

Feynman Integrals, Regulators and Elliptic Polylogarithms

Pierre Vanhove

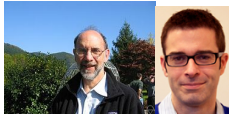


Automorphic Forms, Lie Algebras and String Theory

March 5, 2014

based on [arXiv:1309.5865] and work in progress

Spencer Bloch, Matt Kerr



There are many technics to compute amplitudes in field theory

- ▶ On-shell (generalized) unitarity
- ▶ On-shell recursion methods
- ▶ twistor geometry, Graßmanian, Symbol,...
- ▶ Dual conformal invariance
- ▶ Infra-red behaviour (inverse soft-factors, ...)
- ▶ amplitude relations,
- ▶ String theory,...

These methods indicate that amplitudes have simpler structures than the diagrammatic from Feynman rules suggest

The questions are:

what are the basic building blocks of the amplitudes?

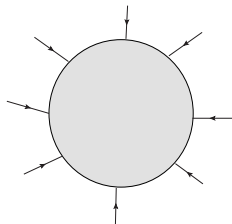
Can the amplitudes be expressed in a basis of integrals functions?

Part I

One-loop amplitudes

The one-loop amplitude

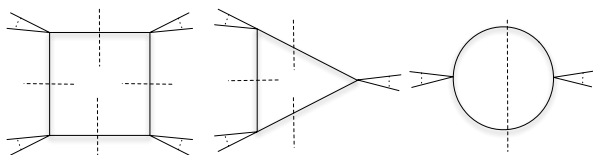
- ▶ In four dimensions any one-loop amplitude can be expressed on a basis of integral functions [Bern, Dixon, Kosower]



$$= \sum_r c_r \text{Integral}_r + \text{Rational terms}$$

The one-loop amplitude

- ▶ The integral functions are the box, triangle, bubble, tadpole



- ▶ The integrals functions are given by dilogarithms and logarithms

$$\text{Boxes, Triangles} \sim \text{Li}_2(z) = - \int_0^z \log(1-t) d \log t$$

$$\text{Bubble} \sim \log(1-z) = \int_0^z d \log(1-t)$$

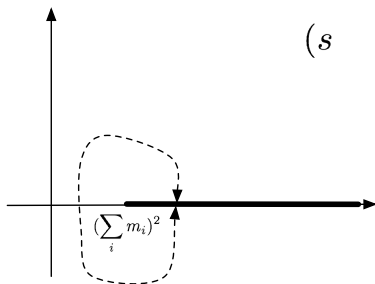
The one-loop amplitude

This allows to characterize in a simple way one-loop amplitudes in various gauge theory

- ▶ Only boxes for $\mathcal{N} = 4$ SYM for one-loop graph
- ▶ No triangle property of $\mathcal{N} = 8$ SUGRA [Bern, Carrasco, Forde, Ita, Johansson; Bjerrum-Bohr, Vanhove]
- ▶ Only box for QED multi-photon amplitudes with $n \geq 8$ photons [Badger, Bjerrum-Bohr, Vanhove]

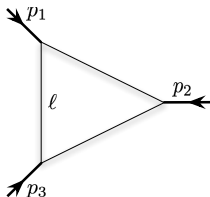
Monodromies, periods

- ▶ Amplitudes are multivalued quantities in the complex energy plane with monodromies around the branch cuts for particle production



- ▶ They satisfy differential equation with respect to its parameters : kinematic invariants s_{ij} , internal masses m_i , ...
- ▶ monodromies with differential equations : typical of periods integrals

A one-loop example I



We consider the 3-mass triangle $p_1 + p_2 + p_3 = 0$ and $p_i^2 \neq 0$

$$I_{\triangleright}(p_1^2, p_2^2, p_3^2) = \int_{\mathbb{R}^{1,3}} \frac{d^4 \ell}{\ell^2 (\ell + p_1)^2 (\ell - p_3)^2}$$

Which can be represented as

$$I_{\triangleright} = \int_{\substack{x \geq 0 \\ y \geq 0}} \frac{dx dy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)}$$

A one-loop example II

and evaluated as

$$I_{\triangleright} = \frac{D(z)}{(p_1^4 + p_2^4 + p_3^4 - (p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2))^{\frac{1}{2}}}$$

z and $\bar{z}$ roots of $(1-x)(p_3^2 - x p_1^2) + p_2^2 x = 0$

- ▶ Single-valued Bloch-Wigner dilogarithm for $z \in \mathbb{C} \setminus \{0, 1\}$

$$D(z) = \Im(Li_2(z)) + \arg(1-z) \log|z|$$

- ▶ The permutation of the 3 masses: $\{z, 1 - \bar{z}, \frac{1}{z}, 1 - \frac{1}{z}, \frac{1}{1-z}, -\frac{\bar{z}}{1-\bar{z}}\}$
this set is left invariant by the $D(z)$
- ▶ The integral has branch cuts arising from the square root since $D(z)$ is analytic

