Preface

The present book collects most of the courses and seminars delivered at the meeting entitled " Frontiers in Number Theory, Physics and Geometry", which took place at the Centre de Physique des Houches in the french Alps March 9-21, 2003. It is divided into two volumes. Volume I contains the contributions on three broad topics: Random matrices, Zeta functions and Dynamical systems. The present volume contains sixteen contributions on three themes: Conformal field theories for strings and branes, Discrete groups and automorphic forms and finally, Hopf algebras and renormalization.

The relation between Mathematics and Physics has a long history. Let us mention only ordinary differential equations and mechanics, partial differential equations in solid and fluid mechanics or electrodynamics, group theory is essential in crystallography, elasticity or quantum mechanics ...

The role of number theory and of more abstract parts of mathematics such as topological, differential and algebraic geometry in physics has become prominent more recently. Diverse instances of this trend appear in the works of such scientists as V. Arnold, M. Atiyah, M. Berry, F. Dyson, L. Faddeev, D. Hejhal, C. Itzykson, V. Kac, Y. Manin, J. Moser, W. Nahm, A. Polyakov, D. Ruelle, A. Selberg, C. Siegel, S. Smale, E. Witten and many others.

In 1989 a first meeting took place at the Centre de Physique des Houches. The triggering idea was due at that time to the late Claude Itzykson (1938-1995). The meeting gathered physicists and mathematicians, and was the occasion of long and passionate discussions.

The seminars were published in a book entitled "Number Theory and Physics", J.-M. Luck, P. Moussa, and M. Waldschmidt editors, Springer Proceedings in Physics, Vol. 47, 1990. The lectures were published as a second book entitled “From Number Theory to Physics”, with C. Itzykson joining the editorial team, Springer (2nd edition 1995).

Ten years later the evolution of the interface between theoretical physics and mathematics prompted M. Waldschmidt, P. Cartier and B. Julia to renew the experience. However the emphasis was somewhat shifted to include in particular selected chapters at the interface of physics and geometry, random matrices or various zeta- and L- functions. Once the project of the new meeting entitled “ Frontiers in Number Theory, Physics and Geometry” received support from the European Union this “High Level Scientific Conference” was organized in Les Houches.

The Scientific Committee for the meeting “Frontiers in Number Theory, Physics and Geometry”, was composed of the following scientists: Frits Beukers, Jean-Benoît Bost, Pierre Cartier, Predrag Cvitanovic, Michel Duflo, Giovanni Gallavotti, Patricio Leboeuf, Werner Nahm, Ivan Todorov, Claire Voisin, Michel Waldschmidt, Jean-Christophe Yoccoz, and Jean-Bernard Zuber.

The Organizing Committee included:
Bernard Julia (LPTENS, Paris scientific coordinator),
Pierre Moussa (SPhT CEA-Saclay), and
Pierre Vanhove (CERN and SPhT CEA-Saclay).

During two weeks, five lectures or seminars were given every day to about seventy-five participants. The topics belonged to three main domains:

1. Dynamical Systems, Number theory, and Random matrices,
with lectures by E. Bogomolny on Quantum and arithmetical chaos, J. Conrey on L-functions and random matrix theory, J.-C. Yoccoz on Interval exchange maps, and A. Zorich on Flat surfaces;

2. Polylogarithms and Perturbative Physics,
with lectures by P. Cartier on Polylogarithms and motivic aspects, W. Nahm on Physics and dilogarithms, and D. Zagier on Polylogarithms;

3. Symmetries and Non-perturbative Physics,
with lectures by A. Connes on Galoisian symmetries, zeta function and renormalization, R. Dijkgraaf on String duality and automorphic forms, P. Di Vecchia on Gauge theory and D-branes, E. Frenkel on Vertex algebras, algebraic curves and Langlands program, G. Moore on String theory and number theory, C. Soulé on Arithmetic groups.

In addition seminars were given by participants many of whom could have given full sets of lectures had time been available. They were: Z. Bern, A. Bondal, P. Candelas, J. Conway, P. Cvitanovic, H. Gangl, G. Gentile, D. Kreimer, J. Lagarias, M. Marcolli, J. Marklof, S. Marmi, J. McKay, B. Pioline, M. Pollicott, H. Then, E. Vasserot, A. Vershik, D. Voiculescu, A. Voros, S. Weinzierl, K. Wendland and A. Zabrodin.

We have chosen to reorganize the written contributions in two halves according to their subject. This naturally led to two different volumes. The present one is the second volume, let us now briefly describe its contents.

