The Ricci Curvature as organizing principle

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Outline of the talk

- Since its introduction in 1904, the role of the Ricci curvature has never stopped to broaden:
    1. the initial geometric motivation;
    2. its central role in General Relativity in the Einstein equations, initially and later on;
    3. its role in controlling the geometry via the volume;
    4. Gromov's precompactness theorem;
    5. the concept of Ricci flow;
    6. the generalisation of the concept in the context of Probability Theory;
    7. the connection between convexity of entropy functionals and lower bounds on the Ricci curvature, in particular in relation to Optimal Transport Theory.
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1. Definition of the Ricci Curvature

Towards the Definition

The context is that of the geometry of a variable metric g,

- initialized by the *Theorema Egregium* of Carl Friendrich GAUSS that legitimates the concept of an *intrinsically curved* space,
- developed by Bernhard RIEMANN, who introduced the Riemann curvature tensor R^g,
- a 4-tensor that measures the deviation of a space from being flat.

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Definition of the Ricci Curvature

- Grigorio RICCI-CURBASTRO understands the absolute derivative (1888) of a Riemannian metric g.
- In *"Direzioni et invarianti principali in una varieta qualunque"* (Atti del Real Inst. Veneto) published in 1904, he introduces the *Ricci curvature* Ric_g;

$$r^g(X,Y) = \mathrm{Trace}(Z \mapsto R^g_{Z,X}Y) \ .$$

- As the title says, the purpose is to introduce privileged directions at each point in connection with the case of a hypersurface

$$r^g = (\mathrm{Trace}_g\, II)\, II - II \circ_g II \ ,$$

where II denotes the second fundamental form of the hypersurface.

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2, General Relativity

General Relativity

- Albert EINSTEIN and Marcel GROSSMANN use the Ricci curvature in 1913 in their first attempt to define a *theory of General Relativity* based on a Lorentzian metric.
- In 1915, Albert EINSTEIN and David HILBERT obtain the correct field equations through a variational principle

  $$g \mapsto \int_{\mathrm{space-time}} s^g \, v_g \ ,$$

  where $s^g = \mathrm{Trace}_g r^g$ is the *scalar curvature* of g.
- The Einstein equations are

  $$r^g - \frac{1}{2} s^g \, g = T \ ,$$

  where T is the stress-energy tensor (in the vacuum $T = 0$).
- Several approaches to discuss the Einstein equations have been used, of particular interest here is the ADM approach.

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General Relativity (continued)

- In the ADM approach, first suggested by Yvonne BRUHAT, the 4-dimensional Lorentzian geometry is translated into that of a curve $t \mapsto (g_t, k_t, \phi_t)$ where g_t is the induced metric on a space hypersurface, k_t its second fundamental form and ϕ_t the lapse function.
- The Einstein equations take the form of *evolution equations*

$$\begin{cases} \frac{\partial g_t}{\partial t} & = -2\, \phi_t\, k_t \\ \frac{\partial k_t}{\partial t} & = -D^{g_t} d\phi_t + \phi_t\,(r_{g_t} + (c_{g_t}(k_t) k_t - 2\, k_t \cdot_{g_t} k_t)\ ; \end{cases}$$

- and *constraint equations*

$$\begin{cases} \delta_{g_t} k_t - d(c_{g_t}(k_t)) & = 0 \\ s_{g_t} - |k_t|^2_{g_t} + (c_{g_t}(k_t))^2 & = 0\ . \end{cases}$$

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3. Geometric Control by the Ricci Curvature

Volume Control by the Ricci Curvature

In the Riemannien context, it is not clear at all that the Ricci curvature controls locally the full curvature. The only geometric information that one can deduce locally from the Ricci curvature is a control of the volume of balls.

Bishop-Gromov Inequality

Let $p \in M^n$, *and* $s \leq s'$. *If* $r^g \geq (n-1)\,k\,g$, *then*

$$\frac{vol(B_p^M(s))}{vol(B_p^M(s'))} \leq \frac{vol(B_{p_k}^{M_k}(s))}{vol(B_{p_k}^{M_k}(s'))}\ .$$

where $B_p^M(s)$ *is the ball around* p *in* M *of radius* s *and* M_k *the simply connected model space with constant curvature* k.

- There is one context where the Ricci curvature becomes the central object, namely Kählerian Geometry.