The triangle graph motive I

$$I_{\triangleright} = \int_{\substack{x \geq 0 \\ y \geq 0}} \frac{dx dy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)}$$

The integral is defined over the domain $\Delta = \{[x, y, z] \in \mathbb{P}^2, x, y, z \geq 0\}$ and the denominator is the quadric

$$C_{\triangleright} := (p_1^2 x + p_2^2 y + p_3^2 z)(xy + xz + yz)$$

Let $L = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ and $D = \{x + y + z = 0\} \cup C_{\triangleright}$

$$\frac{dx dy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)} \in H := H^2(\mathbb{P}^2 - D, L \setminus (L \cup C_{\triangleright}) \cap L, \mathbb{Q})$$

We need to consider the relative cohomology because the domain Δ is not in $H_2(\mathbb{P}^2 - D)$ because $\partial\Delta \cap \neq \emptyset$

The triangle graph motive II

Since $\partial\Delta \cap \mathcal{C}_\triangleright = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ one needs to perform a blow-up these 3 points.

One can define a mixed Tate Hodge structure [Bloch, Kreimer] with weight $W_0H \subset W_2H \subset W_4H$ and grading

$$gr_0^W H = \mathbb{Q}(0), \quad gr_2^W H = \mathbb{Q}(-1)^5, \quad gr_4^W H = \mathbb{Q}(-2)$$

The Hodge matrix and unitarity

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2i\pi & 0 \\ -Li_2(z) & 2i\pi \log z & (2i\pi)^2 \end{pmatrix} \begin{pmatrix} \text{triangle} & 0 & 0 \\ \text{circle} & 0 & 0 \\ \text{triangle} & 0 & 0 \end{pmatrix}$$

- ▶ The construction is valid for all one-loop amplitudes in four dimensions

Part II

Loop amplitudes

Feynman parametrization

- ▶ Typically form of the Feynman parametrization of a graph Γ
- ▶ A Feynman graph with L loops and n propagators

$$I_{\Gamma} \propto \int_0^{\infty} \delta(1 - \sum_{i=1}^n x_i) \frac{\mathcal{U}^{n-(L+1)\frac{D}{2}}}{(\mathcal{U} \sum_i m_i^2 x_i - \mathcal{F})^{n-L\frac{D}{2}}} \prod_{i=1}^n dx_i$$

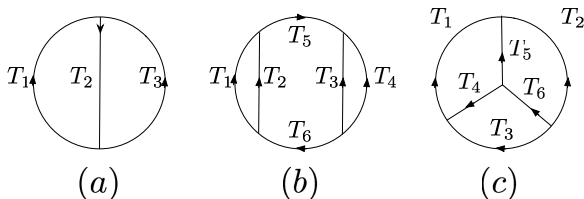
- ▶ $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials [Itzykson, Zuber]
- ▶ $\mathcal{U}$ is of degree L and $\mathcal{F}$ of degree $L+1$

Feynman parametrization and world-line formalism

- Rewrite the integral as

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1 - \sum_{i=1}^n x_i)}{(\sum_i m_i^2 x_i - \widehat{\mathcal{F}})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n dx_i}{\mathcal{U}^{\frac{D}{2}}}$$

- $\mathcal{U} = \det \Omega$ is the determinant of the period matrix of the graph



- The period matrix of integral of homology vectors v_i on oriented loops C_i

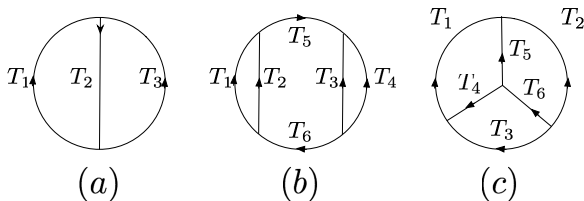
$$\Omega_{ij} = \oint_{C_i} v_j$$

Feynman parametrization and world-line formalism

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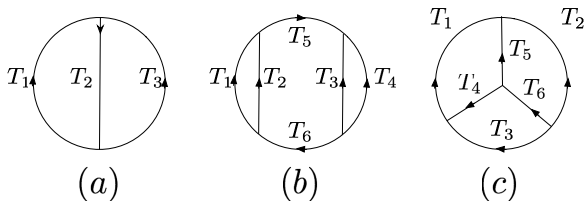
$$\Omega_{2(a)} = \begin{pmatrix} T_1 + T_3 & T_3 \\ T_3 & T_2 + T_3 \end{pmatrix}$$

