Classification results for (1)

Automorphic products of sing weight

work in progress

1. Aut products
2. The unimodular case
3. The prime level case
4. Reflective forms

1. Automorphic products

L even lattice of sign $(n,2)$, $n>2$ with disc form D $(=L'/L)$

F mod form for ρ_D, weight $1-n/2$

holomorphic on H, int princ part

$$\phi(Z) = \int_{\mathcal{F}} \Theta(Z,\tau)\, F(\tau)\, y \,\frac{dx\,dy}{y^2}$$

$$\phi(z) = \log |\Psi(z)|$$

$\Psi(z)$ is an aut form for $O(L,F)^+$ of weight $\frac{1}{2}[F_0](0)$. Its <u>divisors</u> are det by the principal part of F.

$\Psi(z)$ is called the <u>aut prod</u> corre-sponding to L and F.

$\Psi(z)$ has <u>sing weight</u> if it has weight $n/2 - 1$.

smallest possible weight of a holom form

These functions are sometimes the denom functions of infinite dim Lie alg

2. The unimod case

$n = 2 \bmod 8$

L even unimod lattice of sign $(n,2)$, $n > 2$

$\Psi(F)$ holomorphic aut prod of sing weight on L.

F holomorphic on H, weight $1 - n/2$, int principal part

$$F(\tau) = \sum_{m \geq -m_0} c_m q^m \qquad c_{-m_0} \neq 0$$

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{m \geq 1} \sigma_{k-1}(m) q^m$$

Eisenstein series of weight $k = 1 + n/2$

$\Rightarrow \quad k = 2 \bmod 4$

$E_k F$ is a weight 2 form with a pole at ∞

$\Rightarrow$ const coeff of $E_k F$ is 0

res th

$$\Rightarrow \quad 2(k-2) - \frac{2k}{B_k} \sum_{m \geq 1} \sigma_{k-1}(m)\, c_{-m} = 0$$

$\Rightarrow$ $k \equiv 2 \mod 12$:

$B_k (k-2) \in \mathbb{Z}$

von Staudt - Clausen $\quad B_k + \sum_{(p-1) | k} \frac{1}{p} \in \mathbb{Z}$

$$\underbrace{F}_{2-k} \ \underbrace{\Delta^{(k-2)/12}}_{\text{weight } k-2} \qquad \text{holomorphic on } H, \text{ weight } 0$$

possible pole at ∞

$$\Rightarrow \quad m_0 \geq \frac{k-2}{12}$$

$\psi(\tau)$ holomorphic

$\Rightarrow$ $\underline{\displaystyle\sum_{m\geq 1} \sigma_{k-1}(m)\, c_{-m} \geq m_0^{k-1}}$

$\Rightarrow$ $\underline{\left(\frac{k-2}{12}\right)^{k-1} \leq m_0^{k-1} \leq (k-2)\,\frac{B_k}{k}}$

$\Rightarrow$ $\underline{k=14}$ and equality in this case:

$$\left(\frac{k-2}{12}\right)^{k-1} \leq (k-2)\,\frac{B_k}{k}$$

$$1 \leq \underbrace{\frac{(k-2)^2}{12k}\left(\frac{12}{k-2}\right)^k B_k}$$

$$\sim \frac{1}{6}\sqrt{2\pi}\, e^2\, k^{3/2} \left(\frac{6}{\pi e}\right)^k \xrightarrow[k\to\infty]{} 0$$

$$B_k \sim 2\,\frac{k!}{(2\pi)^k}\,, \quad k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$$

hence finitely many sol

$\Rightarrow \quad n = 26, \quad m_0 = 1$

$F = q^{-1} + 24 + \dots$ weight -12

$= 1/\Delta$

Th

Let L be an even unimod lattice of sign $(n,2)$, $n>2$ and $\Psi(F)$ a holom aut prod of sing weight on L.

Then L is isom to $II_{26,2}$ and $F = 1/\Delta$.

The prod exp of $\Psi(F)$ corresponding to a prim norm 0 vector in $II_{26,2}$ is given by

$$e((\rho,z)) \prod_{\alpha \in II_{25,1}^{+}} (1 - e((\alpha,z)))^{[1/\Delta](-\alpha^2/2)}$$

$$= \sum_{w \in W} \det(w)\, e((w\varsigma, z)) \prod_{m \geq 1} \left(1 - e((wm\varsigma, z))\right)^{24}$$

This is the denom id of the fake monster alg

3. The prime level case

Th

Let D be a disc form of prime level p. Then up to isom there are only finitely many lattices of sign $(n,2)$ with $n > 2$ and disc form D carrying a holomorphic aut prod of sing weight. There is an explicit bound on n dep on p. Bound increases with p

5. Reflective forms

L even lattice of sign $(n,2)$, $n > 2$
with disc form D

F mod form for ρ_D, weight $1 - n/2$,
holomorphic on H

F is called <u>reflective</u> if F has only
poles of the form $q^{-1/k}$ coming
from comp F_γ with γ correspon-
ding to roots

In this case the divisors of $\Psi(L,F)$
are zeros of order 1 orth to roots
of L (In part $\Psi(L,F)$ is holom)

F_L mod form for $\rho_{L'/L}$

$M \subset L$ finite index sublattice

Then F_L induces a mod form F_M for $\rho_{M'/M}$

The corresponding aut prod are equal.

(Both live on the set of 2-dim, neg def, pos oriented subspaces of $V = L \otimes \mathbb{R} = M \otimes \mathbb{R}$)

Th (work in progress)

Let L be a lattice of prime level p and F a reflective mod form on L such that $\Psi(F,L)$ has sing weight. Then (F,L) is one of the following functions or induced by one of them

p	L	f
2	$II_{18,2}(2_{II}^{+10})$	$\eta_{1^{-8}2^{-8}}$
	$II_{10,2}(2_{II}^{+2})$, $II_{10,2}(2_{II}^{+10})$	$\eta_{1^{-16}2^{8}}$ $\eta_{1^{8}2^{-16}}$
	$II_{6,2}(2_{II}^{-6})$	$\eta_{1^{4}2^{-8}}$ $(2^{-4}4^{8})$
3	$II_{14,2}(3^{-8})$	$\eta_{1^{-6}3^{-6}}$
	$II_{8,2}(3^{-3})$, $II_{8,2}(3^{-7})$	$\eta_{1^{-9}3^{3}}$ $\eta_{1^{3}3^{-9}}$
	$II_{4,2}(3^{-5})$	$\eta_{1^{1}3^{-3}}$ $(3^{-1}9^{3})$

p	L	f
5	$II_{10,2}(5^{+6})$	$\eta_{1^{-4}5^{-4}}$
	$II_{6,2}(5^{+3})$, $II_{6,2}(5^{+5})$	$\eta_{1^{-5}5^{1}}$ $\eta_{1^{1}5^{-5}}$
7	$II_{8,2}(7^{-5})$	$\eta_{1^{-3}7^{-3}}$
11	$II_{6,2}(11^{-4})$	$\eta_{1^{-2}11^{-2}}$
23	$II_{4,2}(23^{-3})$	$\eta_{1^{-1}23^{-1}}$

All these functions correspond to aut of the Leech lattice

This classifies reflective aut prod of sing weight on lattices of prime level.