MMF & Umbral Moonshine

1. jacobi form and ADE
2. meromorphic mac form & (optimal) mock jacobi forms
3. Niemeier lattices and the umbral moonshine conjecture
4. A theorem on optimal mock jacobi forms

mc - Duncan - Harvey '12, '13

Dabholkar - Murthy - Zagier '12

1. jac form and ADE

Def.

We first discuss Jacobi forms following [35]. For every pair of integers k and m with $m > 0$, we define the m-action of the group $\mathbb{Z}^2$ and the (k, m)-action of the group $SL_2(\mathbb{Z})$ on the space of holomorphic functions $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ as

$$(\phi|_m(\lambda, \mu))(\tau) = e(m(\lambda^2\tau + 2\lambda z))\, \phi(\tau, z + \lambda\tau + \mu) \tag{3.8}$$

$$(\phi|_{k,m}\gamma)(\tau) = e(-m\tfrac{cz^2}{c\tau+d})\ \mathrm{jac}(\gamma, \tau)^k \phi(\gamma(\tau, z)) \tag{3.9}$$

where $\gamma \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$. We say a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is an *(unrestricted) Jacobi form* of weight k and index m for the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ if it is invariant under the above action, $\phi = \phi|_{k,m}\gamma$ and $\phi = \phi|_m(\lambda, \mu)$, for all $\gamma \in SL_2(\mathbb{Z})$ and for all $(\lambda, \mu) \in \mathbb{Z}^2$. In what follows we refer to the transformations (3.8) and (3.9) as the *elliptic* and

· an odd weight Jac. form of ind

$$\phi(\tau,z) = \sum_{r=-m+1}^{m-1} h_r(\tau)\, \theta_{m,r}(\tau,z) \quad , \quad \theta_{m,r}(\tau,z) =$$

with $h_r(\tau) = -h_{-r}(\tau)$

$$\Rightarrow \quad \phi(\tau,z) = \sum_{r=1}^{m-1} h_r(\tau) \left(\theta_{m,r} - \theta_{m,-r}\right)$$

Q: What is the spc $M^{(m)}$ of $2m \times 2m$ matrices Ω s.t.

$$\sum_{r,r' \bmod 2m} h_r \Omega_{r,r'} \theta_{m,r'} \in J_{k,m} \iff \varphi = \sum_r h_r \theta_{m,r} \in J_{k,m} \; ?$$

A: $\dim M^{(m)} = \sigma(m) = \sum_{d \mid m} 1$

spanned by $\{ \Omega(d) \mid d \mid m \}$

$$\sum_{r,r' \bmod 2m} h_r \Omega^{(d)}_{r,r'} \theta_{m,r'} = \frac{1}{d} \sum_{a,b=0}^{d-1} e\left(m\left(\frac{a^2}{d^2}\tau + 2\frac{a}{d}z + \frac{ab}{d^2}\right)\right) \varphi\left(\tau, z + \frac{a}{d}\tau + \frac{b}{d}\right)$$

"Eichler–Zagier map on $J_{k,m}$"

/Atkin–Lehner

k odd

Q: What are the $2m \times 2m$ matrices Ω s.t.

$$\sum_{r,r' \bmod 2m} h_r \Omega_{r,r'} \theta_{m,r'} \in J_{k,m} \iff \varphi = \sum_r h_r \theta_{m,r} \in J_{k,m}$$

$$= \sum_{r=1}^{m-1} h_r (\theta_{m,r} - \theta_{m,-r})$$

and $\Omega_{r,r'} \in \mathbb{Z}_{\geq 0}$ w. $\Omega_{1,1} - \Omega_{1,-1} = 1$

Prop. such matrices $\xleftrightarrow{1-1}$ A-D-E root systems

$m = $ Cox. # of X

$\Omega^X_{r,r} - \Omega^X_{r,-r} = \alpha^X_r =$ multiplicities of the Coxeter exponent r

$\in \{0,1,2\}$

cf. Cappelli-Itzykson-Zuber ('87)

2. meromorphic mac form & (optimal) mock jacobi forms

Generalisation: merom. Jac. form ↕ mock theta series,, ($z \mapsto \varphi(\tau, z)$, as at torsion pnt $z \in \mathbb{Q}\tau + \mathbb{Q}$)

