Exact Results for a Model in Non-Equilibrium Statistical Mechanics

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The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law :

$$P_{\mathrm{eq}}(\mathcal{C}) = \frac{\mathrm{e}^{-E(\mathcal{C})/kT}}{Z},$$

the Partition Function Z being related to the Thermodynamic Free Energy F :

$$F = -kT\, Log\, Z$$

This provides us with a well-defined prescription to analyze systems *at equilibrium* :

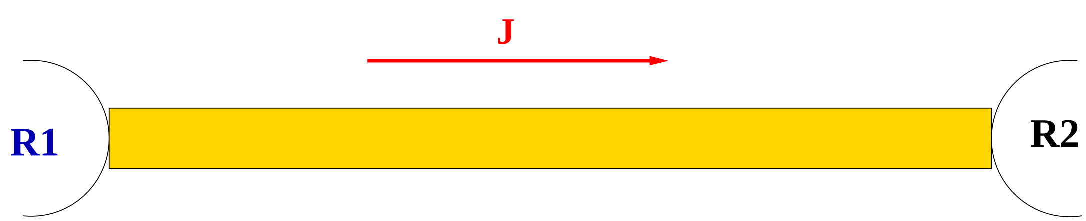
(i) Observables are mean values w.r.t. the canonical measure.

(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

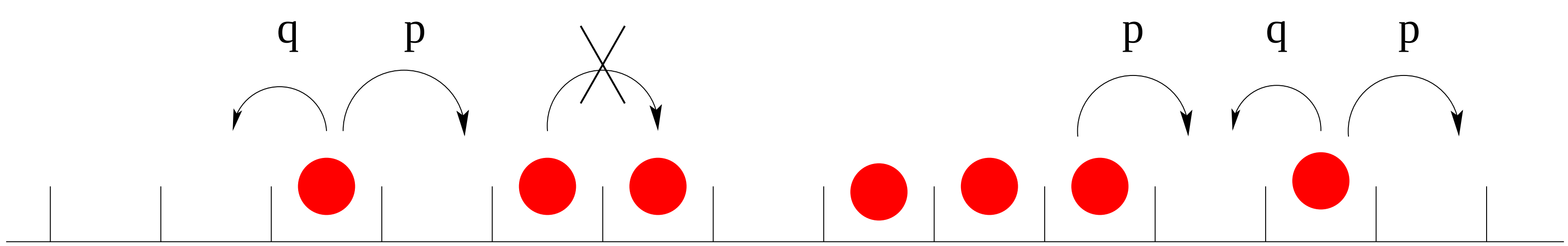
For systems *far from equilibrium* : No fundamental theory is yet available.

- What are the relevant macroscopic parameters ?
- Which functions describe the state of a system ?
- Do Universal Laws exist ? Can one define Universality Classes ?
- Can one postulate a general form for the microscopic measure ?
- What do the fluctuations look like ('non-gaussianity') ?

Example : Stationary driven systems in contact with reservoirs.



ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

EXCLUSION : Hard core-interaction ; at most 1 particle per site.

ASYMMETRIC : External driving ; breaks detailed-balance

PROCESS : Stochastic Markovian dynamics ; no Hamiltonian

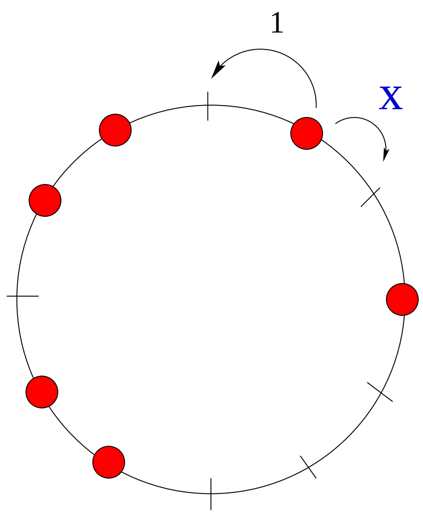
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.

APPLICATIONS

- Traffic flow.
- Sequence matching. Brownian motors.

