Mathematical structures behind supersymmetric dualities

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Based mainly on works with

Grigory Vartanov,
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and works in progress with

Vyacheslav Spiridonov and Grigory Vartanov,
Wei Li and Edvard Musaev,

Simple Motivation

- **Supersymmetric dualities**: Seiberg and Seiberg–like dualities, S–duality and etc.
- Mathematical structures: **Elliptic hypergeometric integrals**

Connection: calculation of **superconformal index**

- The superconformal index of a four–dimensional supersymmetric gauge theory can be expressed in terms of elliptic hypergeometric integrals.
- The identification of superconformal indices of Seiberg dual theories is nothing but Weyl group symmetry transformations for certain elliptic hypergeometric functions.

Recent progress

- SUSY dualities in $\mathcal{N}=1$
- S–duality in $\mathcal{N}=2,4$ theories
- justification of known supersymmetric dualities
- test holographic dualities
- new SUSY dualities
- inclusion of surface and line operators
- 't Hooft anomaly matching
- interesting mathematical structures

Generalization of the Witten index

The SUSY algebra

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \quad \{Q, (-1)^F\} = 0$$

Topological invariant [Witten '82]

$$\mathrm{I}_W = Tr(-1)^F e^{-\beta H}$$

- no β–dependence
- independent of the parameters of the theory
- characterizes SUSY breaking

The superconformal index

The **superconformal index** is a non–trivial generalization of the Witten index:

$$I(\underline{f}) = \mathrm{Tr}(-1)^F e^{-\beta H} \prod f_i^{F_i}$$

- F_i – generators commuting with Q and $Q^\dagger$
- f_i – fugacities (additional regulators)

[Romelsberger 0510060]

[Kinney, Maldacena, Minwalla and Raju 0510251]

Example: $4d$ $\mathcal{N}=1$ theory

Consider the $\mathcal{N}=1$ **superconformal theory** on $S^3 \times S^1$.

The symmetry group of this theory is $SU(2,2|1)$.

which has the following generators

$$J_i,\ \bar{J}_i\text{— Lorentz rotations}$$

$$P_\mu,\ Q_\alpha,\ \bar{Q}_{\dot{\alpha}}\text{— Supertranslations}$$

As in any conformal invariant field theory, we have superconformal generators

$$K_\mu,\ S_\alpha,\ S_{\dot{\alpha}}\text{— Special superconformal transformation}$$

$$H\text{— Dilatations}$$

The action of a supersymmetric theory should also invariant under R–symmetry

$$R\text{— } U(1)_R \text{ rotations.}$$

Example: $4d$ $\mathcal{N} = 1$ theory

Different possibilities to realize relation

$$\{Q, Q^\dagger\} = 2\mathcal{H}$$

Consider, for example, the supercharges $\bar{Q}_1$ and $\bar{S}^1$, which satisfy the following relation

$$2\{Q, Q^\dagger\} = H - \frac{3}{2} R - 2\bar{J}_3$$

- dilatation operator
- R–symmetry on S^3
- Cartans of $SU(2) \times SU(2)$ isometry group of S^3

Example: $4d$ $\mathcal{N} = 1$ theory

Then one defines the superconformal index in the following way

$$I(t, x, f) \;=\; \mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} e^{\sum_i f_i F_i} e^{\sum_j g_j G_j} \;,$$

$\boxed{\textbf{Only states annihilated by } \mathcal{H} \textbf{ contribute to the index.}}$

The superconformal index counts **gauge invariant** operators which satisfy **BPS condition** and which cannot be combined to form long multiplets.

- does not depend on the coupling constants

recent discussion [Gerchkovitz 1311.0487]

- invariant under marginal deformations of the theory

Calculating the superconformal index

Römelsberger introduced a simple procedure for an explicit computation of the superconformal index.

[Romelsberger 0707.3702]

First compute a single letter index

$$ind(\{f_i\})$$

which one gets by summing over all the fields contributing to the index.

