Congruence sheaves and congruence differential equations

Beyond hypergeometric functions

Vasily Golyshev

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Plan of talk

Report on joint work in progress with
Anton Mellit and Duco van Straten

- ☐ Hypergeometric monodromy
- ☐ Calabi-Yau operators
- ☐ Langlands Correspondence
- ☐ Idea of Congruence sheaves
- ☐ Example of D2s
- ☐ Outlook

Motivation: Fano classification

Fano: complex projective variety F with positive $-K$.

Classification:

dim 1: $\mathbb{P}^1$

dim 2: del Pezzo surfaces

dim 3: $H^2(F, \mathbb{Z}) = \mathbb{Z}$: Iskovskikh, 17 families;
higher rank: Mori–Mukai, 105 families

New approach to classification

Classify Fanos by classifying mirror dual objects:
Landau–Ginzburg models; respective Picard–Fuchs differential equations

Golyshev, 'Classification problems and mirror duality'
Corti, Coates, Galkin, Golyshev, Kasprzyk, 'Mirror symmetry and Fano manifolds'

Pioneered for CY by van Straten – van Enckevort.

A famous hypergeometric series

$$\phi(t) := \sum_{n=0}^{\infty} \frac{5n!}{n!^5} t^n$$

Satisfies hypergeometric differential equation

$$\mathcal{P}\phi(t) = 0$$

$$\mathcal{P} := D^4 - 5^5 t(D + \frac{1}{5})(D + \frac{2}{5})(D + \frac{3}{5})(D + \frac{4}{5})$$

$$D := t\frac{\partial}{\partial t}$$

Three singular points $\Sigma := \{0,\ 1/5^5,\ \infty\} \subset \mathbb{P}^1$

Candelas, de la Ossa, Green, Parkes (1991)

- Picard-Fuchs equation of family

$$\pi : Y \longrightarrow \mathbb{P}^1$$

  of Calabi-Yau threefolds.

- $\mathrm{rank} H^3(Y_t) = 4,\ h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$
- Defines symplectic local system $R^3\pi_*(\mathbb{Z})$ on $\mathbb{P}^1 \setminus \Sigma$
- Contains *enumerative information* of general quintic $X \subset \mathbb{P}^4$: numbers of lines, conics, etc on X.

Logarithmic solution:

$$\phi_1(t) := \log(t) \cdot \phi(t) + f_1(t), \quad f_1(t) \in t\mathbb{Q}[[t]]$$

q-coordinate:

$$q := exp(\phi_1(t)/\phi(t) = t + 770t^2 + 101475t^3 + \ldots$$

Express $\mathcal{P}$ in q-coordinate

$$P \sim D^2 \frac{5}{K(q)} D^2$$

$$K(q) = 5 + \sum_d \frac{n_d d^3 q^d}{1 - q^d}$$

$$n_1 = 2875, \quad n_2 = 609250, \quad n_3 = 317206375, \ldots$$

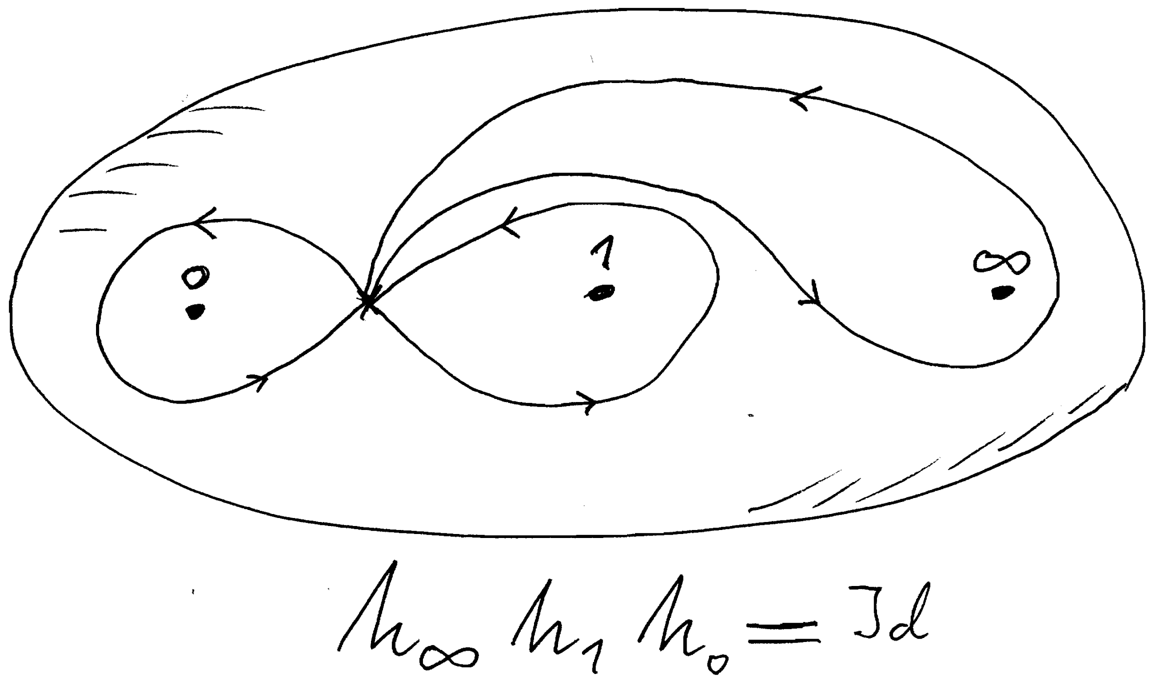
$$D^4 - Nt(D+\alpha_1)(D+\alpha_2)(D+\alpha_3)(D+\alpha_4)$$

