Rankin-Selberg methods for String Amplitudes

Boris Pioline

CERN & LPTHE

Institut Paul Painlevé, Lille, March 5, 2014

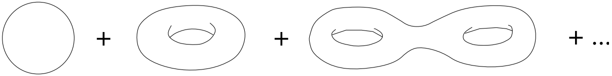
based on work with C. Angelantonj and I. Florakis,

arXiv:1110.5318,1203.0566,1304.4271,1401.4265 and work in progress

String theory in a nutshell I

- Scattering amplitudes of n external states in perturbative string theory have a topological expansion

$$\mathcal{A}_n = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathfrak{M}_{h,n}} \mathrm{d}\mu_{h,n}\, F_{h,n}$$



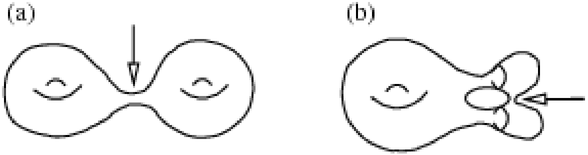
where $F_{h,n}$ is a correlator of n vertex operators (along with ghost insertions) in a certain SCFT on a Riemann surface Σ_h of genus h with n punctures z_i, integrated over the moduli space of super-Riemann surfaces $\mathfrak{M}_{h,n}$.

String theory in a nutshell II

- After integrating over the positions of the punctures and fermionic part of supermoduli, one is left with an integral over the (ordinary) moduli space of Riemann surfaces $\mathcal{M}_h$:

$$\mathcal{A}_h = \int_{\mathcal{M}_h} \mathrm{d}\mu_h\, F_h$$

- There is no canonical way of projecting the supermoduli space onto bosonic moduli space. Different projections differ by total derivatives on $\mathcal{M}_h$, which can in principle be fixed by matching with QFT behavior at the boundaries.



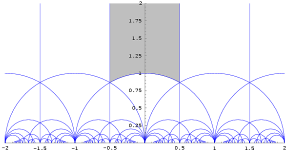
Donagi Witten

String theory in a nutshell III

- The moduli space $\mathcal{M}_h = \mathcal{T}_h / \Gamma_h$ is the quotient of the Teichmüller space $\mathcal{T}_h$ by the mapping class group Γ_h. The integrand is naturally a function on $\mathcal{T}_h$ invariant under Γ_h.
- $\mathcal{T}_h$ is the analog of the space of Schwinger/Feynman parameters in QFT, while Γ_h has no analog in QFT. The quotient by Γ_h is largely responsible for the UV finiteness of string theory.
- For genus $h \leq 3$, the Teichmüller space $\mathcal{T}_h$ is isomorphic to (an open set in) the Siegel-Poincaré upper half plane $\mathcal{H}_h$, parametrized by the period matrix Ω, a complex $h \times h$ symmetric matrix with positive definite imaginary part. The integrand $F_h(\Omega)$ is a Siegel modular form for $\Gamma_h = Sp(2h, \mathbb{Z})$, acting as $\Omega \mapsto (A\Omega + B) \cdot (C\Omega + D)^{-1}$.

String theory in a nutshell IV

- At genus 1, $\mathcal{T}_1$ is the Poincaré upper-half plane, parametrized by $\Omega_{11} \equiv \tau = \tau_1 + \mathrm{i}\tau_2$ and the integrand F_1 is invariant under $SL(2,\mathbb{Z})$. A convenient choice of fundamendal domain is



- τ_2 can be interpreted as a Schwinger parameter while τ_1 (for $\tau_2 > 1$) a Lagrange multiplier projecting the spectrum on level-matched states

String theory in a nutshell V

- E.g. the one-loop vacuum amplitude in bosonic closed string theory in $D = 26$ flat space time is proportional to

$$\mathcal{A}_1 = \int_{\mathcal{F}} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^{1+D/2}} \frac{1}{|\eta|^{2(D-2)}}$$

  where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function ($q = e^{2\pi i \tau}$)
- For genus 2, it takes 25 inequalities to define $\mathcal{F}_2$!
- For genus $h \geq 4$, $\mathcal{T}_h$ is a codimension $\frac{1}{2}(h-2)(h-3)$ locus inside $\mathcal{H}_h$ known as the Schottky locus. It is not clear how to extend F_h to a modular form on $\mathcal{H}_h$.

