Second Quantized Mathieu Moonshine

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Automorphic Forms, Lie Algebras and String Theory

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References

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Plan of the talk

1. (Generalized) Mathieu Moonshine
2. Interpretation of Mathieu Moonshine
3. Second Quantized Mathieu Moonshine

(Generalized) Mathieu Moonshine

Elliptic genus of K3: definition

- 2-dim $\mathcal{N} = (4,4)$ SCFT with central charge $c = 6$
- Non-linear σ-model with target space K3
- The model depends on the choice of metric and B-field (80-dim moduli space of theories).

Elliptic genus

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{F+\tilde{F}}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\, y^{2J_0^3}\right)$$

where $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ and $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$.

J_0^3 is Cartan generator of (left) $su(2)$ in $\mathcal{N} = (4,4)$ SC algebra

Elliptic genus of K3: properties

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{F+\tilde{F}}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\, y^{2J_0^3}\right)$$

- Receives contributions only from right-moving ground states $\rightarrow$ holomorphic in τ and z
- Independent of the metric and B-field
- Elliptic and modular properties:

$$\phi(\tau, z+\ell\tau+\ell') = e^{-2\pi i(\ell^2\tau+2\ell z)}\,\phi(\tau, z) \qquad \ell, \ell' \in \mathbb{Z}$$

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{2\pi i\frac{cz^2}{c\tau+d}}\,\phi(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$\equiv$ (weak) Jacobi form of weight 0 and index 1

- $\phi_{K3}(\tau, z) = 8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right)$

$\mathcal{N} = 4$ characters and Mathieu representations

Decompose ϕ_{K3} into (left) $\mathcal{N} = 4$ characters $\mathrm{ch}_{\Delta,\ell}(\tau, z)$ at $c = 6$

$$\phi_{K3}(\tau, z) = 20\, \mathrm{ch}^{BPS}_{\frac{1}{4},0}(\tau, z) - 2\, \mathrm{ch}^{BPS}_{\frac{1}{4},\frac{1}{2}}(\tau, z) \qquad \text{BPS reps}$$

$$+ \sum_{n=1}^{\infty} A_n\, \mathrm{ch}_{\frac{1}{4}+n,\frac{1}{2}}(\tau, z) \qquad \text{massive reps}$$

where A_n are the multiplicities of massive $\mathcal{N} = 4$ irreps

$$\tfrac{1}{2} A_n = 45, \quad 231, \quad 770, \quad 2277, \quad 5796, \quad 13915, \quad \ldots$$

The Mathieu group M_{24}

M_{24} is a finite simple group of order

$$|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \ \sim \ 2 \times 10^8$$

Properties:

- Subgroup of S_{24}(=permutations of 24 symbols)
- 26 conjugacy classes
- 26 irreducible representations of dimensions

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265,$$
$$1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395$$

$\mathcal{N} = 4$ characters and Mathieu representations

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$$\frac{1}{2}A_n = 45, \quad 231, \quad 770, \quad 2277, \quad 5796, \quad 13915, \quad \ldots$$

Dimensions of irreps of M_{24}:

1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265,
1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395

... and $13915 = 3520 + 10395$
(for higher n, many possible decompositions)
[Eguchi, Ooguri, Tachikawa 1004.0956]

A Mathieu Moonshine?

Conjecture:[EOT]

There are M_{24} reps $R_{\Delta,\ell}$ s.t.

$$\phi_{K3}(\tau, z) = \sum_{(\Delta,\ell)} \dim R_{\Delta,\ell} \; \mathrm{ch}_{\Delta,\ell}(\tau, z)$$

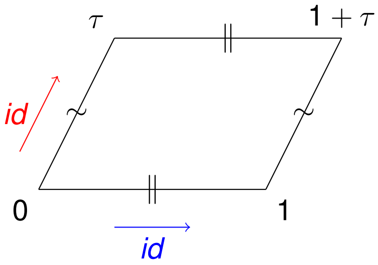
Spectrum $\mathcal{H}$ of right-moving BPS decomposes as

$$\mathcal{H} = \bigoplus_{(\Delta,\ell)} \big(\text{rep. } R_{\Delta,\ell} \text{ of } M_{24}\big) \otimes \big(\mathcal{N} = 4 \text{ irrep with label } (\Delta, \ell)\big)$$

How can we prove this?

