Scattering Amplitudes in Gauge Theories

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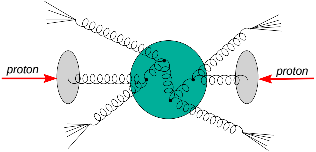
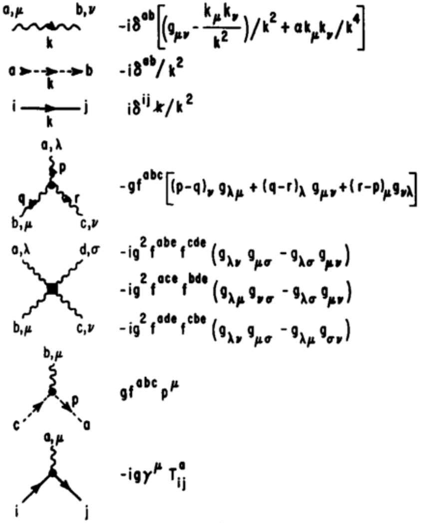
Simplicity of scattering amplitudes

Example: 2 gluons in, $n-2$ gluons out.



$$A(p_1^+, p_2^+; p_3^+, p_4^+, \ldots, p_n^+) = \frac{\langle \lambda_1\ \lambda_2 \rangle^4}{\langle \lambda_1\ \lambda_2 \rangle \langle \lambda_2\ \lambda_3 \rangle \ldots \langle \lambda_{n-1}\ \lambda_n \rangle \langle \lambda_n\ \lambda_1 \rangle}$$

Feynman rules in QCD



[graphic: Wolfram 1978]

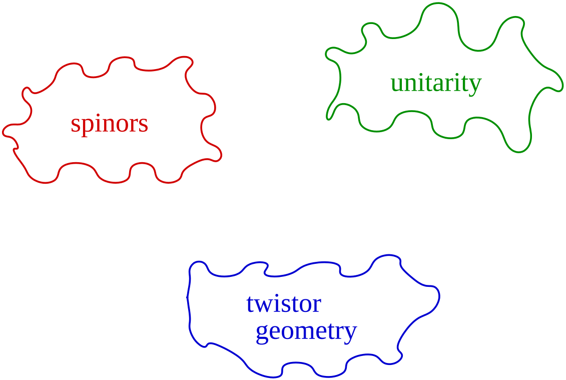
Too many Feynman diagrams

Already at tree level in pure gauge theory:

#gluons	Diagrams
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

Each diagram is a long mathematical expression.

Tools



We are discovering new perspectives.

Construct amplitude from global properties.

From momentum vectors to spinors

Change from Lorentz 4-vector to spinor indices with Pauli matrices:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}}\, p_{\mu} \qquad\qquad a, \dot{a} = 1, 2$$

For a null vector (massless particle):

$$0 = p^2 = \det(p_{a\dot{a}}) \implies p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Lorentz-invariant spinor products:

$$\begin{aligned} \langle \lambda\ \lambda' \rangle &\equiv \epsilon_{ab}\, \lambda^a\, \lambda'^b \\ [\tilde{\lambda}\ \tilde{\lambda}'] &\equiv \epsilon_{\dot{a}\dot{b}}\, \tilde{\lambda}^{\dot{a}}\, \tilde{\lambda}'^{\dot{b}} \end{aligned}$$

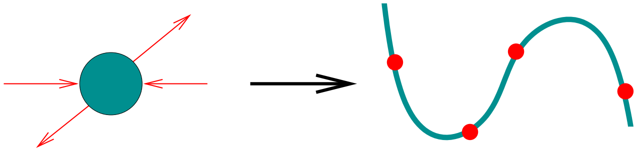
Spinors and Twistors

$$\begin{aligned}(p_1, p_2, p_3, p_4) &\rightarrow (\lambda^1, \lambda^2, \tilde{\lambda}^{\dot{1}}, \tilde{\lambda}^{\dot{2}}) \\ &\rightarrow (\lambda^1, \lambda^2, \mu^{\dot{1}}, \mu^{\dot{2}}),\end{aligned}$$

with

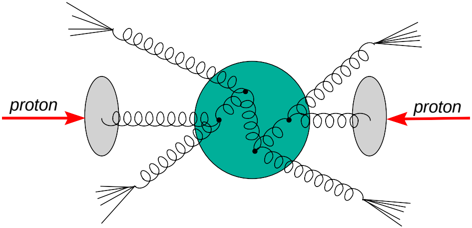
$$\mu^{\dot{a}} \equiv -i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}.$$

Breaks symmetry between λ and $\tilde{\lambda}$.



Simplicity of scattering amplitudes

$$A(p_1^+, p_2^+; p_3^+, p_4^+, \ldots, p_n^+) = \frac{\langle \lambda_1 \ \lambda_2 \rangle^4}{\langle \lambda_1 \ \lambda_2 \rangle \langle \lambda_2 \ \lambda_3 \rangle \ldots \langle \lambda_{n-1} \ \lambda_n \rangle \langle \lambda_n \ \lambda_1 \rangle}$$



- n is arbitrary
- Only half of the spinors are involved (λ but not $\tilde{\lambda}$)
- A line in twistor-space geometry

Aims

1. Exploit the simplicity. LHC applications.
2. Seek deeper structure.

One-loop amplitudes

2006: 6 gluons. Complexity of $2 \to 4$ scattering in QCD at one-loop order

[Ellis, Giele, Zanderighi; Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, RB, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Ita, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu]

Both numerical and analytic results.

Analytic techniques extend readily to larger numbers of particles.

Completed 2009: analytic results for $pp \to$ Higgs + 2 jets.