Then the full index is formed by summing over multiparticle states, i.e. by inserting the single letter index into the so–called "plethystic" exponential and integrating over the gauge group in order to get gauge–invariant quantity

$$\int_G d\mu(g) \ \ \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} ind(\{f_i\}^n)\right),$$

where $\mu(g)$ is the invariant Haar measure.

[Benvenuti, Feng, Hanany and Y.H. He 0608050]

[Feng, Hanany and Y.H. He 0701063]

Romelsberger prescription

Single letter index:

Example: $4d$ $\mathcal{N}=1$ theory

$$\mathrm{ind}(p,q,\underline{z},\underline{y}) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj}(\underline{z}) \\ + \sum_j \frac{(pq)^{R_j/2}\chi_{R_F,j}(\underline{y})\chi_{R_G,j}(\underline{z}) - (pq)^{1-R_j/2}\chi_{\bar{R}_F,j}(\underline{y})\chi_{\bar{R}_G,j}(\underline{z})}{(1-p)(1-q)}$$

Example: $4d$ $\mathcal{N}=4$ SYM

$$\mathrm{ind}(t,y,v,w) \ = \ \frac{t^2(v+1/w+w/v)-t^3(y+1/y)-t^4(w+1/v+v/w)+2t^6}{(1-yt^3)(1-y^{-1}t^3)}\chi_{adj}(G)$$

The full index via a **"plethystic" exponential**

$$I(p,q,\underline{y}) = \int_{G_c} d\mu(g)\ \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{ind}\left(p^n,q^n,\underline{z}^n,\underline{y}^n\right)\right)$$

[Romelsberger 0707.3702]

Localization:
Another way to compute the index is to use localization.

[Nawata 1104.4470]

[Closset and Shamir 1311.2430]

Index can be expressed in terms of integrals over elliptic Gamma functions

[Dolan and Osborn 0801.4947]

the **elliptic Gamma function** is

$$\Gamma(z;p,q)=\prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j},\quad |p|,|q|<1\ .$$

[Ruijsenaars '97]

Example: A chiral multiplet with R–charge r and flavor charge f contributes to the index as a factor:

$$\Gamma((pq)^{r/2}z^f;p,q)$$

The elliptic Gamma function

The elliptic Gamma function appeared earlier in statistical mechanics:

[Baxter '72]
[Jimbo, Miwa and Nakayashiki '93]
[Jimbo, Kedem, Konno, Miwa and Weston '95]
[Davies and Pechel '97]

Properties:

$$\begin{aligned} \text{zeros} \quad & z \in \left\{ p^{i+1} q^{j+1}; i, j \in Z^{\geq 0} \right\}, \\ \text{poles} \quad & z \in \left\{ p^{-i} q^{-j}; i, j \in Z^{\geq 0} \right\}, \\ \text{residue} \quad & \mathrm{Res}_{z=1} \Gamma(z; p, q) = -\frac{1}{(p;p)_\infty (q;q)_\infty}. \end{aligned}$$

Properties of elliptic gamma function

symmetry $\quad \Gamma(z;p,q) = \Gamma(z;q,p)\,,$

functional equations $\quad \Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q),$

$$\Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q)\,,$$

reflection property $\quad \Gamma(z;p,q)\,\Gamma(\tfrac{pq}{z};p,q) = 1\,.$

modular property

$$\Gamma(e^{2\pi i\frac{z}{\sigma}};e^{2\pi i\frac{\tau}{\sigma}},e^{-2\pi i\frac{1}{\sigma}}) = e^{i\pi Q(z,\tau,\sigma)}\Gamma(e^{2\pi i\frac{z-\sigma}{\tau}};e^{-2\pi i\frac{1}{\tau}},e^{-2\pi i\frac{\sigma}{\tau}})\Gamma(e^{2\pi iz};e^{-2\pi i\tau},e^{-2\pi i\sigma})$$

[Felder and Varchenko 9907061]

Seiberg duality

Different theories describe the same physics in their IR fixed points.