Markov Equation for the ASEP



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

x asymmetry parameter

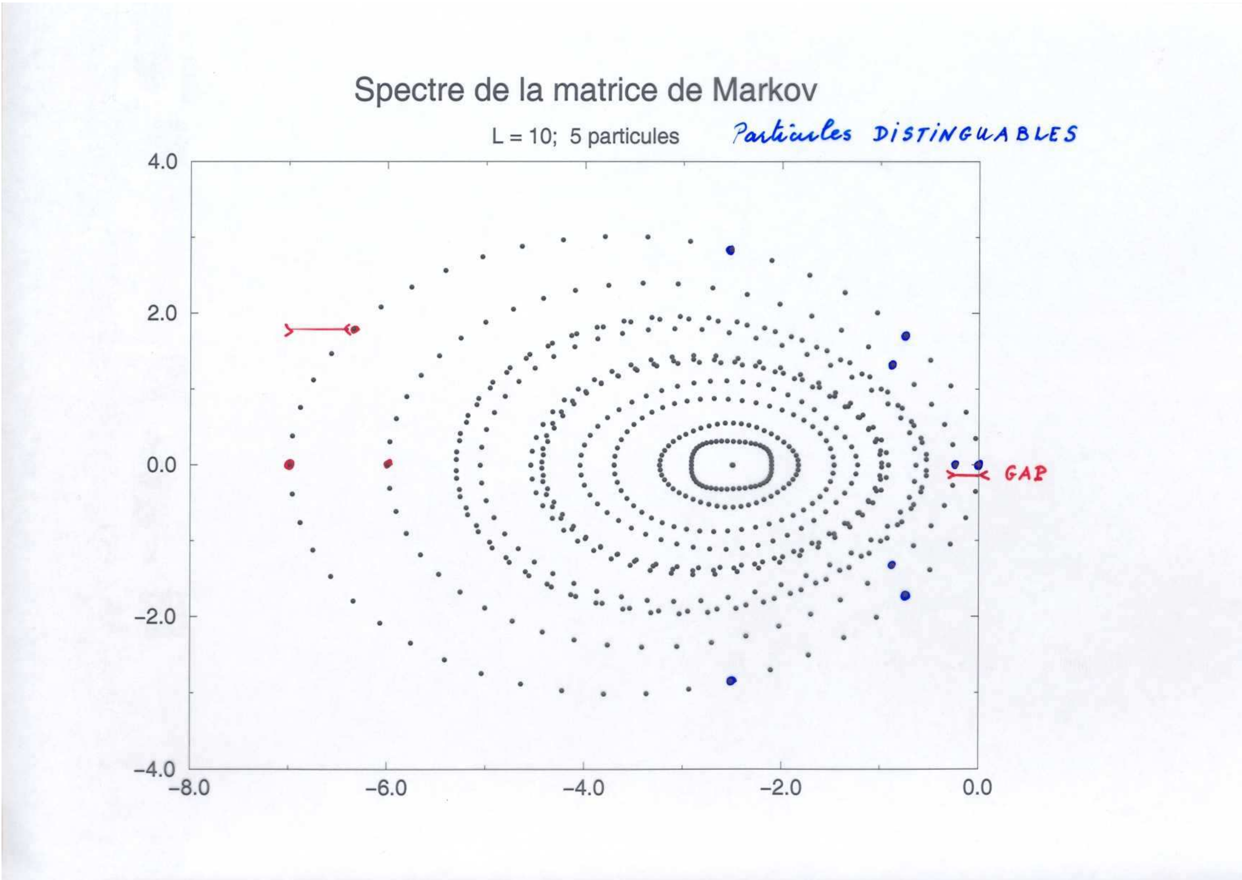
$P_t(x_1, \ldots, x_N)$: Prob. of config. $1 \leq x_1 < \ldots < x_N \leq L$ at time t.

$$\frac{\mathrm{dP_t}}{\mathrm{d}t} = \sum_i \left[P_t(x_1, \ldots, x_i - 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N) \right] = M P_t$$

$(x = 0)$ The sum is restricted to $x_{i-1} < x_i - 1$.

Complex Eigenvalues $M\psi = E\psi$ with $\Re(E) \leq 0$ (Perron-Frobenius)

- Ground State $E = 0$ corresponds to the stationary state.
- Excited States $\rightarrow$ relaxation times.



