Quantum-resistant cryptography from supersingular elliptic curves

David Urbanik

July 12, 2016

Symmetric Ciphers and Shared Secrets

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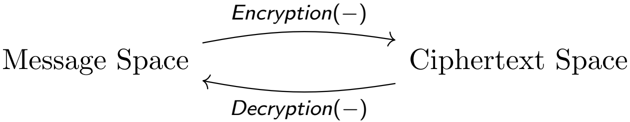

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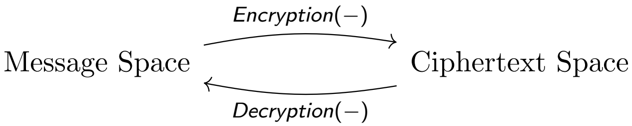

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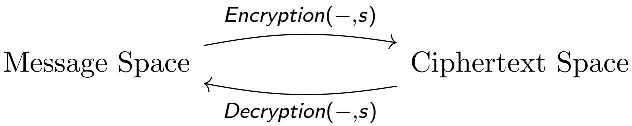

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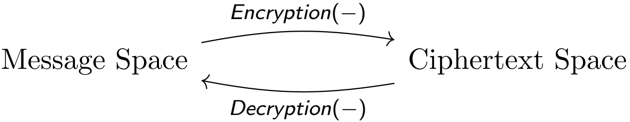

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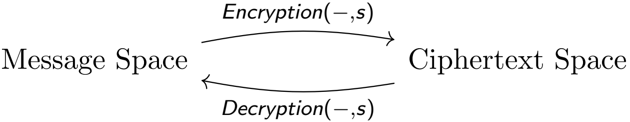

To do encryption, two parties need to agree on s. In 1976, Diffie and Hellman showed it can be done over a public channel.

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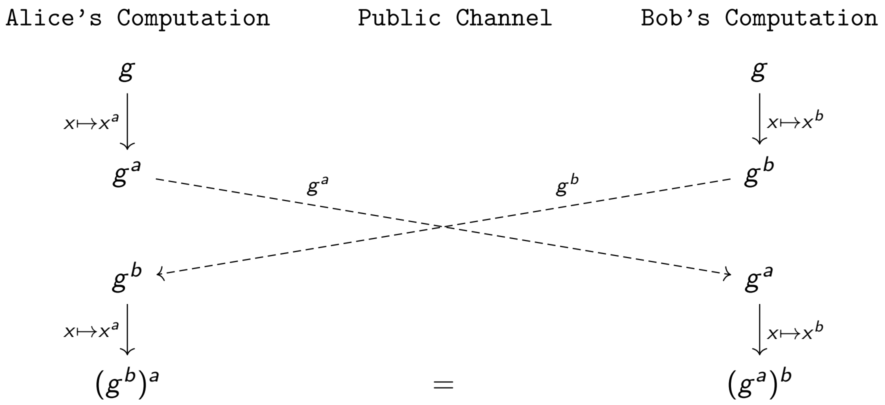

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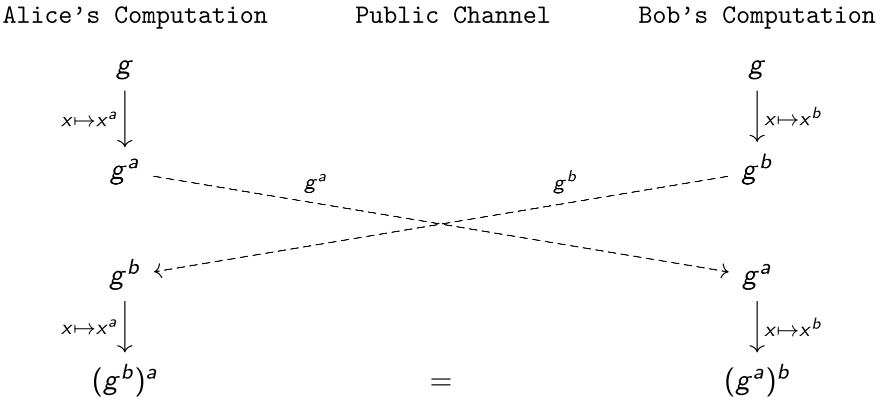

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- Quantum computers can factor integers (breaks RSA), and compute discrete logarithms (breaks all forms of Diffie-Hellman).
- Both are special cases of the Abelian Hidden subgroup problem:

Abelian Hidden Subgroup Problem

Input: Abelian group G, a set X, $H \leq G$, and $f : G \to X$ where $f(g_1) = f(g_2)$ iff $g_1 H = g_2 H$.
Output: A generating set for H.