Path integral

Elliptic genus is obtained from path-integral on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$



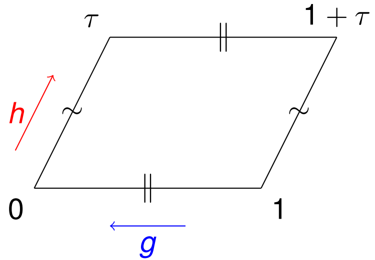
Path integral is $SL(2,\mathbb{Z})$ invariant

$\downarrow$

$\phi(\tau, z)$ is Jacobi form of wt 0 under $SL(2,\mathbb{Z})$

Twisted-twining genera

Twisted-twining genera $\phi_{g,h}$ from path-integral with twisted periodicity conditions



Path integral is invariant under $\Gamma_{g,h} \subseteq SL(2,\mathbb{Z})$

$\downarrow$

$\phi_{g,h}(\tau, z)$ is Jacobi form of wt 0 for $\Gamma_{g,h}$

It only makes sense if $gh = hg$

Operatorial interpretation

- if $g =$ identity (*untwisted* twining genera)

$$\phi_{1,h}(\tau, z) = \mathrm{Tr}_{\mathcal{H}}\left(h\,(-1)^{F+\tilde{F}}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\, y^{2J_0^3}\right)$$
$$= \sum_{(\Delta,\ell)} \mathrm{Tr}_{R_{\Delta,\ell}}(h)\ \mathrm{ch}^{\mathcal{N}=4}_{\Delta,\ell}(\tau, z)$$

- For general $gh = hg$,

$$\phi_{g,h}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_g}\left(\rho_g(h)(-1)^{F+\tilde{F}}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\, y^{2J_0^3}\right)$$
$$= \sum_{(\Delta,\ell)} \mathrm{Tr}_{R^{(g)}{}_{\Delta,\ell}}(\rho_g(h))\ \mathrm{ch}^{\mathcal{N}=4}_{\Delta,\ell}(\tau, z)$$

where

- $\mathcal{H}_g$ is the g-twisted sector
- ρ_g is *projective* representation of

$$C_{M_{24}}(g) := \{h \in M_{24} | gh = hg\}$$

Generalized Mathieu Moonshine

Conjecture: For each commuting $g, h \in M_{24}$ there is a Jacobi form $\phi_{g,h}(\tau, z)$ of wt 0 index 1 (*twisted-twining genera*) such that:

1. $\phi_{1,1} = \phi_{K3}$
2. $\phi_{g,h}(\tau, z) \sim \phi_{k^{-1}gk, k^{-1}hk}(\tau, z) \qquad k \in M_{24}$
3. $\phi_{g,h}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \sim e^{\frac{2\pi i cz^2}{c\tau+d}} \phi_{g^a h^c, g^b h^d}(\tau, z) \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$
4. $\phi_{g,h}(\tau, z) = \sum_{(\Delta,\ell)} \mathrm{Tr}_{R^{(g)}_{\Delta,\ell}}(\rho_g(h))\ \mathrm{ch}^{\mathcal{N}=4}_{\Delta,\ell}(\tau, z) \qquad h \in C_{M_{24}}(g)$

   where $\rho_g^{(\Delta,\ell)}$ is a (projective) rep of $C_{M_{24}}(g)$

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   where $\rho_g^{(\Delta,\ell)}$ is a (projective) rep of $C_{M_{24}}(g)$

Generalized Mathieu Moonshine

- All $\phi_{g,h}$ with expected modular properties have been found
- Verified that $R^{(g)}_{\Delta,\ell}$ exist such that

$$\phi_{g,h}(\tau, z) = \sum_{(\Delta,\ell)} \mathrm{Tr}_{R^{(g)}_{\Delta,\ell}}(\rho_g(h))\ \mathrm{ch}^{\mathcal{N}=4}_{\Delta,\ell}(\tau, z)$$