Rankin-Selberg method / unfolding trick I

- Our goal is to develop methods to compute integrals of Siegel modular forms over a fundamental domain of the Siegel upper-half plane analytically.
- The key idea is to represent the integrand as a Poincaré series,

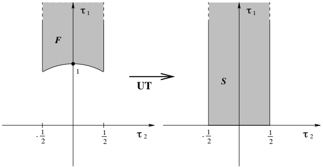
$$F_h(\Omega) = \sum_{\gamma \in \Gamma_{h,\infty} \backslash \Gamma_h} f_h|_\gamma(\Omega)$$

where $f_h|_\gamma(\Omega) = f_h(\gamma \cdot \Omega)$ and the 'seed' $f_h(\Omega)$ is invariant under a subgroup $\Gamma_{h,\infty} \subset \Gamma_h$. Typically, $\Gamma_{h,\infty}$ is the stabilizer of the cusp at infinity, acting by integer shifts of Ω_1.

Rankin-Selberg method / unfolding trick II

- Provided the sum is absolutely convergent, one can exchange the sum and integral and obtain

$$\int_{\Gamma_h\backslash\mathcal{H}_h} \mathrm{d}\mu_h\, F_h(\Omega) = \int_{\Gamma_{\infty,h}\backslash\mathcal{H}_h} \mathrm{d}\mu_h\, f_h(\Omega)\ .$$



- We gain if $\Gamma_{\infty,h}\backslash\mathcal{H}_h$ and f_h are simpler than $\Gamma_h\backslash\mathcal{H}_h$ and F_h !
- This method is limited by our ability to represent the integrand as a Poincaré series. Not much is known in genus $h > 1$. In genus one, any weakly, almost holomorphic modular form of negative weight can be represented as a Poincaré series.

Rankin-Selberg method / unfolding trick III

- We shall focus on a class of one-loop amplitudes of the form

$$\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu \, \varGamma_{d+k,d} \, \Phi(\tau) \, , \quad \mathrm{d}\mu = \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}$$

where $\Phi(\tau)$ is a weakly, almost holomorphic modular form of weight $w = -k/2$ (the elliptic genus) and $\varGamma_{(d+k,d)}$ is a Siegel Theta series (the Narain lattice partition function) for an even self-dual lattice (Γ, B) of signature $(d+k, d)$,

$$\varGamma_{(d+k,d)} = \tau_2^{d/2} \sum_{p \in \Gamma} e^{-\pi \tau_2 \mathcal{M}^2(p) + \pi \mathrm{i} \tau_1 \langle p,p \rangle}$$

- The positive definite quadratic form $\mathcal{M}^2(p)$ is parametrized by the orthogonal Grassmannian

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k) \times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a) \, ,$$

Rankin-Selberg method / unfolding trick IV

- Such modular integrals arise in certain BPS-saturated amplitudes, such as F^2, R^2, F^4, R^4 in type II string theory ($k=0$) or heterotic string ($k=8,16$) compactified on a torus T^d.
- $\mathcal{A}$ is invariant under T-duality, i.e. under the automorphisms of the lattice. Mathematically, $\Phi \mapsto \mathcal{A}$ is a Theta correspondence between $SL(2,\mathbb{Z})$ and $O(\Gamma_{d+k,d})$ automorphic forms.

Borcherds; Kudla Rallis

- In the physics literature, such integrals are typically computed the orbit method, i.e. by applying the unfolding trick to $\Gamma_{(d+k,d)}$. Instead, we shall apply the unfolding trick to $\Phi(\tau)$, which has the advantage of keeping T-duality manifest throughout.

Dixon Kaplunovsky Louis; Harvey Moore

Outline

Outline

Rankin-Selberg method I

- Consider the completed non-holomorphic Eisenstein series

$$E^\star(\tau;s) = \zeta^\star(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \tau_2^s|\gamma = \tfrac{1}{2}\,\zeta^\star(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\,\tau+d|^{2s}}$$

  where $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \zeta^\star(1-s)$.
- $E^\star(\tau;s)$ is convergent for $\mathrm{Re}(s)>1$, and has a meromorphic continuation to all s, invariant under $s\mapsto 1-s$, with simple poles at $s=0,1$ with constant residue:

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)\,,$$

Rankin-Selberg method (cont.)