This volume is itself composed of three parts including each lectures and seminars covering one theme. In the first part, we present the contributions on the theme "Conformal Field Theories (CFT's) for Strings and Branes". They begin with two intertwined sets of lectures by Don Zagier and by Werner Nahm who have had a long personal interaction at the modular border between Mathematics and Physics.

The presentation by Don Zagier starts with a review of the properties of Euler's dilogarithm and of its associated real Bloch-Wigner function. These functions have generalizations to polylogarithms and to some real functions defined by Ramakrishnan respectively. Their importance in Hyperbolic 3-geometry, in Algebraic K_{2m-1}-theory (Bloch group) and their relation to values of Dedekind zeta functions (see volume I) at argument m are explained. On the other hand the modular group appears to be mysteriously related to the Bloch-Wigner function and its first Ramakrishnan generalization. The second chapter of these lectures introduces yet more variants, in particular the Rogers dilogarithm and the enhanced dilogarithm which appear in W. Nahm's lectures, the quantum dilogarithm as well as the multiple (poly)logarithms which depend on more than one argument. Their properties are reviewed, in particular functional equations, relations with modular forms (see also the next contribution), special values and again (higher) K-theory.

In his lectures on "CFT's and torsion elements of the Bloch group" Werner Nahm expresses the conformal dimensions of operators in the (discrete) series of rational two dimensional (2d) Conformal Field Theories as the imaginary part of the Rogers dilogarithm of torsion elements from algebraic K-theory of the complex number field. The lectures begin with a general introduction to conformally invariant quantum field theories or more precisely with a physicist's conceptual presentation of Vertex operator algebras. The "rational" theories form a rare subset in the moduli space of CFT's but one may consider perturbations thereof within the set of totally integrable quantum field theories. The following step is to present a bird's eye view of totally integrable two dimensional quantum field theories and to relate in simple cases the scattering matrix to Cartan matrices of finite dimensional Lie algebras, in particular integrality of the coefficients follows from Bose statistics and positivity from the assumed convergence of partition functions, there are natural extensions to arbitrary statistics.

Then Nahm conjectures and illustrates on many examples that the "modular" invariance of the chiral characters of a rational CFT admitting a totally integrable perturbation implies that all solutions to the integrability conditions (Bethe equations) define pure torsion elements in the (extended) Bloch group of the complex field. The perturbations that can be analyzed are defined by pairs of Cartan matrices of A D E or T type. In fact Nahm gives the general solution of the torsion equation for deformations of (A_m, A_n) type with arbitrary ranks. These conjectures were analyzed mathematically at the end of Zagier's lectures.

After this background comes the seminar by Predrag Cvitanovic on invariant theory and a magic triangle of Lie groups he discovered in his studies of perturbative quantum gauge theories. This structure has been discussed since by Deligne, Landsberg and Manivel... It is different from and does not seem related to similar magic triangles of dualities that contain also the magic square of Tits and Freudenthal in specific real forms and which appear in supergravity and superstring models.

The third series of lectures: "Gauge theories from D-branes", were delivered by Paolo Di Vecchia and written up with Antonella Liccardo. They provide an introduction to string models and the associated D(irichlet)-branes on which open strings may end and they explain the emergence of Yang-Mills gauge theories on these extended objects. They bridge the gap between 2d CFT's and physical models in higher dimensions. Perturbative string theories are particular conformal field theories on the string worldsheet. Most nonperturbative effects in string theory necessitate the inclusion of extended objects of arbitrary spatial dimension p: the p-branes and in particular the Dirichlet D_p branes. Branes allow the computation of the entropy of black holes and permit new dualities between gauge and gravitational theories. For instance the celebrated AdS/CFT duality relates a closed string theory on the product manifold $S^5 \times AdS_5$ to an open string theory ending on a D_3 brane. These lectures start from the worldsheet description of perturbative superstring theory with its BRST invariant (string creation) vertex operators and proceed to describe the "boundary state formalism" that describes the coupling of closed strings to D branes. Then the authors use the latter to compute the interaction between two D-branes, they discuss so-called BPS configurations whose interactions vanish and relate the low energy effective Born-Infeld interactions of massless strings to their couplings to D branes.

One seminar by Katrin Wendland concludes this part: "Superconformal field theories associated to very attractive quartics". The terminology "attractive" was introduced by Greg Moore (see his lectures below) for those Calabi-Yau two-folds whose Picard group is of maximal rank, very attractive is a further restriction on the transcendental lattice. This is a review on the geometrical realization of orbifold models on quartic surfaces and provides some motivation for reading the following chapters.