The Diffie-Hellman Protocol (again)

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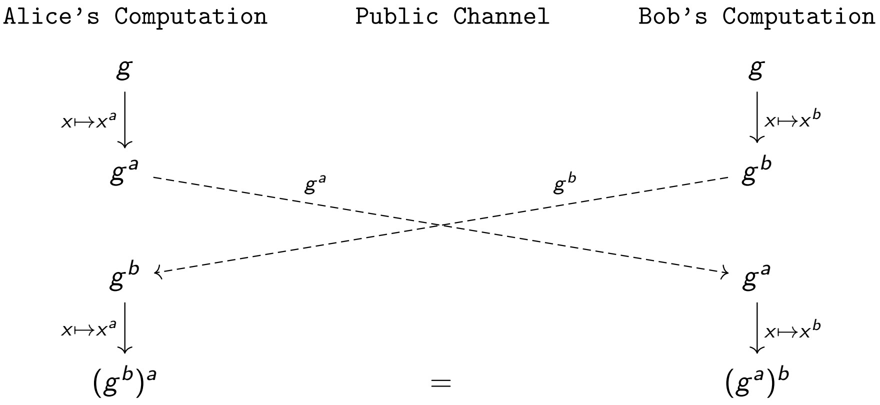

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The Supersingular Isogeny Diffie-Hellman Protocol

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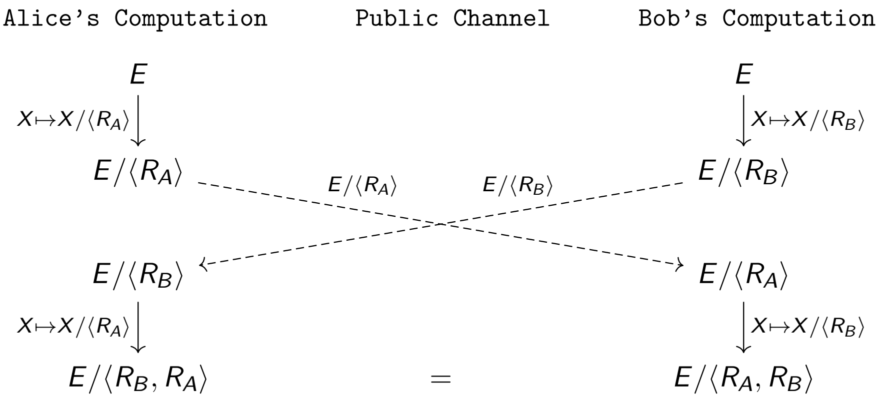

Hard problem: Given E, $E/\langle R_A \rangle$, $E/\langle R_B \rangle$ *, determine $E/\langle R_A, R_B \rangle$.

* Some extra information is also available.

Elliptic Curves

A set of solutions $\{(x, y)\}$ over a field $\mathbb{k}$ to an equation of the form

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After a change of coordinates:

- **Weierstrass Form**

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Elliptic curves also need to be non-singular, i.e. there is a unique tangent line at every point.

Elliptic Curve Examples

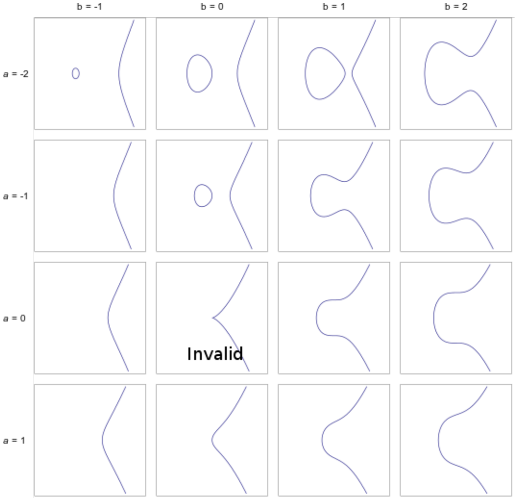

Figure: $y^2 = x^3 + ax + b, \qquad a \in \{-2, -1, 0, 1\}$ and $b \in \{-1, 0, 1, 2\}$.