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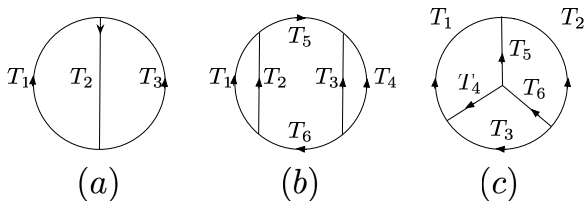
$$\Omega_{3(b)} = \begin{pmatrix} T_1 + T_2 & T_2 & 0 \\ T_2 & T_2 + T_3 + T_5 + T_6 & T_3 \\ 0 & T_3 & T_3 + T_4 \end{pmatrix}$$

Feynman parametrization and world-line formalism

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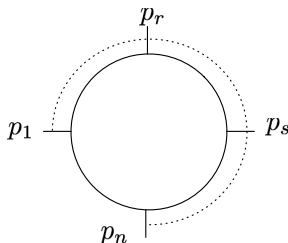
$$\Omega_{3(c)} = \begin{pmatrix} T_1 + T_4 + T_5 & T_5 & T_4 \\ T_5 & T_2 + T_5 + T_6 & T_6 \\ T_4 & T_6 & T_3 + T_4 + T_6 \end{pmatrix}$$

Feynman parametrization and world-line formalism

- Rewrite the integral as

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1 - \sum_{i=1}^n x_i)}{(\sum_i m_i^2 x_i - \hat{\mathcal{F}})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n dx_i}{\mathcal{U}^{\frac{D}{2}}}$$

- $\hat{\mathcal{F}} = \sum_{1 \leq r < s \leq n} k_r \cdot k_s G(x_r, x_s; \Omega)$ sum of Green's function



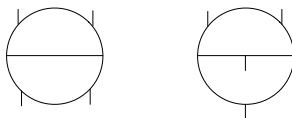
$$G^{1-loop}(x_r, x_s; L) = -\frac{1}{2} |x_s - x_r| + \frac{1}{2} \frac{(x_r - x_s)^2}{T}.$$

Feynman parametrization and world-line formalism

- Rewrite the integral as

$$I_{\Gamma} \propto \int_0^{\infty} \frac{\delta(1 - \sum_{i=1}^n x_i)}{(\sum_i m_i^2 x_i - \hat{\mathcal{F}})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n dx_i}{u^{\frac{D}{2}}}$$

- $\hat{\mathcal{F}} = \sum_{1 \leq r < s \leq n} k_r \cdot k_s G(x_r, x_s; \Omega)$ sum of Green's function



$$G_{\text{same line}}^{2\text{-loop}}(x_r, x_s; \Omega_2) = -\frac{1}{2}|x_s - x_r| + \frac{T_2 + T_3}{2} \frac{(x_s - x_r)^2}{(T_1 T_2 + T_1 T_3 + T_2 T_3)}$$

$$G_{\text{diff line}}^{2\text{-loop}}(x_r, x_s; \Omega_2) = -\frac{1}{2}(x_r + x_s) + \frac{T_3(x_r + x_s)^2 + T_2 x_r^2 + T_1 x_s^2}{2(T_1 T_2 + T_1 T_3 + T_2 T_3)}$$

Periods

- ▶ A Feynman graph with L loops and n propagators

$$I_\Gamma \propto \int_0^\infty \delta(1 - \sum_{i=1}^n x_i) \frac{u^{n-(L+1)\frac{D}{2}}}{(u \sum_i m_i^2 x_i - \mathcal{F})^{n-L\frac{D}{2}}} \prod_{i=1}^n dx_i$$

- ▶ [Kontsevich, Zagier] define periods as follows.

$\mathcal{P} \in \mathbb{C}$ is the ring of periods, is $z \in \mathcal{P}$ if $\Re(z)$ and $\Im(z)$ are of the forms

$$\int_{\Delta \in \mathbb{R}^n} \frac{f(x_i)}{g(x_i)} \prod_{i=1}^n dx_i < \infty$$

with $f, g \in \mathbb{Z}[x_1, \dots, x_n]$ and Δ is algebraically defined by polynomial inequalities and equalities.

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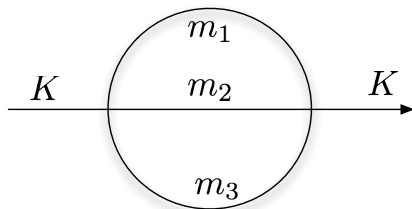
with $f, g \in \mathbb{Z}[x_1, \dots, x_n]$ and Δ is algebraically defined by polynomial inequalities and equalities.