ASEP is integrable by Bethe Ansatz

Eigenvector ψ of M written as a linear combination of plane waves, with pseudo-momenta given by $z_1, \ldots z_N$:

$$\psi(x_1, \ldots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The Bethe Equations provide us with the quantification of the z_i's :

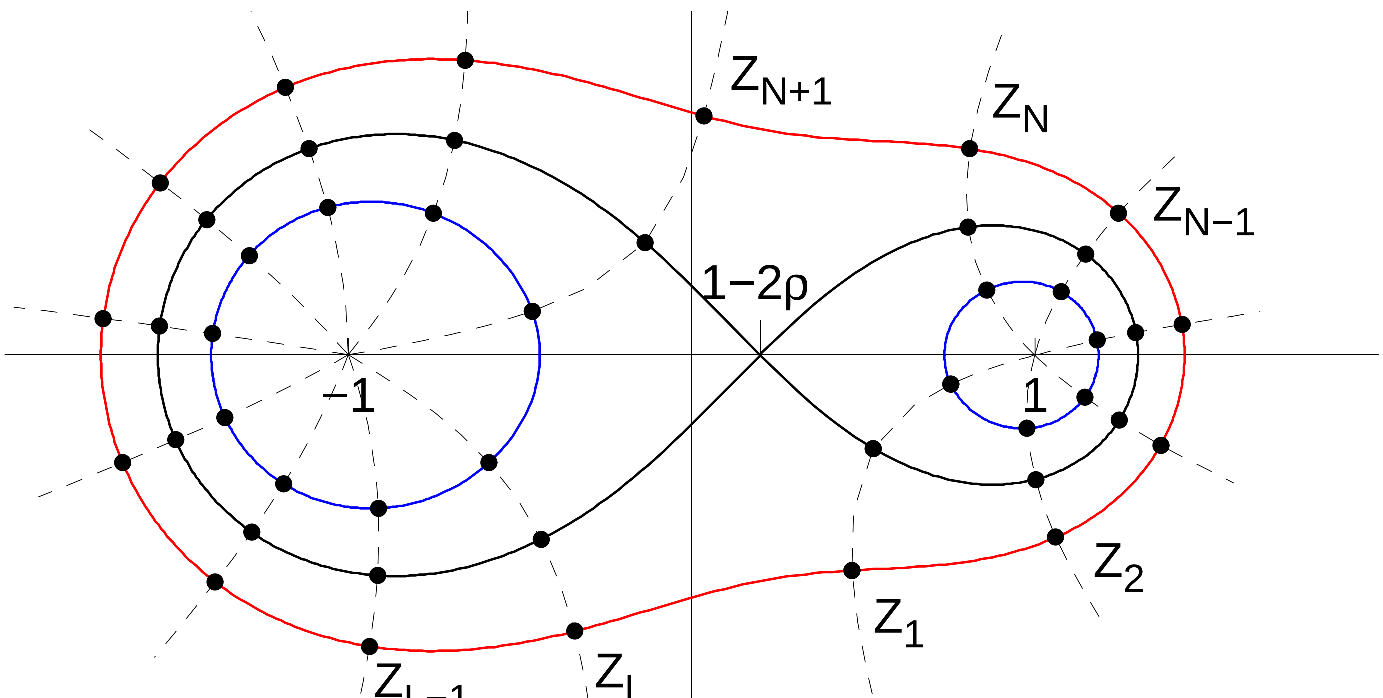
$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x z_i z_j - (1+x) z_i + 1}{x z_i z_j - (1+x) z_j + 1}$$

The corresponding eigenvalue is given by

$$E(z_1, z_2 \ldots z_N) = \sum_{i=1}^{N} \frac{1}{z_i} + x \sum_{i=1}^{N} z_i - N(1+x)\,.$$

Labelling the roots of the Bethe Equations

The loci of the roots (for $x = 0$) are remarquable curves : **Cassini Ovals**



Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ :

$$E_1 = \quad -2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}} \pm \frac{2i\pi(2\rho-1)}{L}\,.$$

RELAXATION OSCILLATIONS

Non-diffusive : Largest relaxation time $T \sim L^z$ with $z = 3/2$.

Oscillations $\rightarrow$ Travelling waves probed by dynamical correlations.

Classification of higher excitations.

Natural symmetries (T, CR) $\rightarrow$ Doublets in the spectrum.

Spectral degeneracies and Invariance of the Bethe Equations

L	N	$m(1)$	$m(2)$	$m(6)$	$m(20)$	$m(70)$
2	1	2				
4	2	4	1			
6	3	8	6			
8	4	16	24	1		
10	5	32	80	10		
12	6	64	240	60	1	
14	7	128	672	280	14	
16	8	256	1792	1120	112	1
18	9	512	4608	4032	672	18

Unexpected multiplets with huge highest degeneracy order $\sim 2^{L/6}$.

