Simplicity of gauge and gravity amplitudes

Pierre Vanhove

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based on [0907.1425], [1003.2403] and work in progress with
N.E.J. Bjerrum-Bohr, P. Damgaard, T. Søndergaard

I just put 1.795372 and 2.204628 together.
And what does that mean?
Four!
(Doctor Who)

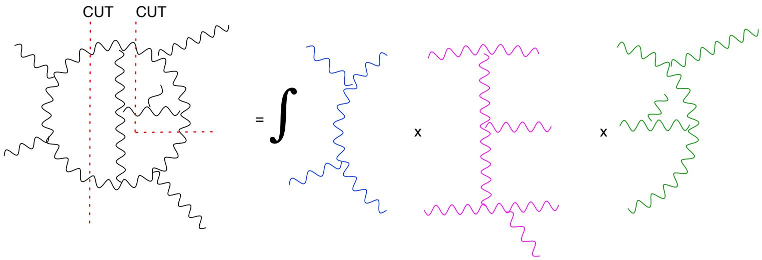
It is crucial for experimental and theoretical reasons to have efficient methods for evaluating amplitudes between physical processes in quantum field theory

- multilegs and multiloop amplitudes for LHC physics
- Quantum gravity: perturbative ultraviolet nature of $\mathcal{N} = 8$ supergravity

Unfortunately the number of individual Feynman graphs rises dramatically with the number of external legs, and tensor reduction methods increase the number of terms even more.

A huge number of cancellation are needed before to get the result leading to

- instabilities due to large numerical cancellations in matrix elements
- obfuscation of the fundamental structure of the interactions: gauge invariance, ultraviolet divergences, infrared singularities, hidden symmetries



One approach is to use unitarity methods (see the talks by Kosower and Britto)

- only on-shell gauge invariant quantities are considered
- use physical properties of amplitudes: factorisation (recursion relations), unitarity, locality, etc.
- Reduce the computation to the phase space integral of product of tree amplitudes

Tree-level amplitudes

In general tree-level amplitudes with many external states are rather involve to compute especially in gravity

Tree-level gauge theory amplitude are decomposed as

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_n / \mathbb{Z}_n} \text{Tr}(\lambda^{\sigma(1)} \cdots \lambda^{\sigma(n)}) \, A_n(\sigma(1), \cdots, \sigma(n))$$

All the information about the amplitude is in color ordered partial amplitude

$$A_n^{\sigma} := A_n(\sigma(1), \cdots, \sigma(n)); \qquad \sigma \in \mathfrak{S}_n$$

The gravity amplitudes

[Kawai-Lewellen-Tye] showed that gravity amplitudes are given by the left/right product of gauge theory amplitudes

$$\begin{aligned} M_4^{\text{tree}} &= -\kappa^2\, s A_4(1,2,3,4)\tilde{A}_4(1,2,4,3) \\ M_5^{\text{tree}} &= \kappa^3 s_{12}s_{34}A_5(1,2,3,4,5)\tilde{A}_5(2,1,4,3,5) \\ &+ \kappa^3 s_{13}s_{24}A_5(1,3,2,4,5)\tilde{A}_5(3,1,4,2,5) \end{aligned}$$

$$M_n^{\text{tree}} = -(-\kappa)^{n-2} \sum_{\sigma,\sigma'\in\mathfrak{S}_n} P_{n-3}^{\sigma,\sigma'}(s_{ij})\, A_n^{\sigma}\tilde{A}_n^{\sigma'}$$

- $P_{n-3}^{\sigma,\sigma'}(s_{ij})$is an homogeneous polynomials of degree $n-3$ in the kinematic invariants to avoid unphysical double poles
- Again all the information about the amplitude is in the color ordered factor A_n^{σ}.

Tree level amplitudes in gauge theory

Feynman rules gives $n!$ different amplitudes but all the partial amplitudes A_n^σ are not independent

- Reflection property

$$A_n(1,\ldots,n) = (-1)^n A_n(n,\ldots,1)$$

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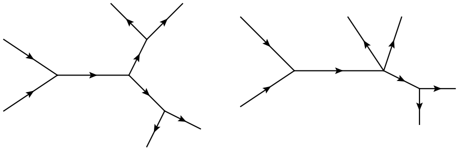
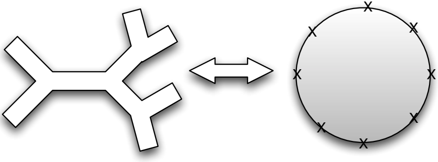
What is the minimal number of independent amplitude to be computed for getting all the tree-level processes?