- ▶ Problem for Feynman graphs $\partial\Delta \cap \{g(x_i) = 0\} \neq \emptyset$
- ▶ Generally the domain of integration is not closed $\partial\Delta \neq \emptyset$
- ▶ Need to consider the relative cohomology and perform blow-ups
[Bloch, Esnault, Kreimer]

Part III

The banana graphs

the two-loop sunset integral



We consider the two-loop sunset integral in two Euclidean dimensions given by

$$\mathcal{I}_{\Theta}^2 \propto \int_{\mathbb{R}^4} \frac{d^2 l_1 d^2 l_2}{(\ell_1^2 + m_1^2)(\ell_2^2 + m_2^2)((\ell_1 + \ell_2 - K)^2 + m_3^2)}$$

- ▶ Related to $D = 4$ by dimension shifting formula [Tarasov; Baikov; Lee]
- ▶ Expression given in term of elliptic function [Laporta, Remiddi; Adams, Bogner, Weinzeirl]

the two-loop sunset integral

The Feynman parametrisation is given by

$$J_{\ominus}^2 = \int_{\substack{x \geq 0 \\ y \geq 0}} \frac{dx dy}{(m_1^2 x + m_2^2 y + m_3^2)(x + y + xy) - K^2 xy} = \int_{\mathcal{D}} \omega.$$

- One has the remarkable representation

$$J_{\ominus}^2 = 2^2 \int_0^\infty x I_0(\sqrt{t}x) \prod_{i=1}^3 K_0(m_i x) dx$$

the two-loop sunset integral

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- ▶ The sunset integral is the integration of the 2-form ω

$$\omega = \frac{z dx \wedge dy + x dy \wedge dz + y dz \wedge dx}{A_{\Theta}(x, y, z)} \in H^2(\mathbb{P}^2 - \mathcal{E}_{K^2})$$

- ▶ The graph is based on the elliptic curve $\mathcal{E}_{K^2} : A_{\Theta}(x, y, z) = 0$

$$A_{\Theta}(x, y, z) := (m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - K^2 xyz.$$

the two-loop sunset integral

The Feynman parametrisation is given by

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$$\omega = \frac{z dx \wedge dy + x dy \wedge dz + y dz \wedge dx}{A_{\Theta}(x, y, z)} \in H^2(\mathbb{P}^2 - \mathcal{E}_{K^2})$$

- ▶ The domain of integration $\mathcal{D}$ is

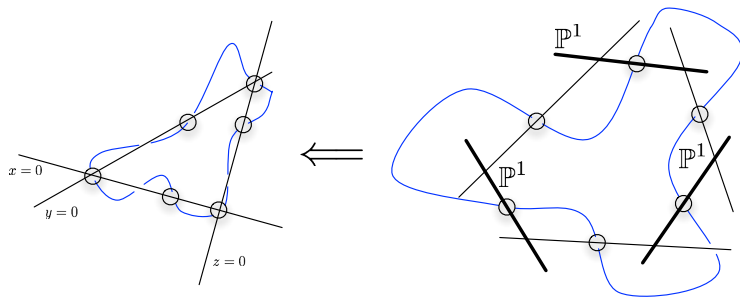
$$\mathcal{D} := \{[x : y : z] \in \mathbb{P}^2 \mid x \geq 0, y \geq 0, z \geq 0\}$$

the sunset graph mixed Hodge structure

- ▶ The elliptic curve intersects the domain of integration $\mathcal{D}$

$$\mathcal{D} \cap \{A_{\Theta}(x, y, z) = 0\} = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$$

- ▶ We need to blow-up work in $\mathbb{P}^2 - \mathcal{E}_{K^2}$



the sunset graph mixed Hodge structure

- ▶ The domain of integration $\mathcal{D} \notin H_2(\mathbb{P}^2 - \mathcal{E}_{K^2})$ because $\partial\mathcal{D} \neq \emptyset$
- ▶ Need to pass to the relative cohomology
- ▶ If $P \rightarrow \mathbb{P}^2$ is the blow-up and $\hat{\mathcal{E}}_{K^2}$ is the strict transform of $\mathcal{E}_{K^2}$
- ▶ Then in P we have resolved the two problems

$$\mathcal{D} \cap \hat{\mathcal{E}}_{K^2} = \emptyset; \quad \mathcal{D} \in H_2(P - \hat{\mathcal{E}}_{K^2}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}_{K^2}))$$

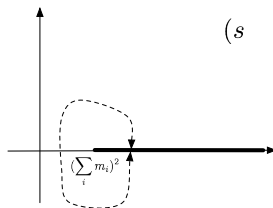
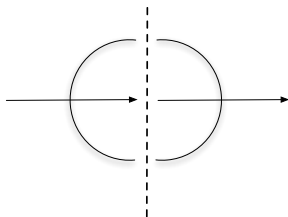
- ▶ We have a variation (with respect to K^2) of Hodge structures

$$H_{K^2}^2 := H^2(P - \hat{\mathcal{E}}_{K^2}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}_{K^2}))$$

- ▶ The sunset integral is a period of the mixed Hodge structure $H_{K^2}^2$

[Bloch, Esnault, Kreimer; Müller-Stach, Weinzeirl, Zayadeh]