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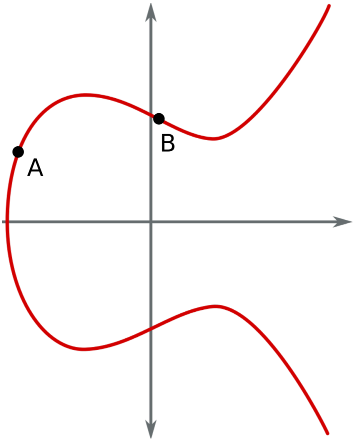

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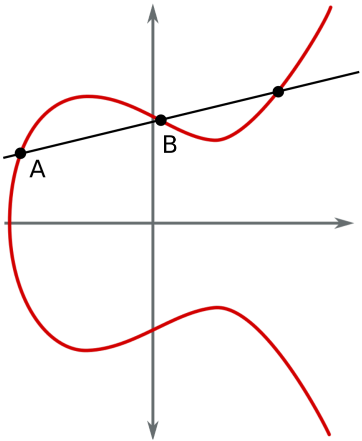

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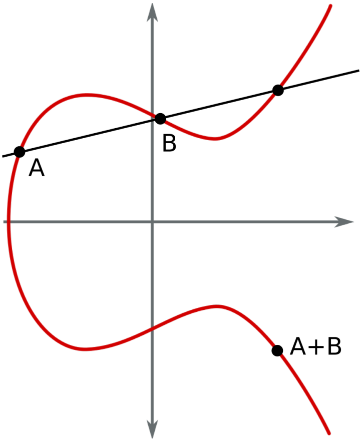

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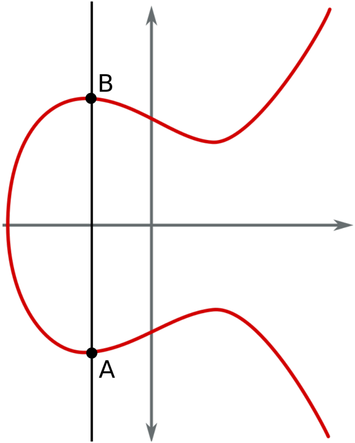

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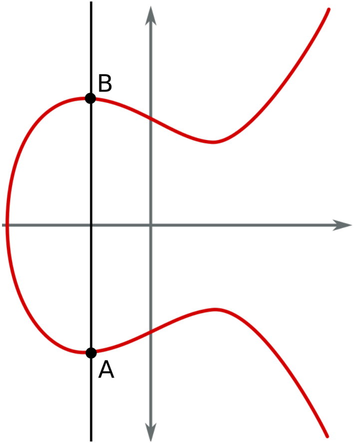

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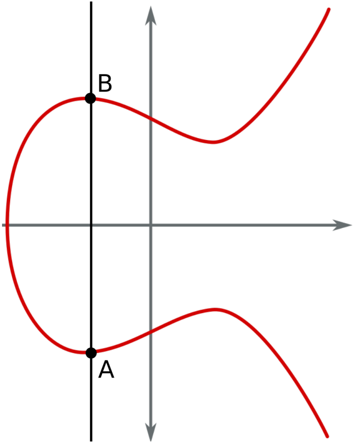

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This way we get an identity element, and also inverses.

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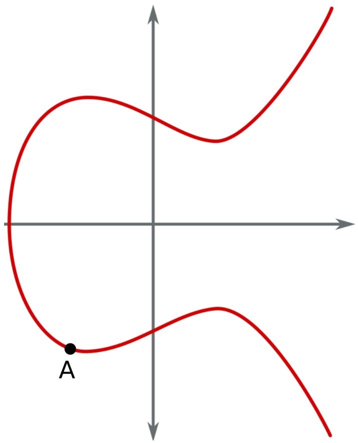

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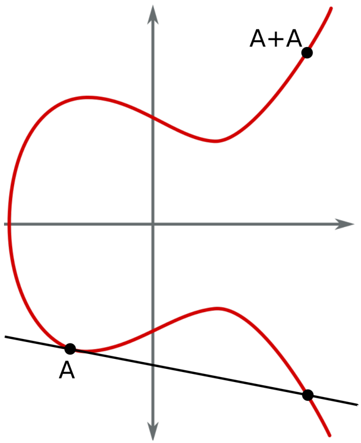

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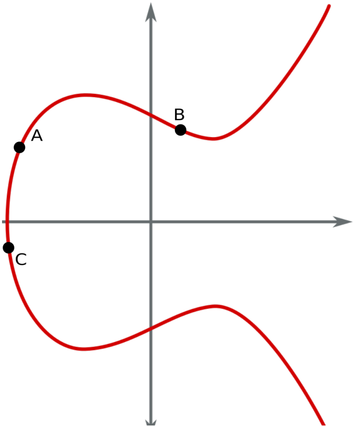

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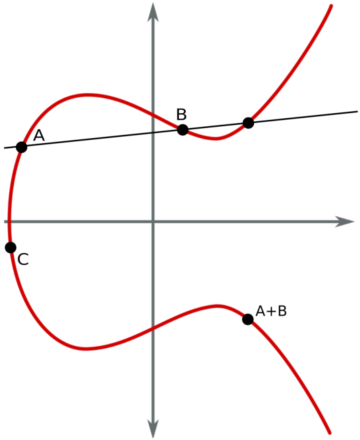