Moduli space for tree-level amplitudes

Instead of considering the sum of the mutilple field theory graph individualy we treat the field theory amplitudes and the infinite tension limit $\alpha' \to 0$ of the tree-level string amplitudes





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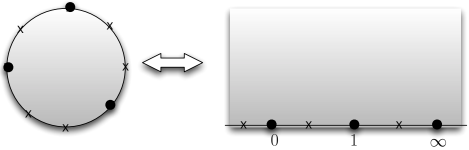
$$A_{\text{SYM}}(1,\dots,n) = \lim_{\alpha'\to 0} \mathfrak{A}(1,\dots,n)$$

$$\mathfrak{A}(1,\dots,n) = \left\langle U^{(1)}(z_1) U^{(n-1)}(z_{n-1}) U^{(n)}(z_n) \prod_{i=2}^{n-2} \int d^2 z_i V^{(i)} \right\rangle$$

This can applied to any string theory formalism (Bosonic, RNS, Green-Schwarz, Pure Spinor, ...) in any spacetime dimensions

Cyclicity: $(n-1)!$ amplitudes



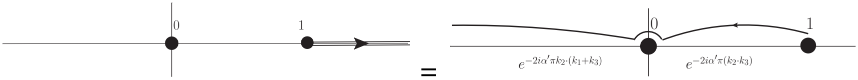
$PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathfrak{A}(1,\dots,n) = \int_{x_1<\cdots<x_n} \prod_{i=2}^{n-2} dx_i \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j} \sum_{(\zeta_j)\in\{0,1,x_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{x_j - \zeta_j}$$

- The L_k depend on the polarisations and external momenta
- The integrals are linear combinaison of the hypergeometric ${}_{n-2}F_{n-3}$ functions that arise as conformal block of correlators for the vertex operators $V_{\vec\alpha}(z) = e^{\vec\alpha\cdot\vec\phi(z)}$ [Dotsenko, Fateev]

Monodromies: Step 1 $(n-2)!$ amplitudes

We deform the contour of integration [Bjerrum-bohr, Damgaard, Vanhove]



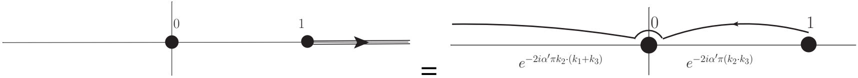
The real part of the monodromy relation leads to the stringy version of the Kleiss-Kuijf relations

$$\mathfrak{A}_n(\beta_1,\dots,\beta_r,1,\alpha_1,\dots,\alpha_s,n) = (-1)^r \times \Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big],$$

$\exp(\alpha,\beta) = \exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta - z_\alpha) > 0$ or 1 otherwise

Monodromies: Step 2 $(n-3)!$ amplitudes

We deform the contour of integration [Bjerrum-bohr, Damgaard, Vanhove]



The imaginary part of the monodromy relation

$$0=\Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})}\sum_{\sigma\subset OP\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)}\mathfrak{A}_n(1,\{\sigma\},n)\Big].$$

Minimal Basis for tree-level amplitudes

- The lead to a linear system of rank $(n-3)!$ in the amplitudes
- All ordered amplitudes can be expanded in the *minimal* basis $\mathfrak{B}_n$

$$\mathfrak{B}_n^{\sigma} = \mathfrak{A}(1, \underbrace{\sigma(2), \ldots, \sigma(n-2)}_{permutation}, n-1, n); \qquad \sigma \in \mathfrak{S}_{n-3}$$

$$\mathfrak{A}_n^{\sigma} = \sum_{\sigma' \in \mathfrak{S}_{n-3}} c_{\sigma'}^{\sigma}\, \mathfrak{B}^{\sigma'}$$

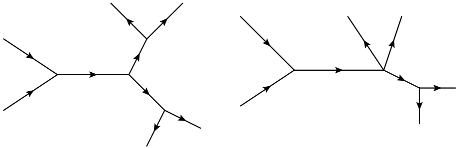
The coefficient $c_{\sigma'}^{\sigma}$ are rational functions of degree 0 in the $\sin(2\alpha'\pi p \cdot q)$