[Badger, Berger, Campbell, Del Duca, Dixon, Ellis, Giele, Glover, Mastrolia, Risager, Sofianatos, Williams, Zanderighi]

Seeking deeper structure

Two recent discoveries:

- Residue theorem $\longrightarrow$ Recursion Relations
- Branch cuts $\longrightarrow$ Loop Constructions

On-shell recursion relation for amplitudes.

Define the following function of a complex variable z:

$$A(z) = A(p_1, \ldots, p_{k-1}, p_k(z), p_{k+1}, \ldots, p_{n-1}, p_n(z))$$

where

$$p_k(z) = p_k - zq, \qquad p_n(z) = p_n + zq,$$

and q satisfies

$$p_k(z)^2 = 0, \quad p_n(z)^2 = 0.$$

The shift preserves momentum conservation while staying on shell.

On-shell recursion relation for amplitudes.

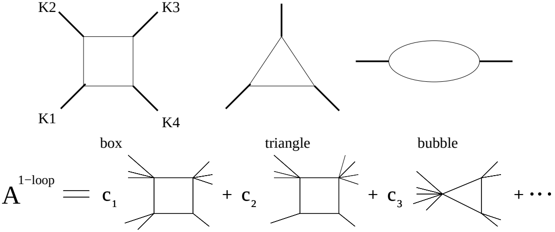
Residue theorem gives formula in terms of smaller, simpler amplitudes.

$$\begin{aligned} A(0) &= \sum_{k=1}^{n-3} A(p_n(z_k), p_1, \ldots, p_k, -P_{n,k}(z_k)) \\ &\quad \times \frac{1}{P_{n,k}^2} A(P_{n,k}(z_k), p_{k+1}, \ldots, p_{n-2}, p_{n-1}(z_k)) \end{aligned}$$

where z_k is the solution to $P_{n,k}(z_k)^2 = 0$.
New results obtained quickly and compactly.

One-Loop Amplitudes

Reduction to “master integrals”:

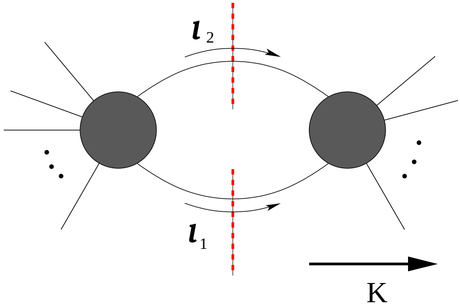


For amplitudes involving many particles, this is not yet enough simplification.

Unitarity Cuts: Loops from Trees

$$\Delta A^{1-\text{loop}} = \int d^4\ell \; A^{\text{tree}}_{\text{Left}} \;\times\; A^{\text{tree}}_{\text{Right}}$$

Cut conditions: $\ell_1^2 = 0, \qquad \ell_2^2 = 0.$

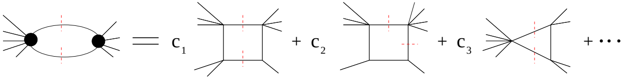


By unitarity, this is the discontinuity of the amplitude across a branch cut.

Amplitudes from unitarity cuts

4-dimensional cuts suffice to determine certain one-loop amplitudes! [Bern, Dixon, Dunbar, Kosower 1994]
Match cuts of amplitudes with cuts of master integrals from Passarino-Veltman reduction: essence of loop momentum integral is done once and for all.

$$\Delta A^{1-\mathrm{loop}} = \sum c_i \Delta I_i$$



Spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change loop momentum to spinor variables in unitarity cut integral.

$$\ell \to \lambda, \tilde{\lambda}$$

- Each term of integrand takes the form:

$$\frac{(K^2)^{n+1}}{\langle\lambda|K|\tilde{\lambda}]^{n+2}} \frac{\prod_{i=1}^{n+k}\langle\lambda|R_i|\tilde{\lambda}]}{\prod_{j=1}^{k}\langle\lambda|Q_j|\tilde{\lambda}]}$$

- Evaluate with residue theorem.
- Identify expressions with cuts of basis integrals and read off coefficients.
- We have given formulas for the resulting coefficients.

Close to four dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} \quad = \quad \frac{(4\pi)^{\epsilon}}{\Gamma(-\epsilon)} \int_0^1 du\ u^{-1-\epsilon} \int d^4\widetilde{\ell}.$$

where $\ell^2_{-2\epsilon} = \frac{K^2}{4}u$.

The integral over u will remain. The u-dependence is controlled:

$$\Delta A \quad = \quad \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell \;\; \delta(\ell^2)\, \delta(\sqrt{1-u}\ K^2 - 2K\cdot\ell)$$

Recognize and perform the 4-d integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov)

Incorporating Masses

Cut amplitude:

$$\int_0^1 du\; u^{-1-\epsilon} \int \langle \ell\; d\ell \rangle [\ell\; d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \ell | K | \ell]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j | \ell]}{\prod_{i=1}^{k} \langle \ell | Q_i | \ell]}$$

- For scalar particles, the formalism/formulas for integral coefficients will look the same. [See also: Kilgore]
- Integral coefficients are polynomials in u.
- New element: tadpole and massless bubble integrals. Coefficients of tadpoles might be found formally by cutting an artificial extra propagator.
- Self-energy and mass renormalization contributions may require gauge fixing. [Ellis, Giele, Kunszt, Melnikov].

Currently...

Implementing analytic algorithm in Mathematica. Seeking optimal order of operations.

ZZggg with massless fermion loop, comparison with Feynman-diagram derivation. Starting with 4d cuts.

Investigating issues of masses.

Closing comments

- Conceptual breakthroughs in quantum field theory have come from tackling specific, difficult calculations.
- Searches for new high-energy physics at LHC demand such calculations.
- Current ideas lead to both formal and LHC-driven advances.