In the second part: "Discrete groups and automorphic forms", the theme is arithmetic groups and some of their applications. Christophe Soulé 's lectures "Introduction to arithmetic groups" set the stage in a more general context than was considered in the lectures by E. Bogomolny in volume I of this book. They begin with the classical reduction theory of linear groups of matrices with integral coefficients and the normal parameterization of quadratic forms. Then follows the general (and intrinsic) theory of algebraic Lie groups over the rationals and of their arithmetic subgroups; the finite covolume property in the semi-simple case at real points is derived, it may be familiar in the physics of chaos. The second chapter deals with presentations and finite or torsion free and finite index subgroups. The third chapter deals with rigidity: the congruence subgroup property in rank higher than one, Kazhdan's property T about invariant vectors and results of Margulis in particular the proof of the Selberg conjecture that arithmeticity follows from finite covolume for most simple non-compact Lie groups. Automorphic forms are complex valued functions defined over symmetric domains and invariant under arithmetic groups, they arise abundantly in string theory.

Boris Pioline expanded his seminar with Andrew Waldron to give a physicists' introduction to "Automorphic forms and Theta series". It starts with the group theoretical and adelic expression of non holomorphic Eisenstein series like $E_{3/2}$ which has been extensively studied by M.B. Green and his collaborators and also theta series. From there one studies examples of applications of the orbit method and of parabolic induction. Among recent applications and beyond the discrete U-duality groups already considered in the previous lectures they discuss the minimal representation of SO(4,4) which arises also in string theory, the E_6 exceptional theta series expected to control the supermembrane interactions after compactification from 11 to 8 dimensions on a torus, new symmetries of chaotic cosmology and last but not least work in progress on the description of black hole degeneracies and entropy computations. M-theory is the name of the unifying, hypothetical and polymorphic theory that admits limits either in a flat classical background 11-dimensional spacetime with membranes as fundamental excitations, in 10 dimensions with strings and branes as building blocks etc...

Gregory Moore wrote up two of his seminars on "Strings and arithmetic" (the third one on the topological aspects of the M-theory 3-form still leads to active research and new developments). The first topics he covers is the Black hole's Farey tail, namely an illustration of the $AdS_3 \times S^3 \times K3$ duality with a two dimensional CFT on the boundary of three dimensional anti-de-Sitter space. One can compute the elliptic genus of that CFT as a Poincaré series that is interpretable on the AdS (i.e. gravity or string) side as a sum of particle states and black hole contributions. This can serve as a concrete introduction to many important ideas on Jacobi modular forms, Rademacher expansion and quantum corrections to the entropy of black holes.

The second chapter of Moore's lectures deals with the so called attractor mechanism of supergravity. After compactification on a Calabi-Yau 3-fold X one knows that its complex structure moduli flow to a fixed point if one approaches the horizon of a black hole solution. This attractor depends on the charges of the black hole which reach there a particular Hodge decomposition. In the special case of $X = K3 \times T^2$ one obtains the notion of attractive $K3$ already mentioned. The main point here is that the attractors turn out to be arithmetic varieties defined over number fields, their periods are in fact valued in quadratic imaginary fields. Finally two more instances of the importance of attractive varieties are presented. Firstly the 12 dimensional so-called "F-theory" compactified on a K3 surface is argued to be dual (equivalent) to heterotic string theory compactified modulo a two-dimensional CFT also down to 8 dimensions. It is striking that this CFT is rational if and only the K3 surface is attractive. Secondly string theory compactification with fluxes turns out to be related to attractive Calabi-Yau 4-folds.

The next contribution is a seminar talk by Matilde Marcolli on chaotic (mixmaster model) cosmology in which she relates a geodesics on the modular curve for the congruence subgroup $\Gamma_0(2)$ to a succession of Kasner four dimensional spacetimes. The moduli space of such universes is highly singular and amenable to description by noncommutative geometry and $\mathbb{C}^*$ algebras.

John McKay and Abdellah Sebbar introduce the concept and six possible applications of "Replicable functions". These are generalizations of the elliptic modular j function that transform under their Faber polynomials as generalized Hecke sums involving their "replicas". In any case they encompass also the monstrous moonshine functions and are deeply related to the Schwarzian derivative which appears in the central generator of the Virasoro algebra.