- For any modular function $F(\Omega)$ of rapid decay, consider the Rankin-Selberg transform

$$\mathcal{R}^\star(F,s) = \int_{\mathcal{F}} \mathrm{d}\mu\, E^\star(\tau;s)\, F(\tau)$$

- By the unfolding trick, $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, F(\tau)$,

$$\begin{aligned}\mathcal{R}^\star(F;s) =& \zeta^\star(2s) \int_{\mathbb{R}^+\times[-\frac12,\frac12]} \mathrm{d}\mu\, \tau_2^s\, F(\tau)\\ =& \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, F_0(\tau_2)\,,\end{aligned}$$

Rankin-Selberg method (cont.)

- It inherits the meromorphicity and functional relations of $E^\star$, e.g. $\mathcal{R}^\star(F;s) = \mathcal{R}^\star(F;1-s)$.
- Since the residue of $E^\star(\tau;s)$ at $s=0,1$ is constant, the residue of $\mathcal{R}^\star(F;s)$ at $s=1$ is proportional to the modular integral of F,

$$\mathrm{Res}_{s=1}\mathcal{R}^\star(F;s) = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\mu\, F$$

- This was extended by Zagier to the case where $F^{(0)}$ is of power-like growth $F^{(0)}(\tau) \sim \varphi(\tau_2)$ at the cusp:

$$\mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = 2\,\mathrm{Res}_{s=1}\mathcal{R}^\star(F;s) + \delta$$

  where δ is a scheme-dependent correction which depends only on the leading behavior $\varphi(\tau_2)$.

Epstein series from modular integrals

- The RSZ method applies immediately to $\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}(g,B)$:

$$\mathcal{R}^\star(\Gamma_{d,d};s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s+d/2-2} \sum_{\langle p,p\rangle=0}' e^{-\pi\tau_2\,\mathcal{M}^2(p)}$$

$$= \zeta^\star(2s)\, \frac{\Gamma(s+\frac{d}{2}-1)}{\pi^{s+\frac{d}{2}-1}}\, \mathcal{E}^d_V(g,B;s+\tfrac{d}{2}-1)$$

where $\mathcal{E}^d_V(g,B;s)$ is the constrained Epstein series

$$\mathcal{E}^d_V(g,B;s) \equiv \sum_{\langle p,p\rangle=0}' \left[\mathcal{M}^2(p)\right]^{-s},$$

a.k.a. degenerate Langlands-Eisenstein series with infinitesimal character $\rho - 2s\alpha_1$

Epstein series and BPS state sums I

- This is identified as a sum over all BPS states of momentum m_i and winding n^i, with mass

$$\mathcal{M}^2(p) = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

  subject to the BPS condition $\langle p, p \rangle = m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

- The constrained Epstein Zeta series $\mathcal{E}^d_V(g, B; s)$ converges absolutely for $\mathrm{Re}(s) > d$. The RSZ method shows that it admits a meromorphic continuation in the s-plane satisfying

$$\mathcal{E}^{d\star}_V(s) = \pi^{-s}\,\Gamma(s)\,\zeta^\star(2s - d + 2)\,\mathcal{E}^d_V(s) = \mathcal{E}^{d\star}_V(d - 1 - s)\,,$$

  with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$ (double poles if $d = 2$).

Epstein series and BPS state sums II

- The residue at $s = \frac{d}{2}$ produces the modular integral of interest:

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}(g,B) = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\, \mathcal{E}^{d}_{V}\left(g,B;\tfrac{d}{2}-1\right)$$

rigorously proving an old conjecture of Obers and myself (1999).

- For $d = 2$, the BPS constraint $m_i n^i = 0$ can be solved, leading to

$$\mathcal{E}^{2\star}_{V}(T,U;s) = 2\, E^{\star}(T;s)\, E^{\star}(U;s)$$

hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\int_{\mathcal{F}} \left(\Gamma_{2,2}(T,U) - \tau_2\right)\, d\mu = -\log\left(T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right) + \text{cte}$$

Outline

Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon, $\Phi(\tau) \sim 1/q^\kappa + \mathcal{O}(1)$ with $\kappa = 1$.
- In mathematical terms, $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ is an almost, weakly holomorphic modular form with weight $w = -k/2 \leq 0$.
- The RSZ method fails, however the unfolding trick could still work provided $\Phi(\tau)$ can be represented as a uniformly convergent Poincaré series with seed $f(\tau)$ is invariant under $\Gamma_\infty : \tau \to \tau + n$,

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

- Convergence requires $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \to 0$. The choice $f(\tau) = 1/q^\kappa$ works for $w > 2$ but fails for $w \leq 2$.