Current statistics as an eigenvalue problem

Statistics of Y_t : total distance covered by all the particles between 0 and t.

Deformation of the Markov Matrix M by adding a jump-counting fugacity $\gamma : M(\gamma) = M_0 + \mathrm{e}^{\gamma} M_+ + \mathrm{e}^{-\gamma} M_-$

In the long time limit, $t \to \infty$

$$\langle \mathrm{e}^{\gamma Y_t} \rangle \simeq \mathrm{e}^{E(\gamma) t}$$

$E(\gamma)$ eigenvalue of $M(\gamma)$ with maximal real part.

Equivalently, $F(j)$, the large-deviation function of the current

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-tF(j)}$$

is the Legendre transform of $E(\gamma)$.

Functional Bethe Ansatz

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$Q(T)R(T) = e^{L\gamma}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)$$

where $Q(T)$ of degree N vanishes at the Bethe roots.

Functional Bethe Ansatz (Baxter's TQ equation) : Restatement of the Bethe Ansatz as a purely algebraic problem.

From $Q(T)$, we deduce $E(\gamma) \rightarrow$ Full statistics of the current and large deviations ($x = 0$ case : B. Derrida and J. L. Lebowitz).

Non-vanishing Skewness (Third cumulant) E_3 $\rightarrow$ Non Gaussian fluctuations.

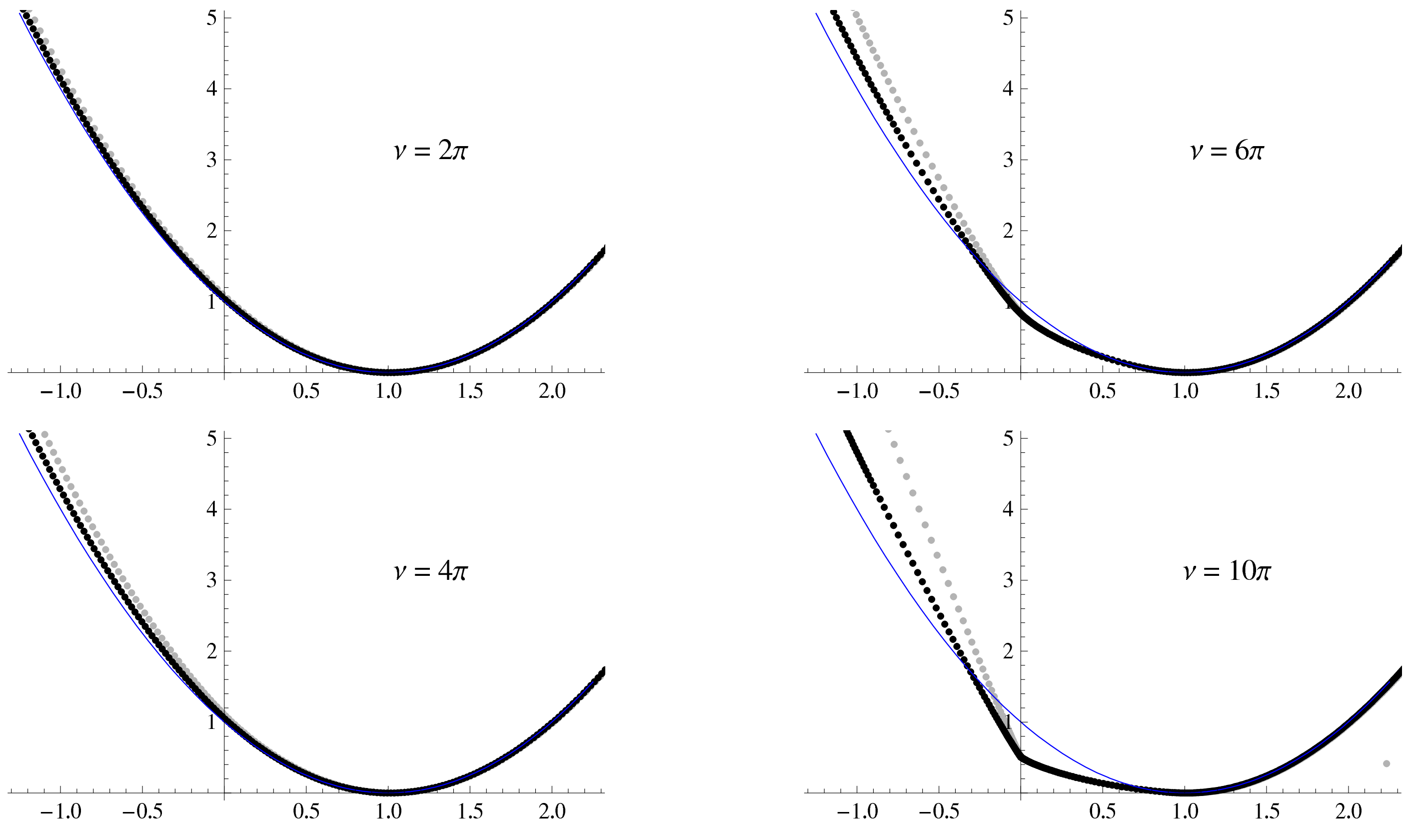
The weakly symmetric case $x = 1 - \frac{\nu}{L}$