$$\alpha_1 + \alpha_4 = 1 = \alpha_2 + \alpha_3$$

Mirror to complete intersections in weighted projective spaces.

α_1	α_2	α_1	α_2
$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{2}{5}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$
$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Hypergeometric Monodromy



$$G = \langle h_0, h_1, h_\infty \rangle \subset GL_n(\mathbb{C})$$

Differential equations D3

Differential operator of type D3 can be written as

$$
\begin{gathered}
D^3 + t\left(D + \tfrac{1}{2}\right)\left(a_3(D^2 + D) + b_1\right) \\
+t^2(D+1)\left(a_2(D+1)^2 + b_0\right) \\
+a_1 t^3(D+2)\left(D + \tfrac{3}{2}\right)(D+1) \\
+a_0 t^4(D+3)(D+2)(D+1).
\end{gathered}
$$

Th. [–, Przyjalkowski] Regularized quantum differential equations of rank 1 Fano threefolds of index d and degree $2d^2N$ are exactly (N,d) – modular D3's: given by the expansions of level N weight 2 modular forms with respect to the d'th root of the Hauptmodul on $X_0(N)^{+N}$.

Calabi-Yau Operators

Abstraction of Picard-Fuchs operators for families

$$Y \longrightarrow \mathbb{P}^1$$

of Calabi-Yau 3-folds/motives defined over $\mathbb{Z}$.

Definition: $\mathcal{P} \in \mathbb{Z}[t, D]$ is called *Calabi-Yau*, when:

- □ Fuchsian of order four.
- □ MUM (maximal unipotent monodromy) at 0.
- □ Monodromy $\subset Sp_4(\mathbb{Z})$.
- □ Integrality: $\phi(x) \in \mathbb{Z}[[t]]$, $n_d \in \mathbb{Z}$.

Beyond hypergeometrics

Simplest family of operators with *four* singular points

$$D^4+t(aD^4+2aD^3+bD^2+cD+e)+ft^2(D+\alpha_1)(D+\alpha_2)(D+\alpha_3))(D+\alpha_4)$$

singularities are at the roots of $1+at+ft^2=0$.
b, c, e: *accesory parameters*.

A project

Find all such operators with

- *coming from geometry.*
- or: with monodromy $\subset Sp_4(\mathbb{Z})$
- or: with an *integral solution* $\phi \in \mathbb{Z}[[x]]$

Seems very difficult:

- ☐ we do not know position of the the four points: j*-invariant*
- ☐ we do not know the value of the accesory parameters
- ☐ ....but we know many examples (~ 85 cases). List complete?