Niebur-Poincaré series I

- A very convenient basis is provided by the Niebur-Poincaré series

$$\mathcal{F}(s,\kappa,w) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\tau)|_w\,\gamma$$

where the seed $f(\tau) = |4\pi\kappa\tau_2|^{-\frac{w}{2}} M_{-\frac{w}{2}\mathrm{sgn}(\kappa),s-\frac{1}{2}}(4\pi|\kappa|\tau_2) e^{-2\pi\mathrm{i}\kappa\tau_1}$ is chosen so that

$$f(\tau) \sim_{\tau_2\to 0} \tau_2^{s-\frac{w}{2}} e^{-2\pi\mathrm{i}\kappa\tau_1} \qquad f(\tau) \sim_{\tau_2\to\infty} \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})} q^{-\kappa}$$

- $\mathcal{F}(s,\kappa,w)$ converges absolutely for $\mathrm{Re}(s) > 1$ and satisfies

$$\left[\Delta_w + \tfrac{1}{2}(s-\tfrac{w}{2})(1-s-\tfrac{w}{2})\right] \mathcal{F}(s,\kappa,w) = 0$$

Niebur; Hejhal; Bruinier Ono Bringmann...

Niebur-Poincaré series II

- Under raising and lowering operators,

$$D_w = \frac{\mathrm{i}}{\pi}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right) , \qquad \bar{D}_w = -\mathrm{i}\pi\,\tau_2^2\partial_{\bar\tau} ,$$

  the NP series transforms as

$$D_w \cdot \mathcal{F}(s,\kappa,w) = 2\kappa\,(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2) ,$$
$$\bar{D}_w \cdot \mathcal{F}(s,\kappa,w) = \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2) .$$

- Under Hecke operators,

$$H_{\kappa'} \cdot \mathcal{F}(s,\kappa,w) = \sum_{d|(\kappa,\kappa')} d^{1-w}\,\mathcal{F}(s,\kappa\kappa'/d^2,w) .$$

- For congruence subgroups of $SL(2,\mathbb{Z})$, one can similarly define NP series $\mathcal{F}_{\mathfrak{a}}(s,\kappa,w)$ for each cusp.

Niebur-Poincaré series III

- For $s = 1 - \frac{w}{2}$, the value relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2-w)\left(q^{-\kappa} - \bar{q}^{\kappa}\sum_{\ell=0}^{-w}\frac{(4\pi\kappa\tau_2)^{\ell}}{\ell!}\right)$$

- For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain, but $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is in general NOT holomorphic, but rather a weakly harmonic Maass form,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m\, q^m + \sum_{m=1}^{\infty} m^{w-1}\,\bar{b}_m\,\Gamma(1-w,4\pi m\tau_2)\,q^{-m}$$

- For any such form, $\bar{D}\Phi = \tau_2^{2-w}\bar{\Psi}$ where $\Psi = \sum_{m\geq 1} b_m q^m$ is a holomorphic cusp form of weight $2-w$, the shadow of the Mock modular form $\Phi^{-} = \sum_{m=-\kappa}^{\infty} a_m\, q^m$.

Niebur-Poincaré series IV

- If $|w|$ is small enough, the negative frequency coefficients b_m vanish and Φ is in fact a weakly holomorphic modular form:

w	$\mathcal{F}(1-\frac{w}{2},1,w)$
0	$j+24$
-2	$3!\,E_4E_6/\Delta$
-4	$5!\,E_4^2/\Delta$
-6	$7!\,E_6/\Delta$
-8	$9!\,E_4/\Delta$
-10	$11!\,\Phi_{-10}$
-12	$13!/\Delta$
-14	$15!\,\Phi_{-14}$

Here Φ_{-10} and Φ_{-14} are genuine harmonic Maass forms with shadow $2.8402... \times \Delta$ and $1.3061... \times E_4\,\Delta$.

Niebur-Poincaré series V

- Theorem (Bruinier) : any weakly holomorphic modular form of weight $w \leq 0$ with polar part $\Phi = \sum_{0<m\leq\kappa} a_{-m}\, q^{-m} + \mathcal{O}(1)$ is a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{0<m\leq\kappa} a_{-m}\, \mathcal{F}(1-\tfrac{w}{2}, m, w) + a'_0\, \delta_{w,0}$$

  (The same holds for congruence subgroups of $SL(2,\mathbb{Z})$, including contributions from all cusps)

- Weakly almost holomorphic modular forms of weight $w \leq 0$ can similarly be represented as linear combinations of $\mathcal{F}(1-\frac{w}{2}+n, m, w)$ with $0 < m \leq \kappa, 0 \leq n \leq p$ where p is the depth. This fails for positive weight, as such forms are not necessarily harmonic !

