Kählerian K3 surfaces and Niemeier lattices

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The talk is based on my papers:

[1] Finite groups of automorphisms of Kählerian K3 surfaces (1979), (ann. (1976)).

[2] Integral symmetric bilinear forms and some their geometric applications (1979).

[3] Preprint arXiv:1109.2879v7 (last variant in August 2013).

During the 30 years gap, important papers by Mukai (1988), Xiao (1996), Kondō (1998), Hashimoto (2010), others were published.

But, from my point of view, some important general point was missing. It is why I decided to write the preprint.

Today, I shall also consider important applications of results of this preprint.

Kälerian K3 X is a compact complex surface with $K_X = 0$ ($(\omega_X) = 0$), $q(X) = 0$.

$H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$.

$H^2(X, \mathbb{Z}) \cong L_{K3} = 3U \oplus 2E_8$.

$H^{2,0}(X) = \mathbb{C}\omega_X \subset H^2(X, \mathbb{Z}) \otimes \mathbb{C}$.

Picard lattice

$$S_X = \{x \in H^2(X, \mathbb{Z}) \mid x \cdot \omega_X = 0\} = H^2(X, \mathbb{Z}) \cap H^{1,1}(X).$$

$S_X \subset H^2(X, \mathbb{Z})$ is a primitive sublattice.

Three Cases:

a) $S_X < 0$ (general case);

b) $S_X \leq 0$ with 1-dimensional Kernel;

c) S_X is hyperbolic (sign. $(1, t_{(-)})$ (X is algebraic.)

For periods, Global Torelli Theorem, we usually consider *Marked K3:* $(X,\ \alpha)$

$$\alpha : H^2(X, \mathbb{Z}) \cong L_{K3}.$$

Further, *General Case:* a) $S_X < 0$:

The main idea of preprint [3]: Introduce ADDITIONAL MARKING

$$\tau : S_X \subset N_i$$

τ is prim. embedding, N_i is one of Niemeier lattices: even, < 0, unimodular lattices of rank 24 (Niemeier, 1973):

$N_1 = N(D_{24})$, $N_2 = N(D_{16} \oplus E_8)$,
$N_3 = N(3E_8)$, $N_4 = N(A_{24})$,
$N_5 = N(2D_{12})$, $N_6 = N(A_{17} \oplus E_7)$,
$N_7 = N(D_{10} \oplus 2E_7), N_8 = N(A_{15} \oplus D_9)$,
$N_9 = N(3D_8)$, $N_{10} = N(2A_{12})$,
$N_{11} = N(A_{11} \oplus D_7 \oplus E_6), N_{12} = N(4E_6)$,

$N_{13} = N(2A_9 \oplus D_6), N_{14} = N(4D_6),$
$N_{15} = N(3A_8), N_{16} = N(2A_7 \oplus 2D_5),$
$N_{17} = N(4A_6), N_{18} = N(4A_5 \oplus D_4),$
$N_{19} = N(6D_4), N_{20} = N(6A_4),$
$N_{21} = N(8A_3), N_{22} = N(12A_2),$
$N_{23} = N(24A_1),\ N_{24} = N(\emptyset) = Leech.$

This marking does exist. It follows from the general result from Nik, [2].

Notations: *Discriminant quadratic form* of an even lattice S:

$$q_S : A_S = S^*/S \to \mathbb{Q}/2\mathbb{Z}.$$

$l(A)$ is the *minimal number of generators* of a finite Abelian group A.

A_p is the *p-component* of A.

$K(q_p)$ is a *p-adic lattice (over $\mathbb{Z}_p$)* of the rank $l(A_{q_p})$ with the discriminant form q_p on a p-group A_{q_p}.

