# A $\mathbb{Z}^N$-graded generalization of the Witt algebra

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# 1 Generalized Witt Algebras

Let $\Lambda$ be a $\mathbb{Z}$-lattice of rank $N > 0$ and $\mathfrak{g} = \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda$ be a $\Lambda$-graded Lie algebra, i.e., $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. $\mathfrak{g}$ is said to be **simple-graded** if $\dim \mathfrak{g} \geq 2$ and there is no non-trivial proper graded ideal. We assume that each homogeneous component is of finite dimension. Here, our ground field is always fixed as an algebraically closed field $k$ of characteristic 0.

## 1.1 Background

The problem on classification of $\Lambda$-graded Lie algebras that are *simple-graded* is an old problem which is still open. When the rank of $\Lambda$ is 1, the problem had been completely resolved by O. Mathieu [M] in 1992. In the case when the rank of $\Lambda$ is greater than 1, only a partial solution was known and given by

K. I. and O. Mathieu in our recent paper [IM] . There still has no conjectural form for general cases.

To be precise, in [IM], we have classified $\Lambda$-graded *simple-graded* Lie algebras $\mathfrak{g} = \bigoplus_{\lambda\in\Lambda} \mathfrak{g}_\lambda$ where $\dim \mathfrak{g}_\lambda = 1$ for any $\lambda \in \Lambda$,[1] these consists of two classes:

1. $\mathfrak{g}$ is of type $A_1^{(2)}$ or $A_2^{(2)}$,
2. $\mathfrak{g}$ is a generalization of the Witt algebra.

Let us explain the second case in detail.

## 1.2 A generalization of the Witt algebra

Let $\langle \cdot, \cdot \rangle$ be a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^2$ ;

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \det \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix},$$

and set $\rho = (1, 1)$. Let $\mathcal{L}$ be the Lie algebra of the symbols of twisted pseudo differential operators, i.e.,

$$\mathcal{L} = \bigoplus_{\lambda\in\mathbb{C}^2} \mathbb{C} L_\lambda$$

as vector space with its Lie bracket $[\cdot, \cdot]$ satisfying

$$[L_\lambda, L_\mu] = \langle \lambda + \rho, \mu + \rho \rangle L_{\lambda+\mu}.$$

Let $\pi : \Lambda \hookrightarrow \mathbb{C}^2$ be an inclusion and $W_\pi$ be the subalgebra of $\mathcal{L}$ generated by $\{L_\lambda\}_{\lambda\in\pi(\Lambda)}$. This is clearly a $\Lambda$-graded Lie algebra with multiplicity 1.

**Remark 1.1.** *The next identity is the key identity to verify the Jacobi identity of the above Lie bracket*

$$\langle \lambda, \mu \rangle \nu + \langle \mu, \nu \rangle \lambda + \langle \nu, \lambda \rangle \mu = 0,$$

*which is true for any* $\lambda, \mu, \nu \in \mathbb{C}^2$.

The next lemma is easy to verify:

[1] In case when $\dim \mathfrak{g}_\lambda \leq 1$, nothing about the classification, even its conjectural form, is known !

**Lemma 1.2** (cf. [IM])**.** *$W_\pi$ is simple-graded iff* $\operatorname{Im}\pi \not\subset \mathbb{C}\rho$ *and* $2\rho \notin \operatorname{Im}\pi$*.*

When $\operatorname{Im}\pi$ is contained in a one-dimensional subspace of $\mathbb{C}^2$, the commutation relation simplifies as

$$[L_\lambda, L_\mu] = \langle \rho, \mu - \lambda \rangle L_{\lambda+\rho}.$$

In such a case, $W_\pi$ is called a generalized Witt algebra by W. T. Yu Ruppert. In this sense, $W_\pi$ for a generic $\pi$ is a `generalized generalized Witt algebra`.....

**Remark 1.3.** *It can be shown that* $\dim H_2(\mathcal{L}) \leq 1$ *and the equality holds iff* $\mathcal{L}$ *is a generalized Witt algebra.*

# 2 Some Representations

Let us look at representations of the Lie algebra $\mathfrak{g} = W_\pi$. To simplify the notation, we identify $\Lambda$ with its image in $\mathbb{C}^2$ via $\pi$. Here, we describe '$\Lambda$-graded' $\mathfrak{g}$-module $M$ whose multiplicity is a constant, say $C \in \mathbb{N}^*$.

## 2.1 The case $\Lambda = \mathbb{Z}$

Let us recall the known result due to I. Kaplansky and L. J. Santharoubane [KS] for the Witt algebra, i.e., when $\Lambda = \mathbb{Z}$;

$$\mathbf{W} = \mathbb{C}[z^{\pm 1}]\frac{d}{dz}, \qquad [L_m, L_n] = (n-m)L_{m+n},$$

where we set $L_m := z^{m+1}\dfrac{d}{dz}$. Here are examples:

1. For $(u,\delta) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$, $\Omega_u^\delta := \bigoplus_{x \in u} \mathbb{C}e_x^\delta$ with
$$L_m.e_x^\delta := (m\delta + x)e_{x+m}^\delta.$$