Example: Coming from a Calabi-Yau 3-fold in $Grass(3,6)$:

$$D^4 - t(65D^4 + 130D^3 + 105D^2 + 40D + 6) + 4^3 t^2 (D+1/4)(D+1)^2(D+3/4)$$

Solution holomorphic at 0:

$$\phi(x) = \sum_{n=0}^{\infty} a_n t^n$$

$$\begin{aligned} a_n &= \textstyle\sum_{k,l,m} \binom{n}{k}\binom{n}{l}\binom{m}{k}\binom{m}{l}\binom{k+l}{k}\binom{n}{m}^2 \\ &= \textstyle\sum_{k,l} \binom{n}{k}^2\binom{n}{l}^2\binom{k+l}{n}^2 \end{aligned}$$

A step back

Special Heun equations:

$$D^2 + t(aD^2 + aD + \lambda) + bt^2(D+1)^2$$

singularities are at the roots of $1 + at + bt^2 = 0$, λ: accesory parameter.
Basically *four* cases; studied by **F. Beukers**, **A. Beauville**, **D. Zagier** and others.
New systematic approach: **Congruence sheaves**.
Works very nicely for this case.

Number field case:

$$K \supset \mathbb{Q}$$

$$\boxed{Gal(\overline{K}/K)\text{-representations}} \leftrightarrow \boxed{\text{Automorphic representations}}$$

$$\boxed{\rho : Gal(\overline{K}/K) \longrightarrow G} \leftrightarrow \boxed{\pi_\rho \in L^2({}^L G(\mathbb{A})/{}^L G(\mathbb{Q}))}$$

Example: Elliptic curve $E/\mathbb{Q}$:

$$\begin{array}{rcl} H^1(E, \mathbb{Q}_l) & \leftrightarrow & \text{Hecke Eigenform } \sum_{n=0}^{\infty} a_n q^n \\ Tr(F_p) & \leftrightarrow & a_p \end{array}$$

Function field case:

$$K = k(C), \quad C \quad \text{curve over} \quad k := \mathbb{F}_p$$

$\boxed{Gal(\overline{K}/K)\text{-representations}}$ $\leftrightarrow$ $\boxed{\text{Automorphic representations}}$

$\boxed{l\text{-adic sheaves on } C}$ $\leftrightarrow$ $\boxed{\text{Hecke-Eigensheaves on Bun.}}$

Example: *Family* of elliptic curves $\pi : \mathcal{E} \longrightarrow C = \mathbb{P}^1/\mathbb{F}_p$

$$\mathcal{L} = R^1\pi_*(\mathbb{Q}_l)$$

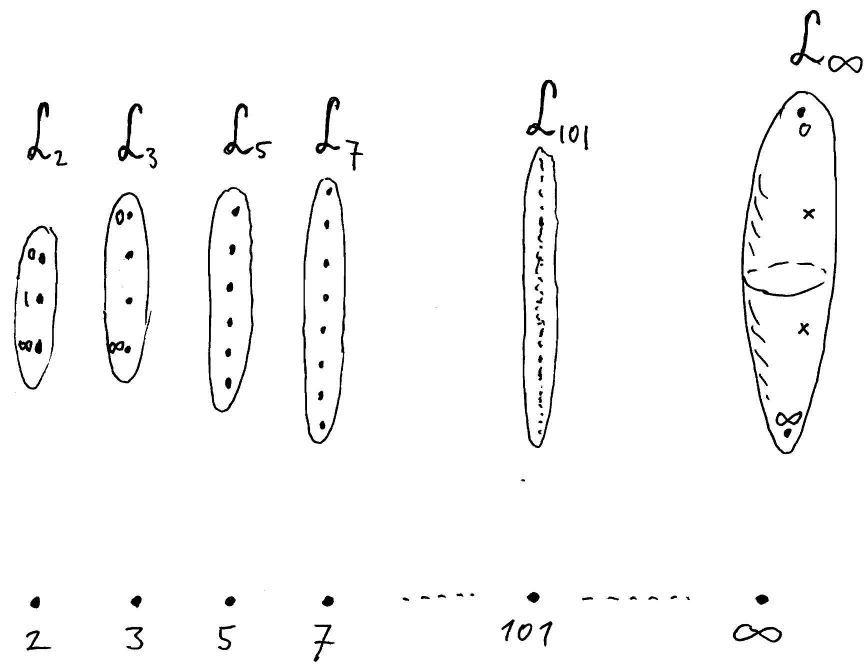
is an l-adic sheaf:

- $x : Spec(k) \longrightarrow C$ point (in the scheme sense), $|k| = q = p^r$.
- $x^*(\mathcal{L})$ is $Gal(\overline{k}/k)$-representation.
- Frobenius $F_x \in Gal(\overline{k}/k)$ (inverse to) $a \longrightarrow a^q$.
- $F_x \in Aut(x^*(\mathcal{L}))$.