Unfolding the modular integral

- By Bruinier's thm, any modular integral is a linear combination of

$$\mathcal{I}_{d+k,d}(s,\kappa) = \mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{d+k,d}(G,B,Y)\, \mathcal{F}(s,\kappa,-\tfrac{k}{2})$$

- Using the unfolding trick, one arrives at the BPS state sum

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& (4\pi\kappa)^{1-\frac{d}{2}}\, \Gamma(s+\tfrac{2d+k}{4}-1) \\ & \times \sum_{\substack{p\in\Gamma \\ \langle p,p\rangle=\kappa}} {}_2F_1\left(s-\tfrac{k}{4}\,,\, s+\tfrac{2d+k}{4}-1\,;\, 2s\,;\, \tfrac{4\kappa}{p_{\mathrm{L}}^2}\right) \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}}\end{aligned}$$

Bruinier; Angelantonj Florakis BP

where $p_{\mathrm{L}}^2 = \mathcal{M}^2(p) + 4\langle p,p\rangle$. This converges absolutely for $\mathrm{Re}(s) > \frac{2d+k}{4}$ and can be analytically continued to $\mathrm{Re}(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$.

Unfolding the modular integral

- For $s = 1 - \frac{w}{2} + n$, the values relevant for almost holomorphic modular forms, the summand can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1+\tfrac{k}{4},\kappa) = -\,\Gamma(2+\tfrac{k}{2}) \sum_{\mathrm{BPS}} \left[\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell} \right]$$

- The result is manifestly $O(\Gamma_{d+k,d})$ invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.

Fourier-Jacobi expansion I

- For $d=2, k=0$, the Fourier expansion in T_1 (or U_1) can be obtained by solving the BPS constraint. E.g. for $\kappa=1$, all solutions to $m_1 n^1 + m_2 n^2 = 1$ are

$$\begin{cases} m_1 = b + dM, \; n^1 = -c \\ m_2 = a + cM, \; n^2 = d \end{cases} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2,\mathbb{Z}) \, , M \in \mathbb{Z}$$

- After Poisson resumming over M, the sum over γ neatly produces Niebur-Poincaré series in U,

$$\begin{aligned} \mathcal{I}(s,1) =& 2^{2s}\sqrt{4\pi}\,\Gamma(s-\tfrac12)\, T_2^{1-s}\mathcal{E}(U;s) \\ & +4\sum_{N>0}\sqrt{\tfrac{T_2}{N}}\,K_{s-\frac12}(2\pi N T_2)\left[e^{2\pi \mathrm{i} N T_1}\underbrace{\mathcal{F}(s,N,0;U)}_{=H_N\cdot\mathcal{F}(s,1,0;U)}+\mathrm{cc}\right] \end{aligned}$$

Fourier-Jacobi expansion II

- The same result is obtained by the usual orbit method. In fact, both methods end up computing the same integral,

$$\int_{\mathcal{H}} \mathrm{d}\mu \, e^{-\pi T_2 \frac{|U-\tau|^2}{\tau_2 U_2}} \, \mathcal{F}(\tau) = 2\, T_2^{-1/2} e^{2\pi T_2} \, K_{s-\frac{1}{2}}(2\pi T_2) \, \mathcal{F}(U) \,,$$

  where $\mathcal{F}(\tau)$ is the seed of the NP series in the unfolding method, or the full NP series $\mathcal{F}(s, \kappa, 0; \tau)$ in the old orbit method.

  Bachas Fabre Kiritsis Obers Vanhove

- This formula works for any solution of $[\Delta + \frac{1}{2}s(1-s)]\mathcal{F}(\tau) = 0$, irrespective of modular invariance. It generalizes the average value property of harmonic functions.