Example:

- **electric theory**: with SU(2) gauge group and quark superfields in the fundamental representation of the SU(6) flavour group.
- **magnetic theory**: does not have gauge degrees of freedom, the matter sector contains meson superfields in 15-dimensional antisymmetric SU(6)-tensor representation of the second rank

[Seiberg 9402044]

$$\frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{j=1}^{6} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q) \Gamma(z^{-2}; p, q)} \frac{dz}{2\pi i z} = \prod_{1 \leq i < j \leq 6} \Gamma(t_i t_j; p, q)$$

with the balancing condition $\prod_{j=1}^{6} t_j = pq$.

[Dolan and Osborn 0801.4947]

Theorem 1. *The elliptic extension of the contour q-beta integral in* [1], [2], *or the elliptic beta function, has the form*

$$\frac{1}{2\pi i}\int_T \Delta(z)\frac{dz}{z} = \frac{2\prod_{0\le m<s\le 4}\Gamma(t_m t_s;p,q)}{(q;q)_\infty (p;p)_\infty \prod_{m=0}^{4}\Gamma(At_m^{-1};p,q)}, \tag{2}$$

where $A = \prod_{m=0}^{4} t_m$ *and*

$$\Delta(z) = \frac{\prod_{m=0}^{4}\Gamma(zt_m, z^{-1}t_m;p,q)}{\Gamma(z^2, z^{-2}, zA, z^{-1}A;p,q)}. \tag{3}$$

Here T *denotes the unit circle on* $\mathbb{C}$ *with the counter-clockwise orientation, and the complex parameters* t_m *are in* $\mathcal{D}$, *where the domain* $\mathcal{D}\subset\mathbb{C}^5$ *is defined by the conditions* $|t_m|<1$, $|pq|<|A|$.

V. P. Spiridonov, Russ. Math. Surv. 56 185 (**2001**)

Interestingly, this integral identity arises in different fields of theoretical physics, particularly, it is a star–triangle relation of an integrable lattice model.

[Bazhanov and Sergeev 1006.0651; 1106.5874]

Also note that limits of the beta integral lead to many identities for hypergeometric integrals, for instance, the limit $p \to 0$ gives the Nassrallah–Rahman q–beta integral

[Nassrallah and Rahman '85]

[Rahman '86]

Susy dualities via elliptic hypergeometric integrals

The identification of superconformal indices of Seiberg dual theories is nothing but Weyl group symmetry transformations for certain elliptic hypergeometric functions.

Example:

$4d$ $\mathcal{N}=1$ SYM theory with $SU(2)$ gauge group and 4 flavors has many duals whose superconformal indices are equal due to the Weyl group symmetry $W(E_7)$.

[Spiridonov and Vartanov 0811.1909]

The index of electric theory

$$I(t_1,\ldots,t_8;p,q)=\frac{(p;p)_\infty(q;q)_\infty}{2}\int_{\mathbb{T}}\frac{\prod_{j=1}^{8}\Gamma(t_jz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{2\pi iz}$$

with the balancing condition $\prod_{j=1}^{8}t_j=(pq)^2$.

$+$ indices of daul magnetic theories

The equality of the indices of all dual theories follows from the following identity

$$I(t_1,\ldots,t_8;p,q)=\prod_{1\leq j<k\leq 4}\Gamma(t_jt_k;p,q)\Gamma(t_{j+4}t_{k+4};p,q)\,I(s_1,\ldots,s_8;p,q),$$

where the complex variables s_j, $|s_j|<1$, are given in terms of t_j $(j=1,\ldots,8)$,

$$s_j=\rho^{-1}t_j,\ j=1,2,3,4,\quad s_j=\rho t_j,\ j=5,6,7,8,$$
$$\rho=\sqrt{\frac{t_1t_2t_3t_4}{pq}}=\sqrt{\frac{pq}{t_5t_6t_7t_8}}\ .$$

This is a manifestation of a boundary $5d/4d$ model with the enhanced E_7 global symmetry group.

$$I(\{s_i\}) = \prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_i s_j)^{-1}; p, q\right)_\infty} \frac{(p;p)_\infty (q;q)_\infty}{2} \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^{8} \Gamma(\sqrt[4]{pq} s_i z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q)}.$$

[Dimofte and Gaiotto 1209.1404]

where the term

$$\prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_i s_j)^{-1}; p, q\right)_\infty}$$

corresponds to a $5d$ hypermultiplet.