Frobenius F_x $\leftrightarrow$ Hecke-operator H_x

The glueing problem

Families over $\pi : X \longrightarrow \mathbb{P}^1 := \mathbb{P}^1_{\mathbb{Z}}$



For p prime get $X \mod p \longrightarrow \mathbb{P}^1 \mod p$ and thus l-adic sheaves $\mathcal{L}_p$.
Relation between $\mathcal{L}_p$?

Provisional definition:
Condition on $P_x(t) = det(F_x - t)$ characteristic polynomial of F_x.
For $\mathcal{L}$ of **rank** 2 **and weight** 1:

$$P_x(t) = (t - \alpha_0)(t - \alpha_1)$$

$$\alpha_0 = 1 \mod N, \quad \alpha_1 = p \mod N$$

If this happens for *almost all* p and $x \in C(\mathbb{F}_p)$, we say:
$\mathcal{L}$ is a N-*congruence sheaf.*

$$1 + p - Tr(F_x) \equiv 0 \mod N$$

For $\mathcal{L}$ of **rank** 4 **and weight** 3:

$$P(t) = (t-\alpha_0)(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)$$

$$\alpha_0 = 1 \mod N_1, \quad \alpha_3 = p^3 \mod N_1$$
$$\alpha_1 = p \mod N_2, \quad \alpha_2 = p^2 \mod N_2$$

If this happens for *almost all* p and $x \in C(\mathbb{F}_p)$, we say:
$\mathcal{L}$ is a (N_1, N_2)-*congruence sheaf*.

Works as glue between various $\mathcal{L}_p$

A special Hecke-algebra

Kontsevich (2005) *Notes on motives in finite characterisic*

Makes Langlands correspondence **explicit** for

- SL_2 -local systems
- on $\mathbb{P}^1 \setminus \{\text{four points}\}$
- with unipotent local monodromies

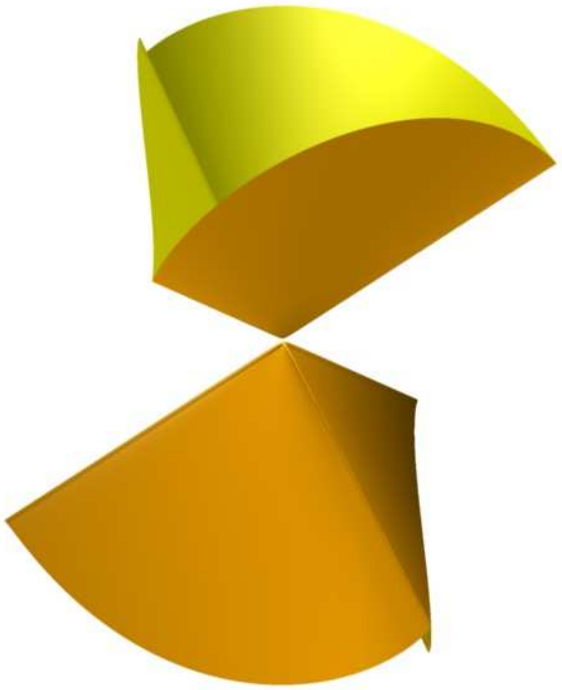
From data $\Sigma = \{\text{roots of } t^3 + at^2 + bt + c = 0, \infty\}$ get a polynomial in three variables:

$$P(x, y, z) = (b - xy - yz - xz)^2 - 4(xyz + c)(x + y + z + a)$$

It is the **discriminant** of the polynomial

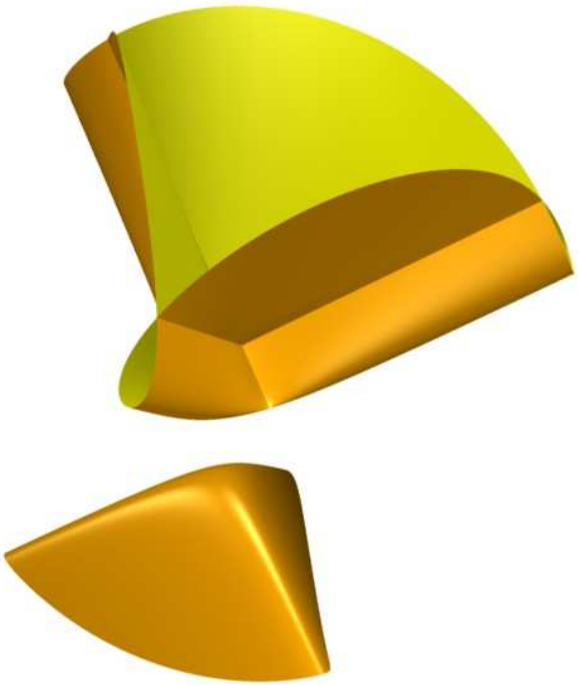
$$(t^3 + at^2 + bt + c) - (t - x)(t - y)(t - z)$$