  Fay

Fourier-Jacobi expansion III

- For $s = 1$, using $\mathcal{F}(1,1,0;U) = j(U) + 24$ one finds

$$\mathcal{A} = 8\pi\,\mathrm{Res}_{s=1}\left[T_2^{1-s}\mathcal{E}(s;U)\right] + 2\sum_{N>0}\left[\frac{q_T^N}{N}\,H_N^{(U)}\cdot[j(U)+24] + \mathrm{cc}\right]$$
$$= -24\log\left[T_2U_2|\eta(T)\eta(U)|^4\right] - \log|j(T)-j(U)|^4$$

consistently with Borcherds product

$$q_T\;[j(T)-j(U)] = \prod_{M>0,N\in\mathbb{Z}}(1-q_T^M q_U^N)^{c(MN)}\;,\quad j = \sum c(M)q^M$$

Borcherds; Harvey Moore

Fourier-Jacobi expansion IV

- For $s = 1 + n$, relevant for almost holomorphic modular forms of depth $p \geq n$, one can express $\mathcal{I}_{2,2}(n+1,1)$ as the iterated derivative of a generalized prepotential,

$$\mathcal{I}_{2,2}(n+1,1) = 4\,\mathrm{Re}\left[\frac{(-D_T D_U)^n}{n!} f_n(T,U)\right]$$

where f_n is holomorphic in T but harmonic in U,

$$\begin{aligned} f_n(T,U) =& 2\,(2\pi)^{2n+1}\,\mathcal{E}(n+1,-2n;U) \\ & + \sum_{N>0} \frac{2q_T^N}{(2N)^{2n+1}} \mathcal{F}(n+1,N,-2n;U) \end{aligned}$$

Fourier-Jacobi expansion V

- One can turn f_n into a holomorphic function $\tilde{f}_n(T, U)$ by replacing the Eisenstein series $\mathcal{E}(n+1, -2n; U)$ by its analytic part

$$\tilde{E}(n+1, -2n; \tau) = \frac{\zeta(2n+2)\,(2\pi \mathrm{i}\tau)^{2n+1}}{(-4\pi^2)^{n+1}} + \frac{1}{2}\zeta(2n+1) + \sum_{N\geq 1} \sigma_{-1-2n}(N)\, q^N$$

  without affecting the real part of its iterated derivative.
- The generalized holomorphic prepotential becomes

$$\tilde{f}_n(T, U) = \sum_{N,M} c_n(NM)\, \mathrm{Li}_{2n+1}(q_T^M\, q_U^N) + \Gamma(2n+2)\, \mathrm{Li}_{2n+1}\left(\frac{q_T}{q_U}\right)$$
$$+ \frac{(-1)^n\,(2\pi)^{2n+2}}{2\zeta(2n+2)} \left[\zeta(2n+1) + \frac{\zeta(-2n-1)}{\Gamma(2n+2)}(2\pi \mathrm{i} U)^{2n+1}\right]$$

  where $\mathcal{F}(n+1, 1, -2n) = \sum_{M\geq -1} c_n(M) q^M$.

Fourier-Jacobi expansion VI

- $\tilde{f}_n(T, U)$ now transforms as an Eichler integral of weight $(-2n, -2n)$ under $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \ltimes (T \leftrightarrow U)$,

$$(cU+d)^{2n}\,\tilde{f}_n\left(T, \frac{aU+b}{cU+d}\right) = \tilde{f}_n(T,U) + P_\gamma(U)\,,$$

  where $P_\gamma(U)$ is a computable polynomial of degree $\leq 2n$.
- The case $n=1$ describes the standard prepotential appearing in string vacua with $\mathcal{N}=2$ supersymmetry.

Antoniadis, Ferrara, Gava, Narain, Taylor; Harvey Moore

- The case $n=2$ has appeared in the context of 1/4-BPS amplitudes in Het/K_3. Higher n has not come up in physics yet, but is suggestive of CY_{2n+1}-fold.

Lerche Stieberger Warner

Outline

Rankin-Selberg method at higher genus I

- String amplitudes at genus $h \le 3$ take the form

$$\mathcal{A}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, \varGamma_{d+k,d,h}(G,B,Y;\Omega)\, \Phi(\Omega) \ , \quad \mathrm{d}\mu_h = \frac{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2}{[\det \Omega_2]^{h+1}}$$

  - $\mathcal{F}_h$ is a fundamental domain of the action of $\Gamma = Sp(2h,\mathbb{Z})$ on Siegel's upper half plane $\{\Omega = \Omega^t \in \mathbb{C}^{h\times h}, \Omega_2 > 0\}$
  - $\varGamma_{d+k,d,h}$ a Siegel-Narain theta series of signature $(d+k,d)$

$$\varGamma_{d+k,d,h} = [\det \Omega_2]^{d/2} \sum_{p_\alpha \in \Gamma_{d+k,d}, \alpha=1\dots h} e^{-\pi \Omega_2^{\alpha\beta} \mathcal{M}^2(p_\alpha,p_\beta) + 2\pi \mathrm{i} \Omega_1^{\alpha\beta} \langle p_\alpha, p_\beta \rangle}$$

  - $\Phi(\Omega)$ a Siegel modular form of weight $-k/2$.