2. The *A-family* $(A_{a,b})_{(a,b)\in\mathbb{C}^2}$. Here, $A_{a,b}$ is the $\mathbf{W}$-module with basis $\{e_n^A\}_{n\in\mathbb{Z}}$ and the action given by the formula:
$$L_m.e_n^A := \begin{cases} (m+n)e_{m+n}^A & n \neq 0, \\ (am^2 + bm)e_m^A & n = 0. \end{cases}$$

3. The *B-family* $(B_{p,q})_{(p,q)\in\mathbb{C}^2}$. Here, $B_{p,q}$ is the $\mathbf{W}$-module with basis $\{e_n^B\}_{n\in\mathbb{Z}}$ and the action gien by the formula:

$$L_m.e_n^B := \begin{cases} ne_{m+n}^B & m+n \neq 0, \\ (pm^2+qm)e_0^B & m+n=0. \end{cases}$$

Essentially, these modules exhaust all such $\mathbf{W}$-modules for $C=1$, i.e., it is known (cf. [KS] and [M]) that

1. if $M$ is indecomposable and $C=1$, then $M$ is isomorphic to one of the above three modules, and
2. if $M$ is irreducible $\mathbf{W}$-module, then $C=1$ and it is given by 1.

**Remark 2.1.** *1. The A-family is a deformation of* $\Omega_0^1$.

*2. The B-family is a deformation of* $\Omega_0^0$.

## 2.2 The case $\Lambda = \mathbb{Z}^N$ $(N>1)$

In this case, fix $\alpha \in \Lambda \subset \mathbb{C}^2$ such that $\langle \rho, \alpha\rangle \neq 0$ and set $\mathfrak{a} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_{n\alpha}$. The Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is isomorphic to $\mathbf{W}$. The results explained here will be explained in a paper in preparation with O. Mathieu.

A natural generalization of the $\mathbf{W}$-modules of type $\Omega_u^\delta$ is given as follows: for $u \in \mathbb{C}^2/\Lambda$, we set $M_u := \bigoplus_{\mu\in u}\mathbb{C}L_\mu \subset \mathcal{L}$. Then, $M_u$ is naturally a $\mathfrak{g}$-module by the adjoint action:

$$L_\lambda.L_\mu = \langle \lambda+\rho, \mu+\rho\rangle L_{\lambda+\mu},$$

with $\lambda \in \Lambda$ and $\mu \in u$. It is easy to see that

1. $M_u$ is irreducible iff $u \cap \{-\rho, -2\rho\} = \emptyset$,
2. For each $\mathbb{Z}\alpha$-coset $\gamma \subset u$, $M_u[\gamma] := \bigoplus_{\mu\in\gamma}\mathbb{C}L_\mu$ is an $\mathfrak{a}$-submodule isomorphic to $\Omega_u^\delta$ for some $(u,\delta) \in \mathbb{C}/\mathbb{Z}\times\mathbb{C}$.

Secondly, we consider the case $-\rho \in u$. For $(a,b) \in \mathbb{C}^2$, we set

$$\mathcal{A}_{a,b} := \bigoplus_{\substack{\gamma\subset u \\ -\rho\notin\gamma}} M_u[\gamma] \oplus A_{a,b}.$$

One can introduce a structure of $\mathfrak{g}$-module on $\mathcal{A}_{a,b}$ which extends its $\mathfrak{a}$-module structure. It an indecomposable $\mathfrak{g}$-module iff $(a,b) \neq (0,0)$.

Finally, we consider the case $-2\rho \in u$. For $(p,q) \in \mathbb{C}^2$, we set

$$\mathcal{B}_{p,q} := \bigoplus_{\substack{\gamma \subset u \\ -2\rho \notin \gamma}} M_u[\gamma] \oplus B_{p,q}.$$

One can introduce a structure of $\mathfrak{g}$-module on $\mathcal{B}_{p,q}$ which extends its $\mathfrak{a}$-module structure. It is an indecomposable $\mathfrak{g}$-module iff $(p,q) \neq (0,0)$.

We have

**Theorem 2.2.** *Assume that $\rho \notin \operatorname{Im} \pi$. Then, any indecomposable $\Lambda$-graded $\mathfrak{g}$-module $M$ with multiplicity 1 is isomorphic to one of the above three modules.*

So now, we assume that $\rho \in \operatorname{Im} \pi$. In this case, in addition to the above three types of $\mathfrak{g}$-modules, there is one another class which we define below. Suppose that $\rho, 2\rho \in u$. For $(a,b), (p,q) \in \mathbb{C}^2$, we set

$$\mathcal{AB}_{a,b;p,q} = \bigoplus_{\substack{\gamma \in u \\ -\rho, -2\rho \notin u}} M_u[\gamma] \oplus A_{a,b} \oplus B_{p,q}.$$

One can introduce a structure of $\mathfrak{g}$-module on $\mathcal{AB}_{a,b;p,q}$ which extends its $\mathfrak{a}$-module structure. It is an indecomposable $\mathfrak{g}$-module iff $(a,b), (p,q) \neq (0,0)$.

We can also show the next theorem:

**Theorem 2.3.** *Assume that $\rho \notin \operatorname{Im} \pi$. For each $C \in \mathbb{N}$ such that $C \geq 3$, there is a $\Lambda$-graded irreducible $\mathfrak{g}$-module $M$ whose multiplicity is $C$.*

One can also consider an analogue of Verma modules. But, in general, such modules are reducible !

# References


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