Minimal Basis for tree-level amplitudes

Using the KLT relation we obtain the gravity tree-level amplitude (one can use monodromy on the sphere with three marked points $\mathbb{P}^1\backslash\{z_1 = 0, z_{n-1} = 1, z_n = \infty\}$)

$$\mathfrak{M}_n^{\text{closed}} = \frac{\kappa^{n-2}}{(\alpha')^{n-3}} \sum_{\sigma,\tilde\sigma\in\mathfrak{S}_{n-3}} g_{\sigma\tilde\sigma}\,\mathfrak{B}^{\sigma}\widetilde{\mathfrak{B}}^{\tilde\sigma}$$

where the matrix $g_{\sigma\tilde\sigma}$ and the coefficient are rational function of degree $n-3$ in the kinematic quantities $\sin(2\pi\,\alpha'\,p\cdot q)$



One can consider an expansion of the tree amplitudes based on cubic interactions only [Bern, Carrasco, Johansson]

$$A_n = \sum_i \frac{n_i}{s_1 \cdots s_{n-3}}; \qquad M_n = \sum_i \frac{n_i \tilde{n}_i}{s_1 \cdots s_{n-3}}$$

- The monodromy relation imply that the numerator n_i are not independent and satisfy generalized Jacobi relations

[Tye, Zhang,'10] [Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove, '1

$$n_i - n_j + n_k = P_n(s_{ij})$$

- $P_n(s_{ij})$ polynomial of degree $n-4$ in the kinematic s_{ij} invariants

Application: one-loop 6-point amplitude in $\mathcal{N} = 4$ SYM

One-loop amplitude in $\mathcal{N} = 4$ SYM can be decomposed on a basis of scalar box functions [Bern, Dunbar, Dixon, Kosower]

$$\mathcal{A}_n^{1-\text{loop}} = \sum_i \left(\widehat{b}_i I^{1m;i} + \widehat{c}_i I^{2m\,e} + \widehat{d} I^{2m\,h;i} + \widehat{g}_i I^{3m;i} + \widehat{f}_i I^{4m;i} \right) .$$

The coefficients are determined from the quadruple cut

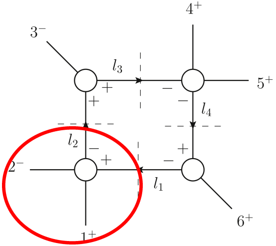
[Britto, Cachazo, Feng]

$$\widehat{a}_\alpha = \frac{1}{2} \sum_{S,J} n_J A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} .$$

The monodromy relations in the cuts gives relations between the coefficients of amplitude with different helicity distribution

Application: one-loop 6-point amplitude in $\mathcal{N}=4$ SYM

$\widehat{c}_1 =$

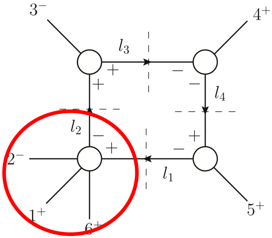


The monodromies relate the coefficients of the 2-mass-easy box I^{2me} for split-helicity $A_{6;1}(1^+,2^-,3^-,4^+,5^+,6^+)$ and mixed-helicity loop amplitudes $A_{6;1}(2^-,1^+,3^-,4^+,5^+,6^+)$

$$\widehat{c}_1 = -\frac{\langle 16\rangle\langle 23\rangle}{\langle 26\rangle\langle 13\rangle}\,\widehat{c}_1'\,,$$

Application: one-loop 6-point amplitude in $\mathcal{N} = 4$ SYM

$\widehat{b}_{621} =$



$$\widehat{b}_{621} = \frac{(s_{16} + s_{(-l_1)1})\widehat{b}_{612} + s_{(-l_1)1}\widehat{b}_{162}}{s_{l_2 1}},$$

The monodromies relate the one-mass integral I^{1m} coefficient of the mixed-helicity amplitude $A_{6;1}(2^-,1^+,3^-,4^+,5^+,6^+)$ to coefficients of the split-helicity amplitudes $A_{6;1}(1^+,2^-,3^-,4^+,5^+,6^+)$ and $A_{6;1}(6^+,2^-,3^-,4^+,5^+,1^+)$.

Outlook

The monodromy relation between the color ordered tree-level amplitudes

- Best possible reorganization of the tree-level amplitude in gravity and gauge theory
- Find an optimal basis for $\mathcal{N} = 8$ supergravity tree-level amplitude under the large-z BCFW shifts
- Reorganization of the expansion of the one-loop amplitudes
- Extension to higher-loop order (higher genus Riemann surfaces) [Work in progress]