Consider, wt 1 index m merom. Jac form w. simple poles

$$\psi = \psi^P + \psi^F$$

(Dabholkar–Murthy–Zagier)

e.g. poles at $z \in \mathbb{Z}\tau + \mathbb{Z}$

$$\psi^P = c \cdot Av_m\left[\frac{y+1}{y-1}\right],$$

$$\mathrm{Av}_m\left[F(y)\right] = \sum_{k \in \mathbb{Z}} q^{mk^2} y^{2mk} F(q^k y)$$

$$\psi^F = \sum_{r \bmod 2m} H_r(\tau)\,\theta_{m,r}(\tau, z)$$

$$S_{m,r} = \frac{\partial}{2\pi i\,\partial z}\theta_{m,r}\Big|_{z=0}$$

ψ^F is a

$\Rightarrow$ mock Jac. form w. shadow $S_m = (S_{m,r})$

$$\hat{\psi}^P(\tau, \bar z; z) = Av_m\left[\frac{y+1}{y-1}\right] + \left(-\frac{1}{\sqrt{2m}}\frac{1}{(4i)^{1/2}}\sum_{r\ (\mathrm{mod}\ 2m)} \theta_{m,r}(\tau, z)\int_{-\bar\tau}^{i\infty}(\tau' + \tau)^{-1/2}\,\overline{S_{m,r}(-\bar\tau')}\,d\tau'\right)$$

$$\psi(\tau,z) \;:=\; \tau_2^{k-1/2}\,\frac{\partial}{\partial\overline{\tau}}\widehat{\varphi}(\tau,z) \;\doteq\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}} \overline{g_\ell(\tau)}\,\vartheta_{m,\ell}(\tau,z)\,. \tag{7.21}$$

(Here $\doteq$ indicates an omitted constant.) The function $\psi(\tau,z)$ is holomorphic in z, satisfies the same elliptic transformation property (4.2) as φ does (because each $\vartheta_{m,\ell}$ satisfies this), satisfies the heat equation $\big(8\pi i m\,\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial z^2}\big)\psi=0$ (again, because each $\vartheta_{m,\ell}$ does), and, by virtue of the modular invariance property of $\widehat{\varphi}(\tau,z)$, also satisfies the transformation property

$$\psi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) \;=\; |c\tau+d|\,(c\overline{\tau}+d)^{2-k}\,e^{\frac{2\pi i m c z^2}{c\tau+d}}\,\psi(\tau,z) \qquad \forall\,\begin{pmatrix} a\ b\\ c\ d\end{pmatrix}\in SL(2;\mathbb{Z}) \tag{7.22}$$

with respect to the action of the modular group. These properties say precisely that ψ is a *skew-holomorphic Jacobi form* of weight $3-k$ and index m in the sense of Skoruppa [101, 102], and the above discussion can be summarized by saying that we have an exact sequence

$$0 \longrightarrow J^{\text{weak}}_{k,m} \longrightarrow \mathbb{J}^{\text{weak}}_{k|0,m} \xrightarrow{\;\mathcal{S}\;} J^{\text{skew}}_{3-k,m} \tag{7.23}$$

(and similarly with the word "weak" omitted), where $\mathbb{J}^{\text{weak}}_{k|0,m}$ (resp. $\mathbb{J}_{k|0,m}$) denotes the space of weak (resp. strong) pure mock Jacobi forms and the "shadow map" $\mathcal{S}$ sends φ to ψ.

Def. $\varphi \in J^{wk}_{1,m}$ is called optimal when $q^{\frac{1}{4m}} H_r(\tau) = O(1)$

$\quad \parallel$

$\sum_{r \in \mathbb{Z}/2m} H_r \cdot \theta_{m,r}$ $\qquad \forall\, r \in \mathbb{Z}/2m.$

Thm. If $\varphi_1, \varphi_2 \in J^{wk}_{1,m}$ are 2 optimal forms w. the same shadow, then $\varphi_1 = \varphi_2$.

Q: For what $\phi \in J^{skew}_{2,m}$ does the pre-image contain an optimal form ??