Theorem 1 (from Nik, [2]). For even lattice S with $(t_{(+)}, t_{(-)}, q_S)$, integers $(l_{(+)}, l_{(-)})$, a primitive embedding $S \subset L$ into one of unimodular even L of signature $(l_{(+)}, l_{(-)})$ exists iff
(1) $l_{(+)} - l_{(-)} \equiv 0 \bmod 8$;
(2) $l_{(+)} - t_{(+)} \geq 0$, $l_{(-)} - t_{(-)} \geq 0$, $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} \geq l(A_S)$;
(3) $(-1)^{l_{(+)} - t_{(+)}} |A_S| \equiv \det K(q_{S_p}) \bmod ({\mathbb{Z}_p}^*)^2$ for each odd prime p such that $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_p})$;
(4) $|A_S| \equiv \pm \det K(q_{S_2}) \bmod ({\mathbb{Z}_2}^*)^2$, if $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_2})$ and $q_{S2} \ncong q_\theta^{(2)}(2) \oplus q'$.

If the last inequality in (2) is strict, one does not need (3) and (4). If $q_{S2} \cong q_\theta^{(2)}(2) \oplus q'$, one does not need (4).

From Theorem 1, we obtain at once:

Theorem 2. Any Kählerian K3, X has marking by one of Niemeier lattices

$$N_1, N_2, \dots, N_{24}.$$

Actually by

$$N_1, N_2, \dots, N_{23}$$

(we can exclude difficult Leech lattice) if we apply Kondō's trick.

Proof: Primitively $S = S_X \subset L_{K3}$. $\operatorname{rk} L_{K3} = 22$. Thus (by Theorem 1),

$$\operatorname{rk} S + l(A_S) \leq 22.$$

Then

$$\operatorname{rk} S + l(A_S) < 24, \quad \operatorname{rk} S \leq 19 < 24\,.$$

Then (by Theorem 1), S has primitive $S \subset N_i$ for one of $i = 1, 2, \dots, 24$.

Kondō's trick. Repeat the same for

$$S_1 = S \oplus \langle -2 \rangle.$$

$\operatorname{rk} S_1 + l(A_{S_1}) \leq 22+2 = 24$, $\operatorname{rk} S_1 \leq 19+1 < 24.$

$\operatorname{rk} S_1 + l((A_{S_1})_p) \leq 22 < 24,\ if\ \ p \neq 2.$

$$\operatorname{rk} S_1 + l((A_{S_1})_2) \leq 22 + 2 = 24,$$

$$S_1 \otimes \mathbb{Z}_2 = S \otimes \mathbb{Z}_2 \oplus K_\theta(2)$$

and primitively $S \subset N_i$ where N_i has elements with square (-2). Therefore, N_i is different from the Leech lattice.

END of proof.

Opposite: by epimorph. of Torelli map:

Theorem 3. Primitive $S \subset N_i$ $(i = 1, 2, \ldots, 24)$ is Picard lattice of a K3 surface $X \iff$

there exists a K3 X with marking $S_X = S \subset N_i$ above

iff exists prim. embedd. $S \subset L_{K3}$ $\iff$

S satisf. Th 1 for $(l_{(+)} = 3, l_{(-)} = 19)$.

The main necessary conditions are: $\operatorname{rk} S \le 19$, $\operatorname{rk} S + l(A_S) \le 22$.

Main Idea: Depending on which N_i, $i = 1, \ldots, 24$, gives marking of X, it has different geometry, arithmetic.

Theorem 4. (preprint Nik, [3]) For each $i = 1, \ldots, 23$, $i \neq 4, 10$, $\exists\, X$ which can be marked by N_i only. All Niemeier lattices are important for Kählerian K3.

Why marking $S_X \subset N_i$ important for X?
(Nik, [1], [2, Rem. 1.14.7], (1979)).

(A) *Using this marking, we can find all non-singular rational curves on X.*

Classes of non-singular rational curves $E \subset X$ correspond to basis $P(S_X) = P(X)$ of the root system $\Delta(S_X)$ of (-2)-roots $\delta \in S_X$, that is $\delta^2 = -2$. They generate the reflection group $W(S_X)$ and $P(S_X)$ is the set of $\perp$ roots to a fundamental chamber $\mathcal{M}$ of $W(S_X)$.