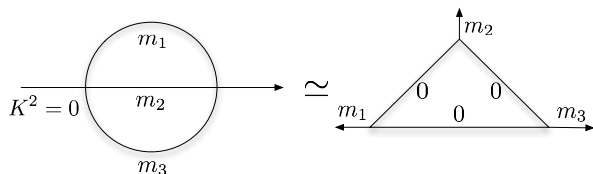
Special values



$$\mathcal{I}_{\Theta}^2(K^2, m_i^2) = \int_{\substack{x \geq 0 \\ y \geq 0}} \frac{dx dy}{(m_1^2 x + m_2^2 y + m_3^2)(x + y + xy) - K^2 xy}$$

- Branch cut at the 3-particle threshold $K^2 = (m_1 + m_2 + m_3)^2$ so $K^2 \in \mathbb{C} \setminus [(m_1 + m_2 + m_3)^2, \infty[$

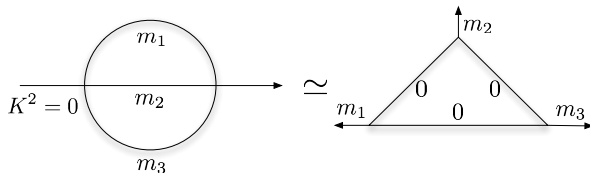
Special values



- At the special fibers $K^2 = 0$ the sunset equal a massive one-loop triangle graph

$$\mathcal{J}_{\Theta}^2(0, m_i^2) \propto \frac{1}{m_1^2} \frac{D(z)}{z - \bar{z}} \quad (1-x)(m_3^2 - m_1^2 x) - m_2^2 x = m_1^2 (x-z)(x-\bar{z})$$

Special values



- ▶ At the special fibers $K^2 = 0$ the sunset equals a massive one-loop triangle graph
- ▶ For equal masses $m_i = m$ then $z = \zeta_6$ such that $(\zeta_6)^6 = 1$

$$\mathcal{J}_{\Theta}^2(0) \propto \frac{1}{m^2} \frac{D(\zeta_6)}{\Im m(\zeta_6)}$$

the elliptic curve of the sunset integral

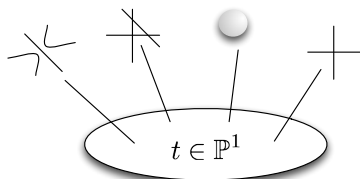
- ▶ With all mass equal $m_i = m$ and $t = K^2/m^2$ the integral reduces to

$$\mathcal{I}_\Theta(t) = \frac{1}{m^2} \int_0^\infty \int_0^\infty \frac{dx dy}{(x+y+1)(x+y+xy) - txy}.$$

$$\mathcal{E}_t: (x+y+1)(x+y+xy) - txy = 0.$$

- ▶ Special values

- At $t = 0$, $t = 1$ and $t = +\infty$ the elliptic curve factorizes.
- At $t = 9$ we have the 3-particle threshold $t \in \mathbb{C} \setminus [9, +\infty[$.



the elliptic curve of the sunset integral

- ▶ With all mass equal $m_i = m$ and $t = K^2/m^2$ the integral reduces to

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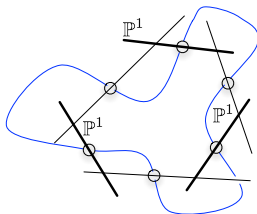
- ▶ Family of elliptic curve surface with 4 singular fibers leads to a K_3 pencil

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\ f \downarrow & & \overline{f} \downarrow \\ X_1(6) & \longrightarrow & X_1(6) \cup \{\text{cusps}\} \end{array}$$

- ▶ This is a universal family of $X_1(6)$ modular curves with a point of order 6

The sunset integral and the motive

$$\begin{array}{ccc}
 \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\
 f \downarrow & & \bar{f} \downarrow \\
 X_1(6) & \longrightarrow & X_1(6) \cup \{\text{cusps}\}
 \end{array}$$



- ▶ Elliptic local system V on $X_1(6)$ with fibre $\mathbb{Q}^2$
- ▶ For $s_1, s_2 \in \mathfrak{h} \cap \mathcal{E}_t$ then the divisor $s_1 - s_2$ on $\mathcal{E}_t$ is of torsion of order 6
- ▶ therefore $H_t^2 = \mathbb{Q}(0)^3 \oplus \hat{H}_t^2$

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \hat{H}_t^2 \longrightarrow H^1(\mathcal{E}_t, \mathbb{Q}(-1)) \rightarrow 0$$

the picard-fuchs equation of the sunset integral

$$\mathcal{E}_t : (x + y + 1)(x + y + xy) - txy = 0$$

- ▶ The picard-fuchs operator is

$$L_t = \frac{d}{dt} \left(t(t-1)(t-9) \frac{d}{dt} \right) + (t-3)$$

- ▶ Acting on the integral we have

$$L_t \mathcal{J}_\Theta^2(t) = \int_{\mathcal{D}} d\beta = - \int_{\partial \mathcal{D}} \beta \neq 0$$