Picture of the surface $P(x, y, z) = 0$



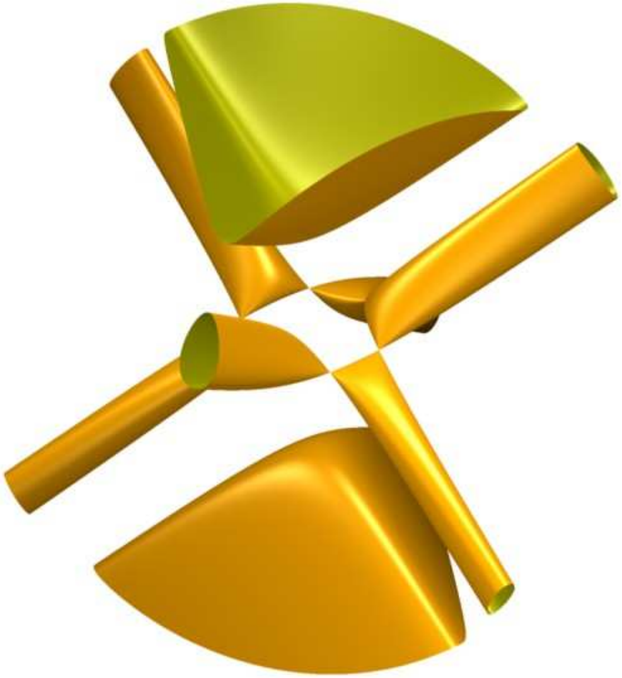
$$a = b = c = 0$$

Picture of the surface $P(x, y, z) = 0$



$a > 0, b = c = 0$

Picture of the surface $P(x, y, z) = 0$



Now take a prime number p and a point $x \in \mathbb{P}^1(\mathbb{F}_p)$.

Define a $(p+1) \times (p+1)$-matrix H_x by

$$(H_x)_{yz} = 2 + \#\{w \mid w^2 = P(x,y,z)\} + \text{correction}$$

For $x, y \in \mathbb{P}^1(\mathbb{F}_p)$ one has (miracle!)

$$[H_x, H_y] = 0$$

Hecke-algebra: algebra generated by $H_x, x \in \mathbb{P}^1(\mathbb{F})$.

Example $p = 7$, $a = 3, b = 4, c = 0$.

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hecke-algebra acts on **functions on** $\mathbb{P}^1(\mathbb{F}_p)$, written as vectors

$$v = (v_x), \quad x \in \mathbb{P}^1(\mathbb{F}_p)$$

$$(H_x v)_y = \sum_z (H_x)_{yz} v_z$$

Drinfelds theorem tells: $SL_2(\mathbb{F}_p)$-automorphic representations correspond to normalised common eigenvectors

$$H_x v = v_x . v, \quad v_x v_y = \sum_z H_{xyz} v_z$$

and hence to l-adic local systems $\mathcal{L}$ on $\mathbb{P}^1 \setminus \Sigma$.

$$\mathcal{L} \leftrightarrow v = (Tr(F_x)_{x \in \mathbb{P}^1})$$

Strategy

- Run over $p = 3, 5, 7, 11, \ldots ....$
- ...and all $f = t^3 + at^2 + bt + c \in \mathbb{F}_p[t]$.
- Determine rational common eigenvectors for Hecke-algebra.
- Sort them for various p using congruences.