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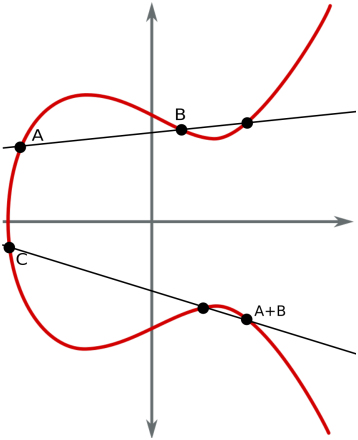

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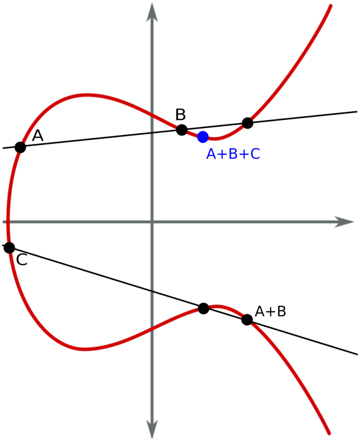

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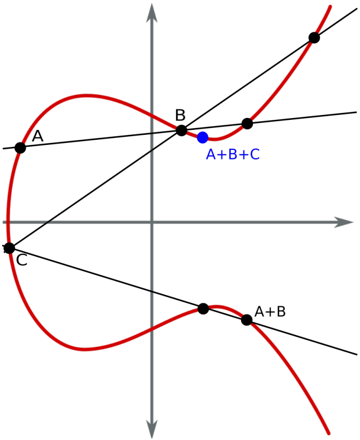

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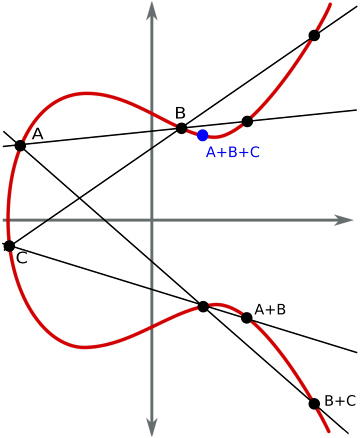

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- If we write E as a cubic homogeneous polynomial $f(x, y, z)$ in the projective plane, then E is supersingular iff the coefficient of $(xyz)^{p-1}$ in $f(x, y, z)^{p-1}$ is zero.

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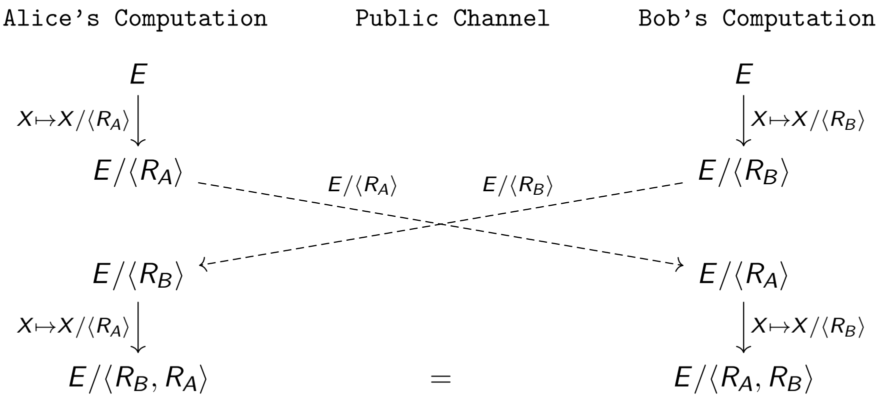

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- Bob then sends $\phi_B(P_A)$ and $\phi_B(Q_A)$ to Alice, and then Alice can compute

$$\langle m_A \phi_B(P_A) + n_A \phi_B(Q_A) \rangle = \langle \phi_B(m_A P_A + n_A Q_A) \rangle = \langle \phi_B(R_A) \rangle \quad .$$

My Work

- The current leading implementation of SIDH was developed by researchers at Microsoft and released in April of 2016. I've been optimizing the finite field arithmetic used for 64-bit ARM architectures. Because the original algorithm used for finite field operations on this platform was very generic, using hand-coded 64-bit ARM assembly I was able to improve the performance by about a factor of 10.
- Of all the quantum-resistant key-exchange protocols, SIDH has by far the smallest key sizes, which can be made smaller with compression. I am currently working on implementing key compression and decompression algorithms which can make the key sizes of SIDH comparable with those of existing quantum-vulnerable algorithms.