Let $P(N_i)$ is basis of the root system R_i of N_i.

Changing marking $\tau : S_X \subset N_i$ to $w\tau$ by $w \in W(N_i)$, we get:

$\tau(\Gamma(P(S_X)) \subset \Gamma(P(N_i))$ (*we can require*).

E.g., for $N_1 = N(D_{24})$, $N_2 = N(D_{16} \oplus$

$E_8)$, $\Gamma(P(S_X)) \subset \mathbb{D}_{24}$, $\mathbb{D}_{16}\mathbb{E}_8$.

(B) *Using marking* $S_X \subset N_i$, *we can find the finite symplectic automorphism group* Aut X.

We assume that $S = S_X < 0$. Then Aut X is finite and symplectic: $\phi^* \omega_X = \omega_X$.

$G =$ Aut X acts in $S = S_X$ without kernel. Coinvariant sublattice $S_G = (S^G)^{\perp}_S$ has properties:

1) $S_G < 0$,
2) S_G has no (-2)-roots,
3) $G|(S_G^*/S_G)$ is trivial,
4) $(S_G)^G = \{0\}$ (obvious).

Such S_G was called in Nik [2] *Leech type lattice.*

The same is valid for any K3 X and finite symplectic $G \subset$ Aut X. $H^2(X,\mathbb{Z})_G = (S_X)_G < 0$ and is Leech type lattice.

(B) We can calculate Aut X from marking
$S = S_X \subset N_i$.
We use

$$O(N_i) = W(N_i) \rtimes A(N_i)$$

where $A(N_i)$ permutes the root basis $P(N_i)$, the Dynkin diagram.

Theorem 5. (Nik, [1],[2, Rem. 1.14.7])
(1) $G = \text{Aut } X =$
$\{\phi \in A(N_i) \mid \phi|(S)^{\perp}_{N_i} = id\}$;
(2) $G \subset A(N_i)$ is KahK3 subgroup (comes from (1) for some K3 X marking by N_i)
$\Longleftrightarrow$ Coinvariant sublattice of G
$(N_i)_G = ((N_i)^G)^{\perp}_{N_i}$ has
primitive $(N_i)_G \subset L_{K3}$ $\Longleftrightarrow$
$(N_i)_G$ satisfies Thm 1 for
$(l_{(+)} = 3,\ l_{(-)} = 19)$.

In preprint Nik. [3], for all

$$N_1, N_2, \ldots, N_{23}$$

all these KahK3 subgroups (conjugacy classes) $G \subset A(N_i)$ are calculated.

Example 1. $N_1 = N(D_{24})$, $N_2 = N(D_{16} \oplus E_8)$ have trivial $A(N_1)$, $A(N_2)$.

If X is marked by N_1, N_2, then Aut X is trivial and respectively
$\Gamma(P(X)) \subset \mathbb{D}_{24}, \quad \Gamma(P(X)) \subset \mathbb{D}_{16}\mathbb{E}_8.$
Any prim. $S_X = S \subset N_1,\ N_2$ satisfying Thm 1 for $(l_{(+)} = 3, l_{(-)} = 19)$ can be taken for this marking (corresponds to K3).

Example 2. $N_1 = N(D_{24})$, $N_2 = N(D_{16} \oplus E_8)$, $N_4 = N(A_{24})$, $N_5 = N(2D_{12})$, $N_{10} = N(2A_{12})$ have trivial KahK3 subgroups only.

Like for Example 1, if X is marked by one of them, then Aut X is trivial and respectively

$$\Gamma(P(X)) \subset \mathbb{D}_{24},\ \mathbb{D}_{16}\mathbb{E}_8,\ \mathbb{A}_{24},\ 2\mathbb{D}_{12},\ 2\mathbb{A}_{12}.$$

Any primitive
$S_X = S \subset N_i, \quad i = 1, 2, 4, 5, 10,$
satisfying Thm 1 for
$(l_{(+)} = 3, l_{(-)} = 19)$ can be taken for this marking (corresponds to K3).