3. Niemeier lattices and the umbral moonshine conjecture

Focus on "ADE shadows"

$$\psi^F = \sum_{r \bmod 2m} H_r(\tau)\,\theta_{m,r}(\tau,z) = H^T \cdot \theta_m \in J^{wk}_{1,m}$$

$$\hat{\psi}^F = \psi^F - \chi \frac{1}{\sqrt{2m}} \frac{1}{(4i)^{1/2}} \sum_{r \,(\mathrm{mod}\, 2m)} \theta_{m,r}(\tau,z) \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2}\, \overline{S_{m,r}(-\bar{\tau}')}\, d\tau'$$

For a Dynkin rt system Y w. Coxeter # m.

$$\psi^F|W^Y := H^T \cdot \Omega^Y \cdot \theta_m \in J^{wk}_{1,m}$$

$$S(\psi^F|W^Y) = \bar{S}_m \cdot \Omega^Y \cdot \theta_m$$

$$\Omega^{\cup Y_i} = \sum_i \Omega^{Y_i}$$

Thm: (Niemeier '73)

There are exactly 24 even, unimodular (+)-def. lattices of rk 24.

One of them is the Leech lattice (Leech '67) with no roots (lattice vectors w. $\langle\cdot,\cdot\rangle = 2$)

The other 23 have a rk 24 root system X which uniquely specifies the corresponding lattice Λ^X (the "Niemeier lattice").

The 23 root systems (Niemeier root sys) are specified by

① they are rank 24 union of simply-laced (ADE) root systems

② each irred. component has the same Coxeter number h ✓

Q: For a given Niemeier rt system X ($\mathrm{cox}(X) = m$) does there exist (unique) optimal $\psi^X \in J^{wk}_{1,m}$ w. $S(\psi^X) = \bar{S}_m \cdot \Omega^X \cdot \theta_m$?

A: Yes, for all 23 X.

$$
\begin{tikzcd}
 & & \text{Niemeier rt system } X \arrow[dl] \arrow[dr] & \\
J^{wk}_{1,m}(\Gamma_0(N)) \ni \psi^X_g \ (\text{opt}) & J^{wk}_{1,m} \ni \psi^X = H^X \cdot \theta_m \ (\text{opt}) \arrow[l, dashed] & & \text{finite grp } G^X := \mathrm{Aut}(\Lambda^X)/\mathrm{Weyl}(X)
\end{tikzcd}
$$

eg. $X = 24A_1$, $H_1^X = -H_{-1}^X = 2q^{-\frac{1}{8}}(-1 + \underline{45}\,q + \underline{231}\,q^2 + \underline{770}\,q^3 + \cdots)$

$G^X = M_{24} = \mathrm{Aut}(\text{binary Golay code})$ (EOT '10)

cf. $J(\tau) = q^{-1} + \underbrace{196884}_{\substack{\shortparallel \\ 1+196883}}\,q + \underbrace{21493760}_{1+196883+21296876}\,q^2 + \cdots$

UM conjecture: There exists an ∞-dim, $\mathbb{Z} \times \mathbb{Q}$-graded G^X-module $K^X = \bigoplus_r \bigoplus_\alpha K^X_{r,\alpha}$ s.t. H^X_g is the graded character of K^X

i.e. $$H^X_{g,r} = \sum_\alpha q^\alpha \left(\mathrm{Tr}_{K^X_{r,\alpha}} g\right)$$

4. A theorem (and some conjectures) on optimal mock jacobi forms

Thm. (CDH '12)

Consider $\psi_{(m)} \in J^{wk}_{1,m}$ w $S(\psi_{(m)}) = \overline{S}_m \cdot \theta_m = \overline{S}_m \cdot \Omega^{A_{m-1}} \cdot \theta_m$

$\exists$ such $\psi_{(m)}$ that is optimal iff $m-1 \mid 24$

(hence $\exists$ Niemeier rt system $\frac{24}{m-1} A_{m-1}$).

pf. If such opt $\psi_{(m)}$ exists, $L(f,1) = 0$ $\forall f \in S_2(\Gamma_0(m))$.

Conj: optimal wk mock Jac form has a genus 0 property.

eg.

Q: $m \in \{1, 2, 3, \ldots\}$. when does $\mathrm{genus}\left(\widehat{\Gamma_0(m)\backslash\mathbb{H}}\right) = 0$?

A: Iff $\exists$ Niem. rt sys. X w. $\mathrm{Cox}(X) = m$, and

X contains an A-type component.

Similarly for D-type. $\Gamma_0(m) \to \Gamma_0(m) + m/2$