[Kim, Kim and Lee 1206.6781]

By setting all chemical potentials to 1 and redefining $p = t^3 y$, $q = t^3 y^{-1}$ one can easily read off the E_7 symmetry of the index

$$I_{4d/5d,\ N_F=4} = 1 + 56t^3 + 1463t^6 + 3002t^9 y + \dots ,$$

where the coefficients 56 and 1463 are the dimensions of the irreducible representations of E_7 with highest weight $[0,0,0,0,0,0,1]$ and $[0,0,0,0,0,0,2]$

and

$3002 = 1539_{[0,0,0,0,0,1,0]} + 1463_{[0,0,0,0,0,0,2]}$

Similar analysis for 3 flavors leading to the confinement and for $3d$ $\mathcal{N} = 2$ theory by performing dimensional reduction of the 4d theory.

[IG and Vartanov 1303.1443]

Other applications

't Hooft anomaly matching conditions

All 't Hooft anomaly matching conditions for dual theories can be derived from $SL(3,\mathbb{Z})$–modular transformation properties of the kernels of dual superconformal indices.

[Spiridonov and Vartanov 1203.5677]

Integrating out a flavor

Restriction to the flavor fugacities and using reflection identity for elliptic gamma function

Pentagon identities

Citadel of Lille (photo by Dominique Andre)

3d index

The index of a $3d$ $\mathcal{N}=2$ can be expressed in terms of the so–called tetrahedron index

$$\mathcal{I}_q[m,z] \;=\; \prod_{i=0}^{\infty} \frac{1-q^{i-\frac{1}{2}m+1}z^{-1}}{1-q^{i-\frac{1}{2}m}z}, \quad \text{with } |q|<1 \text{ and } m \in Z.$$

This is an index of free chiral multiplet with zero R-charge. The integer parameter m stands for the magnetic charge.

[Dimofte, Gaiotto and Gukov 1108.4389; 1112.5179]

Basic hypergeometric functions

Example: Consider the $\mathcal{N}=2$ $d=3$ supersymmetric field theory with $U(1)$ gauge group and one flavor. The superconformal index of this theory is

$$I_e \;=\; \sum_{m\in Z} q^{|m|/3} \int_{\mathbb{T}} \frac{dz}{2\pi i z} \frac{(q^{5/6+|m|/2}z;q)_\infty (q^{5/6+|m|/2}z^{-1};q)_\infty}{(q^{1/6+|m|/2}z;q)_\infty (q^{1/6+|m|/2}z^{-1};q)_\infty}$$

where the q-Pochhammer symbol is defined as $(z;q)_\infty = \prod_{i=0}^{\infty}(1-zq^i)$ and $\mathbb{T}$ denotes the unit circle with positive orientation.

One can rewrite these indices in terms of tetrahedron indices

$$\sum_{m\in Z} \oint \frac{dz}{2\pi i z} z^{-m}\, \mathcal{I}_q[m;q^{1/6}z^{-1}]\, \mathcal{I}_q[-m;q^{1/6}z]$$

[Krattenthaler, Spiridonov and Vartanov 1103.4075]

[IG and Rosengren 1309.2195]

Connecting $4d$ and $3d$ index

Reduction: $3d$ PF from the $4d$ index. The essential step is scaling the fugacities

$$p = e^{2\pi i v\omega_1}, \quad q = e^{2\pi i v\omega_2}, \quad z = e^{2\pi i v u}, \quad t_i = e^{2\pi i v\alpha_i} \ .$$

Then the $3d$ partition function can be achieved by taking $v \to 0$ limit of the $4d$ superconformal index and by integrating out massive fields.