Example 3. $N = N_3 = N(3E_8)$ has $A(N_3) = \mathfrak{S}_3$ on components $3\mathbb{E}_8$. Only cyclic $C_1, C_2 \subset \mathfrak{S}_3$ are KahK3 subgroups. For $C_2 = [(ij)]$ and $k \neq i, j$ (of course, $\{i, j, k\} = \{1, 2, 3\}$.)

$$(ij) : (\mathbb{E}_8)_k$$

$$(ij) : (\mathbb{E}_8)_i \leftrightarrow (\mathbb{E}_8)_j$$

$N_{[(ij)]} = [\delta_{1i} - \delta_{1j}, \dots, \delta_{8i} - \delta_{8j}] \cong$
$\cong E_8(2)$. ($\delta_{1k}, \dots, \delta_{8k}$ is basis of $(E_8)_k$.)
For marking $S_X = S \subset N_3$, Aut $X =$ C_1 (trivial) if $N_{[(ij)]} \not\subset S_X$ for all
$1 \leq i < j \leq 3$, and
Aut $X = [(ij)] \cong C_2$ otherwise.
Any prim. $S_X = S \subset N_3$, satisf. Th 1 for $(3, 19)$, can be taken (it gives K3).

In general, marking by Niemeier lattices razes the following general interesting question:

What are other, different from (A) and (B), properties of K3, X which follow from marking by Niemeier lattices N_i*?*

I know some.

RELATION TO OTHER RESULTS:

• In Nik, [1], (1976, 1979) I classified *finite Abelian symplectic automorphism groups of Kählerian K3*: 14 groups, action on $H^2(X,\mathbb{Z}) = L_{K3}$ is unique, moduli are connected.

E.g. cyclic groups are C_n, $n = 2,3,4,5,6,7,8$.

• Sh. Mukai (1988) classfied *abstract finite symplectic automorphism groups of Kählerian K3*: 80 groups.

• G. Xiao (1996) gave another proof of Mukai's result.

• Sh. Kondō (1998) gave another proof of Mukai's result using Niemeier lattices (similarly to my considerations today).

• K. Hashimoto (2010) proved uniquiness of action (with 5 exceptions) for non-abelian groups using Niemeier lattices (similarly to my considerations today). This result finalized my uniqueness results for Abelian groups (1979).

• My today's considerations can be also considered as some amplification of these results for K3 depending on their marking by Niemeier lattices:

For a fixed $i = 1, \ldots 23$, *what is finite symplectic automorphism group of K3 marked by* N_i *?*

Physicists (Gaberdiel, Hohenegger,Volpato, others) study symmetries of K3 σ-models.

Roughly speaking, even unimodular lattice $H^2(X,\mathbb{Z}) \cong L_{K3}$ of signature $(3,19)$ is replaced by the even unimodular lattice $H^*(X,\mathbb{Z}) \cong L_{K3}^*$ of signature $(4,20)$.

Picard lattice S_X is replaced by negative definite primitive $S \subset L_{K3}^*$. By Thm 1, then existence of primitive $S \subset L_{K3}^*$ is equivalent to existence of primitive $S \subset N_i$ for one of $1 \le i \le 24$.

Then symmetrices of K3 σ-models are

$$G \subset A(N_i),$$

with $\mathrm{rk}(N_i)_G \le 20 \iff \mathrm{rk}\, N_i^G \ge 4$.

The same, what I did for K3, can be repeated for K3 σ-models. Even, this case looks simpler and more natural.

Look preprint: arXiv:1106.4315

M.R. Gaberdiel,S. Hohenegger,
R. Volpato

Symmetries of K3 sigma models.

2011, 40 pages.

(A) and (B) both can be considered as degeneration (given by (A)) of a Kählerian K3 surface with finite symplectic automorphism group given by (B).