- ▶ We find using the Bessel integral representation

$$\int_0^\infty x I_0(\sqrt{t}x) K_0(x)^3 dx$$

$$\frac{d}{dt} \left(t(t-1)(t-9) \frac{d\mathcal{J}_\Theta(t)}{dt} \right) + (t-3)\mathcal{J}_\Theta(t) = -6$$

the sunset integral as an elliptic dilogarithm

- ▶ The Hauptmodul t is given by [Zagier; Stienstra]

$$t = 9 + 72 \frac{\eta(2\tau)}{\eta(3\tau)} \left(\frac{\eta(6\tau)}{\eta(\tau)} \right)^5$$

- ▶ The period ω_r and ω_c , with $q := \exp(2i\pi\tau(t))$ are given by

$$\omega_r \sim \frac{\eta(\tau)^6 \eta(6\tau)}{\eta(2\tau)^3 \eta(3\tau)^2}; \quad \omega_c = \tau \omega_r$$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{I}_{\Theta}^2(t) = i\pi\omega_r(t)(1 - 2\tau) - \frac{6\omega_r(t)}{\pi} E_{\Theta}(\tau),$$

$$E_{\Theta}(\tau) = \frac{D(\zeta_6)}{\Im(\zeta_6)} - \frac{1}{2i} \sum_{n \geq 0} (\mathrm{Li}_2(q^n \zeta_6) + \mathrm{Li}_2(q^n \zeta_6^2) - \mathrm{Li}_2(q^n \zeta_6^4) - \mathrm{Li}_2(q^n \zeta_6^5))$$

- We have $\mathrm{Li}_2(x)$ and not the Bloch-Wigner $D(x)$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{I}_{\Theta}^2(t) = i\pi\omega_r(t)(1 - 2\tau) - \frac{6\omega_r(t)}{\pi} E_{\Theta}(\tau),$$

$$E_{\Theta}(\tau) = \frac{1}{2} \sum_{n \neq 0} \frac{\psi_2(n)}{n^2} \frac{1}{1 - q^n}$$

► $\psi_2(n)$ is an odd mod 6 character

$$\psi_2(n) = \begin{cases} 1 & \text{for } n \equiv 1 \pmod{6} \\ -1 & \text{for } n \equiv 5 \pmod{6} \end{cases}$$

The sunset integral and the motive

- ▶ The integral is given by

$$\mathcal{I}_{\ominus}^2(t) = \int_0^\infty \int_0^\infty \frac{dx dy}{(x+y+1)(x+y+xy) - txy}$$

- ▶ The 2-form has only **log**-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{I}_{\ominus}^2(t) = \textit{periods} + \varpi_r \left\langle \epsilon_1 \tau + \epsilon_2, \int d\tau \sum_{(m,n) \neq (0,0)} \frac{\psi_2(n)(\epsilon_1 \tau + \epsilon_2)}{(m + n\tau)^3} \right\rangle$$

- ▶ Character $\psi : \textit{Lattice}(\mathcal{E}_t) \rightarrow S^1$. Pairing $\langle \epsilon_1, \epsilon_2 \rangle = -\langle \epsilon_2, \epsilon_1 \rangle = 2i\pi$
- ▶ The amplitude integral is *not* the regulator map which involves a real projection $r : K_2(\mathcal{E}_t) \rightarrow H^1(\mathcal{E}_t, \mathbb{R})$
- ▶ The amplitude is multivalued in t whereas the regulator is single-valued

The sunset integral and the motive

- ▶ The integral is given by

$$\mathcal{I}_{\Theta}^2(t) = \int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(x+y+1)(x+y+xy) - txy}$$

- ▶ The 2-form has only **log**-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{I}_{\Theta}^2(t) = \textit{periods} + \varpi_r \left\langle \epsilon_1 \tau + \epsilon_2, \int d\tau \sum_{(m,n) \neq (0,0)} \frac{\psi_2(n)(\epsilon_1 \tau + \epsilon_2)}{(m + n\tau)^3} \right\rangle$$

- ▶ The regulator is an Eichler integral

$$\mathcal{I}_{\Theta}^2(t) = \textit{periods} + \varpi_r \int_{\tau}^{i\infty} \sum_{(m,n) \neq (0,0)} \frac{\psi_2(n)(\tau - x)}{(m + nx)^3} dx$$

The sunset integral and the motive

- ▶ The integral is given by

$$\mathcal{I}_{\ominus}^2(t) = \int_0^\infty \int_0^\infty \frac{dx dy}{(x+y+1)(x+y+xy) - txy}$$

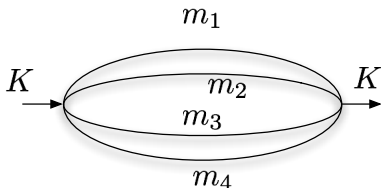
- ▶ The 2-form has only **log**-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{I}_{\ominus}^2(t) = \textit{periods} + \varpi_r \left\langle \epsilon_1 \tau + \epsilon_2, \int d\tau \sum_{(m,n) \neq (0,0)} \frac{\psi_2(n)(\epsilon_1 \tau + \epsilon_2)}{(m + n\tau)^3} \right\rangle$$

