}$? The usual set theoretic construction is to take $0 = \varnothing$, and $n = \{\varnothing, n-1\}$ for all $n \geq 1$. For instance, we then have that

$$3 = \{ \varnothing, \{ \varnothing \}, \{ \varnothing, \{ \varnothing \} \} \} .$$

Perhaps you think this is fine. What, then, are the integers $\mathbb{Z}$? The standard construction is then to "build" the integers out of pairs of natural numbers (a, b) where two such pairs (a, b) and (a', b') are equivalent if and only if $a - b = a' - b'$ (one can write this equation additively if one wishes to avoid subtraction). The resulting structure is a set of equivalence classes of elements of $\mathbb{N}^2$, on which one can define addition and multiplication and show they obey the usual rules.

But is it then true that $\mathbb{N} \subset \mathbb{Z}$? No, because the elements of $\mathbb{N}$ are certainly not equivalence classes of elements of $\mathbb{N}^2$. Neither is it true that $3 \in \mathbb{Z}$, at least not the 3 we defined in the proceeding paragraph. Rather there is now a "new 3" (the one in $\mathbb{Z}$), and a "new $\mathbb{N}$" (the one contained in $\mathbb{Z}$). But even after redefining $\mathbb{N}$ and 3 appropriately, the problem persists. We then find upon constructing $\mathbb{Q}$, again in the usual fashion, that $\mathbb{Z} \not\subset \mathbb{Q}$. Do we make yet another redefinition? Where do we stop, the complex numbers? What if we suddenly fancy quaternions?

The Ancient Greeks used yet a different variant of the natural numbers. In *Euclid's Elements*, numbers are not the abstract quantities we think of them as today, but rather were modelled as *line segments* whose lengths were multiples of a given "unit" segment. Euclid's proof that there are infinitely many prime numbers, for instance, was not an argument about sets but rather an argument about *geometry*, where the natural numbers were to be thought of as geometric objects. Nevertheless, mathematicians correctly attribute to Euclid a proof about $\mathbb{N}$, despite no such concept as *the set of natural numbers* existing in Euclid's day. We recognize that the formalisms we adopt to express mathematical ideas are temporary fixtures, only representing the mathematical objects in our thoughts from moment to moment, and change accordingly when we need them to. There is no one correct formalism for $\mathbb{N}$, but many different ones, each with its own utility, pitfalls, and which needs to be cast aside when appropriate.

And so it is with $f(x, y) = x^2 + y^2$. When we need it to be, f is a function $\mathbb{R}^2 \to \mathbb{R}$, or a function $P \to \mathbb{R}$. When we need them to be, x and y are free variables, elements of $\mathbb{R}$, or co-ordinate maps. And when we need it to, $\frac{\partial}{\partial r}$ differentiates functions with respect to the first co-ordinate, or perhaps asks us to make a substitution first, or maybe is just really a complicated combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ which should really be denoted ∂_1 and ∂_2 anyway, and maybe we should write ∂_r while we're at it.

Don't let the limitations of the formalism bother you, there is none good enough to express what goes on in your head.