For example, a classical example of Kummer surface with $16\mathbb{A}_1$ (16 not intersected non-singular rational curves) can be considered as degeneration of codimension 1 of K3 surfaces with finite symplectic automorphism group $(C_2)^4$.

It is interesting to ask: Do K3 surfaces with finite symplectic automorphism group $(C_2)^4$ have other degenerations of codimension 1 which are different from Kummer?

What are degenerations of codimension 1 (or any codimension) of K3 surfaces with other finite symplectic automorphism groups?

Let me inform you about my recent results in this direction.

I give here only a very particular case, but the same can be done in general for all types of finite symplectic automorphism groups on Kählerian K3 surfaces.

These results can be considered as important application of my preprint Nik [3].

Degenerations of codim 1 of Kählerian K3 surfaces with maximal finite sympl. autom. group and which can be marked by N_{23} only

$n=21$, $H=(C_2)^4$: $\operatorname{rk} N_H=15$: DEGEN $16\mathbb{A}_1$, $4\mathbb{A}_1$, $\operatorname{rk} S=16$;

$n=39$, $H=2^4C_2$, $\operatorname{rk} N_H=17$: DEGEN $16\mathbb{A}_1$, $8\mathbb{A}_1$, $\operatorname{rk} S=18$;

$n=40$, $H=Q_8*Q_8$ ($|H|=32$), $\operatorname{rk} N_H=17$: DEGEN $8\mathbb{A}_1$, $\operatorname{rk} S=18$;

$n=49$, $H=2^4C_3$, $\operatorname{rk} N_H=17$: DEGEN $8\mathbb{A}_1$, $\operatorname{rk} S=18$;

$n=56$, $H=\Gamma_{25}a_1$ ($|H|=64$), $\operatorname{rk} N_H=18$: DEGEN $16\mathbb{A}_1$, $8\mathbb{A}_1$, $\operatorname{rk} S=19$;

$n=65$, $H=2^4D_6$, $\operatorname{rk} N_H=18$: DEG $16\mathbb{A}_1$, $12\mathbb{A}_1$, $8\mathbb{A}_1$, $4\mathbb{A}_1$, $\operatorname{rk} S=19$;

$n=75$, $H=4^2\mathfrak{A}_4$, $\operatorname{rk} N_H=18$: DEGEN $16\mathbb{A}_1$, $\operatorname{rk} S=19$;

Cases

$n = 76,\ H = H_{192}\ (|H| = 192),$

$n = 78,\ H = \mathfrak{A}_{4,4}\ (|H| = 288),$

$n = 80,\ H = F_{384}\ (|H| = 384),$

$n = 81,\ H = M_{20}\ (|H| = 960)$

have $\operatorname{rk} N_H = 19$ and no degenerations since $\operatorname{rk} S \le 19$.

Proof for $n = 21$, $H = (C_2)^4$: By Nik. [3], Kahlerian K3 conjugacy classes in $A(N_{23})$ are

$$H_{21,1} = [(\alpha_2\alpha_{20})(\alpha_3\alpha_{10})(\alpha_5\alpha_6)(\alpha_8\alpha_{11}) (\alpha_9\alpha_{21})(\alpha_{12}\alpha_{22})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{24}), (\alpha_2\alpha_{19})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})(\alpha_8\alpha_9) (\alpha_{11}\alpha_{21})(\alpha_{12}\alpha_{23})(\alpha_{17}\alpha_{22})(\alpha_{20}\alpha_{24}), (\alpha_1\alpha_{16})(\alpha_2\alpha_{20})(\alpha_3\alpha_6)(\alpha_5\alpha_{10}) (\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{18})(\alpha_{17}\alpha_{22})(\alpha_{19}\alpha_{24}), (\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10}) (\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits

$\{\alpha_1, \alpha_{16}, \alpha_{14}, \alpha_{18}\}$, $\{\alpha_2, \alpha_{20}, \alpha_{19}, \alpha_{24}\}$, $\{\alpha_3, \alpha_{10}, \alpha_5, \alpha_6\}$, $\{\alpha_8, \alpha_{11}, \alpha_9, \alpha_{21}\}$, $\{\alpha_{12}, \alpha_{22}, \alpha_{23}, \alpha_{17}\}, \{\alpha_4\}, \{\alpha_7\}, \{\alpha_{13}\}, \{\alpha_{15}\}$.

$$H_{21,2} =$$
$$[(\alpha_1\alpha_3)(\alpha_2\alpha_{23})(\alpha_5\alpha_{14})(\alpha_6\alpha_{16})$$
$$(\alpha_{10}\alpha_{18})(\alpha_{12}\alpha_{20})(\alpha_{17}\alpha_{24})(\alpha_{19}\alpha_{22}),$$
$$(\alpha_1\alpha_2)(\alpha_3\alpha_{23})(\alpha_5\alpha_{17})(\alpha_6\alpha_{12})$$
$$(\alpha_{10}\alpha_{22})(\alpha_{14}\alpha_{24})(\alpha_{16}\alpha_{20})(\alpha_{18}\alpha_{19}),$$
$$(\alpha_1\alpha_{16})(\alpha_2\alpha_{20})(\alpha_3\alpha_6)(\alpha_5\alpha_{10})$$
$$(\alpha_{12}\alpha_{23})(\alpha_{14}\alpha_{18})(\alpha_{17}\alpha_{22})(\alpha_{19}\alpha_{24}),$$
$$(\alpha_1\alpha_{14})(\alpha_2\alpha_{24})(\alpha_3\alpha_5)(\alpha_6\alpha_{10})$$
$$(\alpha_{12}\alpha_{22})(\alpha_{16}\alpha_{18})(\alpha_{17}\alpha_{23})(\alpha_{19}\alpha_{20})]$$

with orbits

$$\{\alpha_1, \alpha_3, \alpha_2, \alpha_{16}, \alpha_{14}, \alpha_{23}, \alpha_6, \alpha_5,$$
$$\alpha_{20}, \alpha_{24}, \alpha_{18}, \alpha_{12}, \alpha_{17}, \alpha_{10}, \alpha_{19}, \alpha_{22}\}$$
$$\{\alpha_4\}, \{\alpha_7\}, \{\alpha_8\}, \{\alpha_9\}, \{\alpha_{11}\}, \{\alpha_{13}\},$$
$$\{\alpha_{15}\}, \{\alpha_{21}\}.$$

For $H_{21,1}$ only orbits

$$\{\alpha_1,\alpha_{16},\alpha_{14},\alpha_{18}\},\ \{\alpha_2,\alpha_{20},\alpha_{19},\alpha_{24}\},$$

$$\{\alpha_3,\alpha_{10},\alpha_5,\alpha_6\},\ \{\alpha_8,\alpha_{11},\alpha_9,\alpha_{21}\},$$

$$\{\alpha_{12},\alpha_{22},\alpha_{23},\alpha_{17}\},$$

are possible (give primitive $S\subset L_{K3}$) by Theorem 1. We get $\mathbb{A}_4$.

For $H=H_{21,2}$ only the orbit

$$\{\alpha_1,\alpha_3,\alpha_2,\alpha_{16},\alpha_{14},\alpha_{23},\alpha_6,\alpha_5,$$

$$\alpha_{20},\alpha_{24},\alpha_{18},\alpha_{12},\alpha_{17},\alpha_{10},\alpha_{19},\alpha_{22}\}$$

is possible (gives primitive $S\subset L_{K3}$) by Theorem 1. We get $\mathbb{A}_{16}$: Kummer surface.

If we add nef polarization $h \in S^{\perp}_{L_{K3}}$, we get the same geometry: finite symplectic automorphism group, non-singular rational curves which are contracting by $|h|$: This will be a codim=1 subfamily in complex family of Kählerian K3 surfaces.