- ▶ The regulator is an Eichler integral

$$\mathcal{I}_{\ominus}^2(t) = \textit{periods} + \varpi_1 \sum_{(m,n) \neq (0,0)} \frac{\psi_2(n)}{n^2(m + n\tau)}$$

three-loop banana graph: integral



We look at the 3-loop banana graph in $D = 2$ dimensions

- ▶ The Feynman parametrisation is given by

$$I_{\ominus}^2(m_i; K^2) = \int_{x_i \geq 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - K^2} \prod_{i=1}^3 \frac{dx_i}{x_i}$$

three-loop banana graph: differential equation

- ▶ The geometry of the 3-loop banana graph is a K_3 surface (Shioda-Inose family for $\Gamma_1(6)^{+3}$) with Picard number 19 and discriminant of Picard lattice is 6

$$(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) \prod_{i=1}^3 x_i - t \prod_{i=1}^3 x_i = 0$$

- ▶ The all equal mass case with $t = K^2/m^2$ satisfies the Picard-Fuchs equation [vanhove]

$$\left(t^2(t-4)(t-16) \frac{d^3}{dt^3} + 6t(t^2-15t+32) \frac{d^2}{dt^2} + (7t^2-68t+64) \frac{d}{dt} + t-4 \right) J_{\oplus}^2(t) = -4!$$

- ▶ One miracle is that this picard-fuchs operator is the symmetric square of the picard-fuchs operator for the sunset graph [Verrill]

three-loop banana graph: solution

- It is immediate to use the Wronskian method to solve the differential equation [Bloch, Kerr, Vanhove]

$$m^2 I_{\oplus}^2(t) = 40\pi^2 \log(q) \varpi_1(\tau) \\ -48\varpi_1(\tau) \left(24\mathcal{L}i_3(\tau, \zeta_6) + 21\mathcal{L}i_3(\tau, \zeta_6^2) + 8\mathcal{L}i_3(\tau, \zeta_6^3) + 7\mathcal{L}i_3(\tau, 1) \right)$$

with $\mathcal{L}i_3(\tau, z)$ [Zagier; Beilinson, Levin]

$$\mathcal{L}i_3(\tau, z) := \text{Li}_3(z) + \sum_{n \geq 1} (\text{Li}_3(q^n z) + \text{Li}_3(q^n z^{-1})) \\ - \left(-\frac{1}{12} \log(z)^3 + \frac{1}{24} \log(q) \log(z)^2 - \frac{1}{720} (\log(q))^3 \right).$$

three-loop banana graph: solution

- ▶ Which can be written using as an Eisenstein series

$$m^2 I_{\oplus}^2(t(\tau)) = \varpi_1(\tau) \left(-4(\log q)^3 + \frac{1}{2} \sum_{n \neq 0} \frac{\psi_3(n)}{n^3} \frac{1+q^n}{1-q^n} \right)$$

where $\psi_3(n)$ is an even mod 6 character

- ▶ Arising from the regulator for $\text{Sym}^2 H^1(\mathcal{E}_{\Theta}(t))$ with $dz = (\epsilon_1 \tau + \epsilon_2)^2$

$$\varpi_1 \left\langle dz^2, \int \sum_{m,n} \frac{d\tau dz^2 \psi_3(n)}{(m+n\tau)^4} \right\rangle$$

- ▶ Again this is given by the Eichler integral

$$\text{period} + \varpi_1 \int_{\tau}^{i\infty} \sum_{m,n} \frac{(x-\tau)^2 \psi_3(n)}{(m+nx)^4} dx$$

Part IV

Higher-loop banana amplitudes

Higher-order bananas I

The integral we have been discussing are given by

$$I_n^2 = \int_{x_i \geq 0} \frac{1}{(1 + \sum_{i=1}^n x_i)(1 + \sum_{i=1}^n x_i^{-1}) - t} \prod_{i=1}^n \frac{dx_i}{x_i}$$

They are given by the following Bessel integral representation

$$I_n^2(t) = 2^{n-1} \int_0^\infty x I_0(\sqrt{t}x) K_0(x)^n dx$$

The Bessel function $K_0(x)$ satisfies a differential equation that implies that

$$\left(\sum_{r=0}^{n+1} p_r(x) \left(x \frac{d}{dx} \right)^r \right) K_0(x)^n = 0$$

This implies that the banana integral satisfies a differential equation

Higher-order bananas II

$$\left(\sum_{r=0}^{n+1} \tilde{p}_r(t) \left(t \frac{d}{dt} \right)^r \right) I_n^2(t) = 0$$