In the limit of large system sizes, $L \to \infty$,

$$E\left(\frac{\gamma}{L}\right) \simeq \frac{\rho(1-\rho)(\gamma^2+\gamma\nu)}{L} - \frac{\rho(1-\rho)\gamma^2\nu}{2L^2} + \frac{1}{L^2}\phi[\rho(1-\rho)(\gamma^2+\gamma\nu)]$$

$$\text{with} \quad \phi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$$

- Leading order (in $1/L$) : Gaussian fluctuations.
- Subleading (in $1/L^2$) : Non-Gaussian correction.
- Phase transition when $\nu \geq \nu_c = \frac{2\pi}{\sqrt{\rho(1-\rho)}}$



ν = 2π
ν = 6π
ν = 4π
ν = 10π

Multispecies Exclusion Models.

N classes of particles and holes with hierarchical priority rules.
During an infinitesimal time step dt, the following processes take place on each bond with probability dt :

$$I\ 0 \to 0\ I \qquad \text{for} \quad I \neq 0$$

$$I\ J \to J\ I \qquad \text{for} \quad 1 \leq I < J \leq N$$

Particles can always overtake holes (= 0-th class particles).
First-class particles have highest priority etc...
There are P_I particles of class I. Total number of configurations :

$$\Omega = \frac{L!}{P_0!P_1!P_2!\dots P_N!}$$

Stationary Measure ?

Matrix Ansatz for 2 Species

Algebraic description of the Stationary Measure (DEHP, DJLS '93).

Configuration represented by a string e.g. 01220211.

Stationary weight :

$$p(01220211) = \frac{1}{Z}\mathrm{Tr}(EDAAEADD)$$

$0 \to E$, $1 \to D$ and $2 \to A$, operators in a *quadratic algebra*

$$\begin{aligned} DE &= D + E \\ DA &= A \\ AE &= A \end{aligned}$$

e.g. $p(01220211) \propto \mathrm{Tr}(D^2EA^3) = \mathrm{Tr}((D^2 + D + E)A^3) \propto 3\mathrm{Tr}(A^3)$

Stationary state properties (currents, correlations, fluctuations).

Tensor Products of Quadratic Algebra

Hierarchical construction of representations of 'nested algebras' using the D, A and E matrices and the shift operators $\delta = D - 1$ and $\epsilon = E - 1$.

For the 3-species case :

$$
\begin{aligned}
\hat{\mathbf{P}}_0 &= \mathbf{1} \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \epsilon \otimes A + \epsilon \otimes \mathbf{1} \otimes D\,. \\
\hat{\mathbf{P}}_1 &= \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E \\
\hat{\mathbf{P}}_2 &= A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E \\
\hat{\mathbf{P}}_3 &= A \otimes A \otimes E
\end{aligned}
$$

CONCLUSIONS

The asymmetric exclusion process can be analyzed through a variety of techniques : Bethe Ansatz, Quadratic Algebras, Young tableaux combinatorics, Orthogonal polynomials, Random Matrices, stochastic differential equations, hydrodynamic limits.

Exact solutions : paradigms for the behaviour of systems far from equilibrium in low dimensions : Dynamical phase transitions, Large deviations, Non-Gibbsean measures, Fluctuations Theorems...

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