- We would like to generalize the previous methods to the case where $\Phi(\Omega)$ is an almost holomorphic modular form with poles inside $\mathcal{F}_h$, such as $1/\chi_{10}$. As a first step, take $k=0$, $\Phi = 1$.

Rankin-Selberg method at higher genus II

- The genus h analog of $\mathcal{E}^\star(s;\tau)$ is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}_h^\star(s;\Omega) = \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} [\det \Omega_2]^s|\gamma$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \right\} \subset \Gamma$.

- The sum converges absolutely for $\mathrm{Re}(s) > \frac{h+1}{2}$ and can be meromorphically continued to the full s plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^\star(2j+1)$

Rankin-Selberg method at higher genus III

- For any modular function $F(\Omega)$ of rapid decay, the Rankin-Selberg transform can be computed by the unfolding trick,

$$\mathcal{R}^\star_h(F;s) = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, F(\Omega)\, \mathcal{E}^\star_h(\Omega,s)$$
$$= \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j) \int_{GL(h,\mathbb{Z})\backslash \mathcal{P}_h} \mathrm{d}\Omega_2 \, |\Omega_2|^{s-h-1} \, F_0(\Omega_2)$$

  where $\mathcal{P}_h$ is the space of positive definite real matrices, $|\Omega_2| = \det \Omega_2$ and $F_0(\Omega_2) = \int_0^1 \mathrm{d}\Omega_1 F(\Omega)$ is the constant term of F.
- The residue at $s = \frac{h+1}{2}$ is proportional to the average of F,

$$\mathrm{Res}_{s=\frac{h+1}{2}} \mathcal{R}^\star_h(F;s) = r_h \int_{\mathcal{F}_h} F \, .$$

Rankin-Selberg method at higher genus IV

- The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$\Gamma^{(0)}_{d,d,h}(g, B; \Omega) = |\Omega_2|^{d/2} \sum_{(m_i^\alpha, n^{i\alpha}) \in \mathbb{Z}^{2d\times h}, m_i^{(\alpha} n^{i\beta)}=0} e^{-\pi \Omega_{2\alpha\beta} \mathcal{M}^{2;\alpha\beta}}$$

  where

$$\mathcal{M}^{2;\alpha\beta} = (m_i^\alpha + B_{ik} n^{k\alpha}) g^{ij} (m_j^\beta + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

  Terms with $\mathrm{Rk}(m_i^\alpha, n^{i\alpha}) < h$ do not decay rapidly at $\Omega_2 \to \infty$. For $d < h$, this is always the case.
- The Siegel-Eisenstein series $\mathcal{E}^\star_h(\Omega, s)$ similarly has non-decaying constant term of the form $\sum_T e^{-\mathrm{Tr}(T\Omega_2)}$ with $\mathrm{Rk}(T) < h$.

- The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a field theory-type amplitude, with BPS states running in the loops,

$$\mathcal{R}_h(\Gamma_{d,d,h};s) = \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \frac{\mathrm{d}\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\mathrm{BPS}} e^{-\pi \mathrm{Tr}(\mathcal{M}^2\Omega_2)}$$
$$= \Gamma_h(s - \tfrac{h+1-d}{2}) \sum_{\mathrm{BPS}} \left[\det \mathcal{M}^2\right]^{\frac{h+1-d}{2}-s}$$

where

$$\sum_{\mathrm{BPS}} = \sum_{\substack{(m_i^\alpha, n^{i\alpha})\in\mathbb{Z}^{2d\times h}, \\ m_i^{(\alpha} n^{i\beta)}=0, \det\mathcal{M}^2\neq 0}} , \qquad \Gamma_h(s) = \pi^{\frac{1}{4}h(h-1)} \prod_{k=0}^{h-1} \Gamma(s - \tfrac{k}{2})$$

Rankin-Selberg method at higher genus VI

- For $d > h$, this is recognized as the Langlands-Eisenstein series of $SO(d,d,\mathbb{Z})$ with infinitesimal character $\rho - 2(s - \frac{h+1-d}{2})\lambda_h$, associated to $\Lambda^h V$ where V is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d}; s) \propto \mathcal{E}^{SO(d,d)}_{\Lambda^h V}(s - \tfrac{h+1-d}{2}) \qquad (h > d)$$

- The modular integral of $\Gamma_{d,d,h}$ is proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h}; s)$ at $s = \frac{h+1}{2}$, up to a scheme dependent term δ which remains to be computed. For $d < h$, the entire result ought to come from δ.