Because $p_r(x)$ are polynomials in x^2 without constant terms [Borwein, Salvy] then one can factorizes a $(td/dt)^2$ operator and the differential operator for the $n-1$ -loop banana graph is of order the number of loops

$$\left(\sum_{r=0}^{n-1} q_r(t) \left(t \frac{d}{dt} \right)^r \right) I_n^2(t) = S_n + \tilde{S}_n \log(t)$$

Since the banana integral is regular at $t=0$ then $\tilde{S}_n=0$ and the differential equation is

Higher-order bananas III

$$\left(\sum_{r=0}^{n-1} q_r(t) \left(t \frac{d}{dt} \right)^r \right) I_n^2(t) = S_n$$

- ▶ with [vanhove]

$$\begin{aligned} q_{n-1}(t) &= t^{\lfloor \frac{n}{2} \rfloor + \eta(n)} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (t - (n - 2i)^2) \\ q_{n-2}(t) &= \frac{n-1}{2} \frac{dq_{n-1}(t)}{dt} \\ q_0(t) &= t - n. \end{aligned}$$

- ▶ with $\eta(n) = 0$ if $n \equiv 1 \pmod{2}$ and 1 if $n \equiv 0 \pmod{2}$.
- ▶ The inhomogeneous term $S_n = -n!$

Higher-order bananas

- ▶ one-loop banana bubble

$$(t-2)f(t) + t(t-4)f^{(1)}(t) = -2!$$

- ▶ Solution are logarithms

$$I_0^2(m_1, m_2, K^2) = \frac{\log(z^+) - \log(z^-)}{\sqrt{\Delta}}$$

- ▶ $z^\pm$ and Δ are roots and discriminant of the equation

$$(m_1^2 x + m_2^2)(1+x) - K^2 x = m_1^2(x - z^+)(x - z^-) = 0$$

- ▶ four-loop banana bubble

$$\begin{aligned} & (t-5)f(t) + (3t-5)(5t-57)f^{(1)}(t) \\ & + \left(25t^3 - 518t^2 + 1839t - 450\right)f^{(2)}(t) \\ & + \left(10t^4 - 280t^3 + 1554t^2 - 900t\right)f^{(3)}(t) \\ & + t^2(t-25)(t-1)(t-9)f^{(4)}(t) = -5! \end{aligned}$$

- five-loop banana bubble

$$\begin{aligned} & (t-6)f(t) + (31t^2 - 516t + 1020)f^{(1)}(t) \\ & + (90t^3 - 2436t^2 + 12468t - 6912)f^{(2)}(t) \\ & + (65t^4 - 2408t^3 + 19836t^2 - 27648t)f^{(3)}(t) \\ & + (15t^5 - 700t^4 + 7840t^3 - 17280t^2)f^{(4)}(t) \\ & + t^3(t-36)(t-4)(t-16)f^{(5)}(t) = -6! \end{aligned}$$

- ▶ Connection between the Feynman parametrization and the world-line formalism : Possible treatment of field theory graph a la string theory
- ▶ Amplitudes are multivalued functions given by motivic periods. The Hodge matrix corresponds to unitarity (discontinuity, cut).

- At special values $t = 1$ the integrals are *pure* periods related to values of L -functions in the critical band : Deligne's conjecture [Zagier]

$$I_{\odot}(1) = \frac{4\pi}{\sqrt{15}} L(K_3, 2)$$

The new form for the $L(K_3, s)$ [Peeters, Top, van der Vlugt]

$$f(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n} q^{m^2+mn+4n^2} \in S_3(15, \left(\frac{\cdot}{15}\right))$$

1.3 Multiplicative modular forms and eta products

For levels $N < 16$, there are precisely 15 multiplicative modular forms that are products of eta values. Here they are listed with notes on quantum field theory (QFT):

form	weight	level	QFT
$\eta_1^2 \eta_{11}^2$	2	11	Broadhurst talk @ IHES
$\eta_1 \eta_2 \eta_7 \eta_{14}$	2	14	
$\eta_1 \eta_3 \eta_5 \eta_{15}$	2	15	
$\eta_1^3 \eta_2^2$	3	7	BS
$\eta_1^2 \eta_2 \eta_4 \eta_8^2$	3	8	BS
$\eta_2^3 \eta_6^3$	3	12	BS + BBBG + BV : Sections 3 and 4
$\eta_1^4 \eta_5^4$	4	5	BS
$\eta_1^2 \eta_2^2 \eta_3^2 \eta_6^2$	4	6	BS + BB : Sections 3 and 5
$\eta_2^3 \eta_4^3$	4	8	
η_5^3	4	9	
$\eta_1^4 \eta_2^2 \eta_4^4$	5	4	BS
$\eta_1^6 \eta_3^3$	6	3	BS
η_2^{12}	6	4	BS

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- ▶ But there is an obstruction to modularity at higher-loop [Brown]
- ▶ At higher-loop order the integral are *not* elliptic polylogarithm as seen on the Picard-Fuchs equation