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Kählerian Geometry

- On a complex manifold M one considers triples (g, J, ω) (where $\omega(X, Y) = g(JX, Y)$) verifies the constraint $d\omega = 0$.
- The formula
$$\rho_\omega = -i\,\partial\bar{\partial}\log(\det g_{\alpha\bar{\beta}})$$
says that the Ricci form is the curvature of the so-called canonical line bundle $\Lambda^m_{\mathbb{C}} T^*M \longrightarrow M$.
- Kählerian Geometry has been dominated by the question of solving the equation
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connected to the Calabi conjecture, leading to the existence of Calabi-Yau metrics.

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4. Gromov Precompactness Theorem

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Precompactness Theorem

The subset $\mathcal{M}(D, L, k)$ of the set of metric spaces $\mathcal{M}$ consisting of compact Riemannian manifolds (M, g) of dimension D having diameter bounded by L and Ricci curvature bounded from below by k is precompact in the Gromov-Hausdorff topology.

- The closure of $\mathcal{M}(D, L, k)$ in $\mathcal{M}$ consists of metric spaces (X, d) of Hausdorff dimension at most D that are not necessarily manifolds (because of the possible occurence of collapsing phenomena).
- Limit points are rather special metric spaces, as shown in works of Jeff CHEEGER and Tobias COLDING.
- They point out that it is important to consider metric spaces equipped with measures and that the measure be controlled.

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5. The Ricci Flow

The Ricci Curvature as Vector Field in the Space of Metrics

- The setting is that of the infinite dimensional cone of Riemannian (or Lorentzian) metrics on a space (or on a space-time).
- This point of view is implicit in the Hilbert-Einstein approach as the Einstein equations express that the space-time metric is a critical point of a functional on the space of metrics.
- The Ricci curvature r^g can be viewed as a tangent vector at g.
- This suggested to view it as a candidate infinitesimal deformation of the metric in order to improve the metric.
- This was first used by Thierry AUBIN in 1970. I used it too in my thesis in 1974.

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QUESTION: *Does the local flow theorem hold for the vector fields* $r^g - k\, s^g\, g$ *on the space of metrics? What is the global behaviour on their integral curves if they exist.*

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- His main contributions are valid in dimension 3 where he showed that one can control the geometry (and the analysis) beyond a singularity.
- This was the basis for his solution of the Poincaré conjecture to the effect that *"any compact simply connected 3-manifold is differentiably a sphere"*.
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6. Connections to Probability Theory and Transport Theory

Bounds on the Ricci Curvature and Entropy

- A remarkable work initiated by Dominique BAKRY and Michel ÉMERY from a purely probabilistic point of view showed how to make sense of the Ricci curvature for some specific semi-groups of operators on probabilistic spaces.
- The key element that came from this is the principle relating the *k-convexity of the entropy functional $\mathcal{E}$*, a notion that makes sense on a geodesic space, to the fact that *the Ricci curvature is bounded from below by k*.
- Actually, it is in the context of Kählerian geometry that this link was considered for the first time by Toshiki MABUCHI through the notion of "K-energy" on the space of Kähler potentials.

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- Recall that, on a geodesic space $(\mathcal{M}, d)$, a function ϕ is *k-convex* for $k \in \mathbf{R}$ if, along each geodesic $\gamma : [0,1] \longrightarrow \mathcal{M}$,

$$\phi(\gamma_t) \leq (1-t)\,\phi(\gamma_0) + t\,\phi(\gamma_1) - \frac{1}{2} k\, t(1-t)\, d^2(\gamma_0, \gamma_1)\ .$$

- The geodesic space to be considered there is the space $\mathcal{P}(M)$ of probability measures on a metric space M endowed with the so-called Wasserstein metric.

Theorem (Karl-Theodor STURM, Max-Konstantin von RENESSE)

The entropy functional $\mathcal{E}$ defined on $\mathcal{P}(M)$ is k-convex if the Ricci curvature of (M, g) is bounded from below by k.

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- This was recently visited by many authors among which John LOTT, Peter TOPPING to replace the geometric assumption on the bounds of the Ricci curvature by estimates on the evolution of various quantities, most of the time of entropy type, along the Ricci flow (or sometimes the inverse Ricci flow when one knows that it does not blow up).

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The Ricci Curvature of Singular Spaces

- Thanks to the previous works mentioned, one can close the circle of ideas introduced by GROMOV mentioned earlier in the talk and prove that

Theorem

A metric space that is a limit of a family of Riemannian manifolds with diameter bounded by L and Ricci curvature bounded from below by k and volume controlled has also its Ricci curvature bounded from below by k.

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I thank you for your attention.

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