[Dolan, Spiridonov and Vartanov 1104.1787]
[Gadde and Yan 1104.2592], [Imamura 1104.4482]

From the perspective of special functions this reduction brings elliptic Gamma functions to hyperbolic Gamma functions

$$\Gamma(e^{2\pi \mathrm{i} vz}; e^{2\pi \mathrm{i} v\omega_1}, e^{2\pi \mathrm{i} v\omega_2}) \quad \to \quad e^{-\pi \mathrm{i}(2z-(\omega_1+\omega_2))/24v\omega_1\omega_2} \, \gamma^{(2)}(z; \omega_1, \omega_2)$$

To obtain the superconformal index from the partition function of a $3d$ theory, roughly speaking, one should change all hyperbolic Gamma functions to a tetrahedron index

$$\gamma^{(2)}(a \pm b; \omega_1, \omega_2) \quad \to \quad \mathcal{I}_q[m; a\, b^{\pm 1}]$$

[IG and Rosengren 1309.2195]

Reduction of the elliptic beta–integral gives the following result

$$\sum_{m\in Z}\oint\frac{dz}{2\pi iz}\,(-z)^{-3m}\prod_{i=1}^{3}\mathcal{I}_q[-m,q^{\frac{1}{6}}\xi_i z]\,\mathcal{I}_q[m,q^{\frac{1}{6}}\eta_i z^{-1}]=\prod_{i,j=1}^{3}\mathcal{I}_q[0,q^{\frac{1}{3}}\xi_i\eta_j]$$

We introduce the following function

$$\mathcal{B}[m;a,b]=\frac{\mathcal{I}_q[m,a]\,\mathcal{I}_q[-m,b]}{\mathcal{I}_q[0,ab]}$$

Pentagon identity

Then the equality of indices turns out to be the pentagon identity for the function $\mathcal{B}[m; a, b]$

$$\boxed{\sum_{m\in Z} \oint \frac{dz}{2\pi i z} (-z)^{-3m} \prod_{i=1}^{3} \mathcal{B}[m; \xi_i z^{-1}, \eta_i z] = \mathcal{B}[0; \xi_1\eta_2, \xi_3\eta_1]\ \mathcal{B}[0; \xi_2\eta_1, \xi_3\eta_2]}$$

On the left side we have the $3d$ $\mathcal{N}=2$ superconformal field theory with $U(1)$ gauge symmetry and six chiral multiplets, while the mirror partner on the right side has nine chirals.

[Kashaev, Luo and Vartanov 1210.8393]

[IG and Rosengren 1309.2195]

Geometrical interpretation of the pentagon identity

There is a nice relation between $3d$ $\mathcal{N}=2$ supersymmetric gauge theories and 3–manifolds known as "class R".

[Dimofte, Gaiotto and Gukov 1108.4389; 1112.5179]

[Dimofte, Gabella and Goncharov 1301.0192]

In this context the pentagon identity encodes information about the geometry of the corresponding 3–manifolds, namely it corresponds to $3-2$ **Pachner move** for 3–manifolds.

Open problems for mathematicians

SUSY duality implies complicated and amazing integral identities. Most of them are unknown and need to be proven.

For other mathematical aspects see also

[Closset, Dumitrescu, Festuccia and Komargodski 1212.3388; 1309.5876]

and

- "On mathematical conjectures arising from $4d$ $\mathcal{N} = 2$ supersymmetric quantum field theories" by Yuji Tachikawa
- "A pseudo–mathematical pseudo–review on 4d N = 2 supersymmetric quantum field theories" by Yuji Tachikawa

Summary

- The superconformal index is a useful tool in the studying of the nonperturbative characteristics of supersymmetric gauge theories.
- It provides justification of known supersymmetric dualities and and holographic dualities.
- Moreover one can use the index technique to discover new dualities.
- The relation between superconformal indices and elliptic hypergeometric integrals allows the discovery of complicated new integral identities and interesting mathematical structures.

Thank you!