Decomposition of the Hecke-algebra

Example: $p = 7$

a, b, c						
$0, 0, 1$	1	1	2	2	2	
$0, 0, 2$	1	1	6			
$0, 0, 3$	1	1	6			
$0, 1, 0$	1	1	2	2	2	
$0, 1, 1$	1	7				
$0, 1, 3$	1	3	4			
$0, 3, 0$	1	1	1	1	2	2
$0, 3, 1$	1	3	4			
$0, 3, 2$	1	7				
$0, 3, 3$	1	1	2	4		

The 8 non-trivial rational eigenvectors for $p = 7$.

a, b, c	Eigenvector v	N	j
$0, 0, 2$	$[\ -4\ \ 2\ \ 2\ \ -1\ \ 2\ \ -1\ \ -1\ \ 8]$	3	0
$0, 0, 3$	$[\ 5\ \ -4\ \ -4\ \ 2\ \ -4\ \ 2\ \ 2\ \ 8]$	3	0
$0, 1, 0$	$[\ -8\ \ 0\ \ 4\ \ -4\ \ -4\ \ 4\ \ 0\ \ 8]$	4	6
$0, 3, 3$	$[\ -2\ \ 8\ \ -2\ \ 3\ \ 3\ \ -2\ \ -2\ \ 8]$	5	4
$0, 3, 0$	$[\ 8\ \ -4\ \ 8\ \ -4\ \ 2\ \ 8\ \ 2\ \ 8]$	6	6
$0, 3, 0$	$[\ 8\ \ 2\ \ 8\ \ 2\ \ -4\ \ 8\ \ -4\ \ 8]$	6	6
$0, 3, 0$	$[\ 8\ \ 0\ \ -8\ \ 0\ \ 0\ \ -8\ \ 0\ \ 8]$	8	6
$0, 0, 1$	$[\ -1\ \ -1\ \ -1\ \ 8\ \ -1\ \ 8\ \ 8\ \ 8]$	9	0

N: the congruence.

j: j-invariant $\in \mathbb{F}_p$ of the four points .

Observations:

Running over $p = 3, 5, 7, 11, \ldots$ we see:

- There are very few $\mathbb{Q}$-Hecke-eigenvectors.
- All of these have a non-trivial congruence.
- There are no 7, 10, 11, 12, $\ldots$ congruences.
- $3, 4, 5, 6, 8$-congruences are persistent.

Focus on 5-congruence:

p	a, b, c	j	Eigenvector
7	0, 3, 3	4	$[-2, 8, -2, 3, 3, -2, -2, 8]$
11	0, 2, 0	1	$[12, -3, -3, 12, 2, -3, 2, -3, 12, 2, 2, 12]$
	0, 2, 0	1	$[12, 2, 2, 12, -3, 2, -3, 2, 12, -3, -3, 12]$
13	0, 2, 5	4	$[4, 4, -1, -6, -1, -1, -1, 4, 14, 4, 4, -6, -6, 14]$
17	0, 1, 2	1	$[3, -2, -2, 3, -2, -7, -2, -2, -2, 3, -2, -7, 3, 3, 8, 3, 18,$
19	0, 2, 4	15	$[5, 5, -5, 0, 20, 5, 0, -20, -20, 0, 0, 0, 0, -5, 0, 0, 5, -5, -5$

j-invariant for 5-congruence trace function

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59
j	4	1	4	1	15	14	12	0	34	5	2	24	13	44

Reconstruct $j \in \mathbb{Q}$ from reductions mod p (Wang algorithm):

$$j = \frac{2^{14}31^3}{5^3} \quad !$$

Simplest choice: $a = 11, b = -1, c = 0.$

Equation of type $D2$

Special *Heun equation* with Riemann symbol

$$\left\{ \begin{array}{cccc} 0 & * & * & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right\}$$

$$\mathcal{P} = f\partial^2 + f'\partial + t - \lambda$$

where

$$f = t(t^2 + at + b)$$

Singular points: 0, roots of $t^2 + at + b$, ∞.
λ: the accesory parameter.

Multiplication theorem:

Let

$$\phi(t) = \phi_\lambda(t) = 1 + b_1(\lambda)t + b_2(\lambda)t^2 + \dots$$

be solution of differential equation

$$\mathcal{P}\phi(t) = 0$$

Then one has:

$$\phi(x)\phi(y) = \frac{1}{2\pi i}\oint K(x,y,z)\phi(z)\frac{dz}{z}$$

where

$$K(x,y,z) = \frac{b}{\sqrt{P(x,y,b/z)}}$$

Complete analogy:

Langlands for $C = \mathbb{P}^1(\mathbb{F}_p)$

$$v_x v_y = \sum_z H_{xyz} v_z$$

Geometric Langlands for $C = \mathbb{P}^1(\mathbb{C})$:

$$\phi(x)\phi(y) = \frac{1}{2\pi i}\oint K(x,y,z)\phi(z)\frac{dz}{z}$$

Change from point-basis to power-basis Functions on $\mathbb{F}_p \subset \mathbb{P}^1(\mathbb{F}_p)$

$$\delta_x, \quad x \in \mathbb{F}_p \leftrightarrow x^n, \quad n = 0, \ldots, p-1$$

$$v \leftrightarrow \phi_p(x) = \sum_{n=0}^{p-1} a_n x^n$$

Inverse Vandermonde-Transformation

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 2^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & p-1 & \ldots & (p-1)^{p-1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{p-1} \end{pmatrix} = \begin{pmatrix} v_{p-1} \\ \vdots \\ v_1 \\ v_0 \end{pmatrix}$$

$p = 7$:

Eigenvector	8	-2	-2	-2	3	3	-2
Transform	1	3	5	0	5	4	1

$p = 13$:

Eigenvector	14	4	4	-6	-6	4	4	-1	-6	-1	-1	-1	4
Transform	1	3	6	4	3	8	10	5	3	9	6	10	1

$p = 43$:

44	-6	4	-1	-6	4	4	4	-6	9	-1	-11	4	4	...
1	3	19	18	4	30	39	42	39	21	32	24	9	12	...

$$x = 0 \mod 7, \quad x = 4 \mod 13, \quad x = 18 \mod 43$$

$$\Rightarrow x = 147 \mod 7 \cdot 13 \cdot 43$$

We get a series

$$\phi(t) = 1 + 3t + 19t^2 + 147t^3 + 1251t^4 + \ldots$$

with

$$\phi(t) \equiv \phi_p(t) \mod p$$

Apéry numbers

$$1, \quad 3, \quad 19, \quad 147, \ldots$$

$$a_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

Generating function

$$\phi(t) := \sum_{n=0}^{\infty} a_n t^n$$

satisfies the differential equation

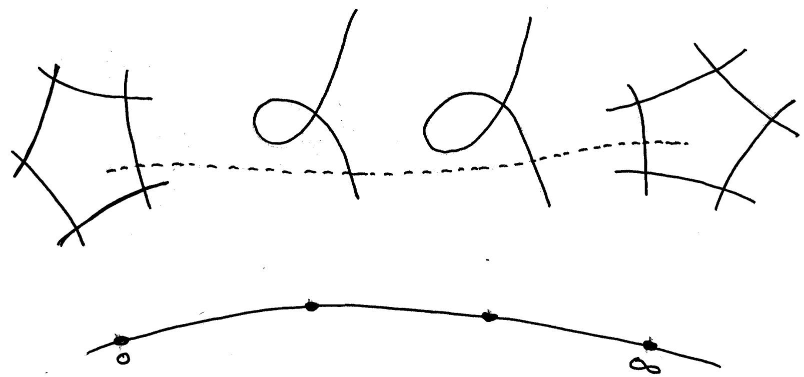
$$(D^2 - t(11D^2 + 11D + 3) + t^2(D+1)^2)\phi(t) = 0$$

F. Beukers (1981) Modular interpretation

Operator arises as Picard-Fuchs operator of modular elliptic surface for $\Gamma_1(5)$.

$$\pi : X \longrightarrow \mathbb{P}^1$$

with singular four singular fibres of type I_1, I_1, I_5, I_5.



Number of points in fibre $X_t(\mathbb{F}_p)$ divisible by 5.

$$1 + p - Tr(F_p) \equiv 0 \mod 5$$

So indeed: $R^1\pi_*(\mathbb{Q}_l)$ is a 5-congruence sheaf! **What did we do?**

- □ Use explicit SL_2-Hecke-algebra.
- □ Found 5-congruence trace-functions.
- □ Found from it Apéry sequence.
- □ Found arithmetic D2-equation.

It works for all Beukers-Zagier-Beauville operators.
Bonus: Get not only differential equation, but also get the Frobenius traces of all fibres!

What one might hope to do

- ☐ Find explicit Hecke-algebras and multiplication theorems for general Heun-equations.
- ☐ Same for $D3$s
- ☐ Find explicit Hecke-algebra for Sp_4-case.
- ☐ Find bi-congruence trace-functions.
- ☐ We claim: is substitute for modularity of D2 and D3!
- ☐ Find corresponding differential operators of CY-type.