Finally part II ends with the lectures by Edward Frenkel "On the Langlands program and Conformal field theory". As summarized by the author himself they have two purposes, first of all they should present primarily to physicists the Langlands program and especially its "geometric" part but on the other hand they should show how two-dimensional Conformal Field Theories are relevant to the Langlands program. This is becoming an important activity in Physics with the recognition that mathematical (Langlands-)duality is deeply related to physical string theoretic S-duality in the recent works of A. Kapustin and E. Witten, following results on magnetic monopoles from the middle seventies and the powerful tool of topological twists of supersymmetric theories which help to connect N=4 super Yang-Mills theory in 4 dimensions to virtually everything else. The present work is actually about mirror symmetry (T-duality) of related 2d supersigma models.

Specifically the lectures begin with the original Langlands program and correspondences in the cases of number fields and of function fields. The Taniyama-Shimura-Weil (modular) conjecture (actually a theorem now) is discussed there. The geometric Langlands program is presented next in the abelian case first and then for an arbitrary reductive group G. The goal is to generalize T duality or Fourier-Mukai duality to the non abelian situation. Finally the conformal blocks are introduced for CFT's, some theories of affine Kac-Moody modules are introduced; at the negative critical level of the Kac-Moody central charge the induced conformal symmetry degenerates and these models lead to the Hecke eigensheaves expected from the geometric Langlands correspondence.

The third and last theme of this volume is "Hopf algebras and renormalization". It leads to promising results on renormalization of Quantum Field Theories that can be illustrated by concrete perturbative diagrammatic computations but it also leads to the much more abstract and conceptual idea of motives like a wonderful rainbow between the ground and the sky. In the first set of lectures Pierre Cartier reviews the historical emergence of Hopf algebras from topology and their structure theorems. He then proceeds to Hopf algebras defined from Lie groups or Lie algebras and the inverse structure theorems. He finally turns to combinatorics instances of Hopf algebras and some applications, (quasi)-symmetric functions, multiple zeta values and finally multiple polylogarithms. This long and pedagogical introduction could have continued into motives so we may be heading towards a third les Houches school in this series.

Then come the series of lectures by Alain Connes; they were written up in collaboration with Matilde Marcolli. The lectures contain the most up-to date research work by the authors, including a lot of original material as well as the basic material in this exciting subject. They have been divided into two parts. Chapter one appeared in the first volume and covered: "Quantum statistical mechanics of $\mathbb{Q}$-lattices" in dimensions 1 and 2. The important dilation operator (scaling operator) that determined the dynamics there reappears naturally as the renormalization group flow in their second chapter contained in this volume with the title: "Renormalization, the Riemann-Hilbert correspondence and motivic Galois theory". It starts with a detailed review of the results of Connes and Kreimer on perturbative renormalization in quantum field theory viewed as a Riemann-Hilbert problem and presents the Hopf algebra of Feynman graphs which corresponds by the Milnor-Moore theorem to a graded Lie algebra spanned by 1PI graphs. Singular cases lead to formal series and the convergence aspects are briefly discussed towards the end.

The whole program is reformulated using the language of categories, algebraic groups and differential Galois theory. Possible connections to mixed Tate motives are discussed. The equivariance under the renormalization group is reformulated in this language. Finally various tantalizing developments are proposed.

Dirk Kreimer discusses then the problem of "Factorization in quantum field theory: an exercise in Hopf algebras and local singularities". He actually treats a toy model of decorated rooted trees which captures the essence of the resolution of overlapping divergences. One learns first how the Hochschild cohomology of the Hopf algebra permits the renormalization program with "locality". Dyson-Schwinger equations are then defined irrespective of any action and should lead to a combinatorial factorization into primitives of the corresponding Hopf algebra.

Stefan Weinzierl in his seminar notes explains some properties of multiple polylogarithms and of their finite truncations (nested sums called Z-sums) that occur in Feynman loop integrals: "Algebraic algorithms in perturbative calculations" and their impact on searches for new physics. Emphasis is on analytical computability of some Feynman diagrams and on algebraic structures on Z-sums. They have a Hopf algebra structure as well as a conjugation and a convolution product, furthermore the multiple polylogarithms do have a second Hopf algebra structure of their own with a shuffle product.

Finally this collection ends with a pedagogical exposition by Herbert Gangl, Alexander B. Goncharov and Andrey Levin on “Multiple logarithms, algebraic cycles and trees”. This work has been extended to multiple polylogarithms and to the world of motives by the same authors. Here they relate the three topics of their title among themselves, the last two are associated to differential graded algebras of algebraic cycles and of decorated rooted trees whereas the first one arises as an integral on hybrid cycles as a generalization of the mixed Tate motives of Bloch and Kriz in the case of the (one-variable) (poly-)logarithms.

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