- For $d = 1$, any h,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}) \,, \quad \mathcal{V}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h = 2 \prod_{j=1}^{h} \zeta^\star(2j)$$

- For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\begin{aligned} \mathcal{R}_2^\star(\Gamma_{2,2}, s) =& 2\zeta^\star(2s)\zeta^\star(2s-1)\zeta^\star(2s-2) \\ & \times [\mathcal{E}_1^\star(T; 2s-1) + \mathcal{E}_1^\star(U; 2s-1)] \end{aligned}$$

hence

$$\mathcal{A}_2 = 2\zeta^\star(2)\,[\mathcal{E}_1^\star(T;2) + \mathcal{E}_1^\star(U;2)]$$

proving the conjecture by Obers and BP (1999).

Outline

Black holes in $D = 4$ and instantons in $D = 3$ I

- 1/4-BPS black holes in $\mathcal{N} = 4$ CHL-type vacua are counted by a Siegel modular form of genus 2 and weight k, where $r = 2k + 8$ is the rank of the lattice of electric charges ($k = 10$ in Het/T^6). Invariance under $G_4 = SL(2, \mathbb{Z}) \times SO(6, r - 6, \mathbb{Z})$ is manifest.

Dijkgraaf Verlinde Verlinde; David Jatkar Sen

- Suitable BPS couplings in $D = 3$ (e.g. $\nabla^2 R^2$) should receive instanton corrections from 1/4-BPS black holes in $D = 4$, along with KK monopoles. Yet they should be invariant under the 3D U-duality group $G_3 = SO(8, r - 4, \mathbb{Z})$.
- 1/4-BPS black holes in $M/K3 \times T^4$ can be represented by M5-branes wrapping around genus 2 curve $\Sigma \subset T^4$. On the heterotic side, one should include all genus 2 wordsheet instantons in T^7, plus NS5 and KK monopoles.

Gaiotto Dabhokar

Black holes in $D = 4$ and instantons in $D = 3$ II

- Thus, it is natural to conjecture

$$f^{D=3}_{\nabla^2 R^2} = \int_{\mathcal{F}_2} \frac{\mathrm{d}^3\Omega \mathrm{d}^3\bar{\Omega}}{(\det \mathrm{Im}\Omega)^3} \, \frac{Z_{8,r-4}(\Omega)\ \hat{E}_2(\Omega)\,(\det \mathrm{Im}\Omega)^4}{\Phi_k}$$

  where $Z_{8,r-4}$ is the partition function of the non-perturbative lattice, and $\hat{E}_2$ is the almost holomorphic Eisenstein series of weight 2.

- At weak heterotic coupling, one should recover the two-loop amplitude, invariant under $SO(7, r-5, \mathbb{Z})$, plus other perturbative corrections.

Black holes in $D = 4$ and instantons in $D = 3$ III

- At large radius, one should recover a sum over 1/4-BPS states, weighted by their entropy, along with KK monopoles

$$f^{D=3}_{\nabla^2 R^2} \stackrel{?}{=} f^{D=4}_{\nabla^2 R^2} + \sum_{\gamma\neq 0} \Omega(\gamma)\, e^{-R\mathcal{M}(\gamma)} + \sum_{k\neq 0} e^{-R^2 k} + \dots$$

Since $\Omega(\ell\gamma) \sim e^{\ell^2}$ while $\mathcal{M}(\ell\gamma) \sim \ell$, the series is at best asymptotic...

Conclusion - Outlook

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- T-duality and singularities from enhanced gauge symmetry are manifest. Fourier-Jacobi expansions can be obtained in some cases by solving the BPS constraint.
- The RSZ method also works at higher genus, at least for $h = 2, 3$. For computing modular integrals with $\Phi \neq 1$ it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as $1/\Phi_{10}$.
- Non-BPS amplitudes where Φ is not almost weakly holomorphic are challenging ! So are amplitudes with $h \geq 4$!