  - complete proof for $g = 1$
    [Cheng 1005.5415, Gaberdiel, Hohenegger, RV 1006.0221, 1008.3778; Eguchi, Hikami 1008.4924; Gannon 1211.5531]
  - for $g \neq 1$, verified only for the first 500 (Δ, ℓ) for each g
    [Gaberdiel, Persson, Ronellenfitsch, Volpato 1211.7074, 1302.5425]
- No ambiguity: $R^{(g)}_{\Delta,\ell}$ are unique
  (but the non-trivial statement is existence)

The conjecture is (partially) proved! But what's the interpretation?

Interpretation of Mathieu Moonshine

Interpretation?

- Obvious idea: is there a K3 model with symmetry group M_{24}?

Interpretation?

- Obvious idea: is there a K3 model with symmetry group M_{24}?

NO

Complete classification of group G of symmetries of K3 models:

- There is no G such that $M_{24} \subseteq G$
- There are some G such that $G \not\subset M_{24}$

[Gaberdiel, Hohenegger, R.V. 1106.4315]

Interpretation?

New approach

- Look for interpretation of elliptic genus as *space-time* supersymmetric index in string theory compactifications
- Use string dualities to explain Moonshine (?)

Examples:

- 1/4 BPS index in type II/K3$\times T^2$
  [Cheng 1005.5415]
- ‘New supersymmetric index’ in Het/K3$\times T^2$
  [Cheng, Dong, Duncan, Harvey, Kachru, Wrase 1306.4981]
  [Harrison, Kachru, Paquette 1309.0510]
  [Wrase 1402.2973]
- 1/2 BPS index in type II/K3$\times S^1$ in NS5 branes background
  [Harvey, Murthy 1307.7717]

Second Quantized Mathieu Moonshine

Four dimensional $\mathcal{N} = 4$

4-dim string compactifications with $\mathcal{N} = 4$ space-time SUSY

$$IIB/(K3 \times T^2) \quad \leftrightarrow \quad IIA/(K3 \times T^2) \quad \leftrightarrow \quad Het/T^6$$

- Moduli space $\mathcal{M}$, with $\dim \mathcal{M} = 134$
- Gauge group (at generic points): $U(1)^{28}$ (= 22 vector multiplets)
- Lattice of electric-magnetic charges: $(P, Q) \in \Gamma^{6,22} \oplus \Gamma^{6,22}$
- Duality group: $SL(2, \mathbb{Z}) \times O(6, 22, \mathbb{Z})$

1/2 BPS states

1/2-BPS states are in the duality orbits of $(0, Q)$ (purely electric)

Index $B_{\frac{1}{2}BPS}(Q)$ counts 'degeneracy' of 1/2 BPS states

Properties of $B_{\frac{1}{2}BPS}(Q)$:

- Independent of the moduli
- Invariant under duality group $SL(2,\mathbb{Z}) \times O(6,22,\mathbb{Z})$

  $\rightarrow$ it depends only on Q^2, i.e $B_{\frac{1}{2}BPS}(Q) = d(\frac{Q^2}{2})$

Derivation in perturbative limit of Het/T^6:

$$\sum_{n\in\mathbb{Z}} d(n)q^n = \frac{1}{\eta(\tau)^{24}} \quad \leftrightarrow \quad d(n) = \oint \frac{e^{-2\pi i n\tau}}{\eta(\tau)^{24}}$$

[Dabholkar, Harvey '89]

1/4 BPS states

1/4-BPS states are dyons (generic charge (P, Q))

Index $B_{\frac{1}{4}BPS}(P, Q)$ counts ‘degeneracy’ of 1/4 BPS states

Properties of $B_{\frac{1}{4}BPS}(P, Q)$:

- Locally constant on moduli space, but can ‘jump’
  $\rightarrow$ wall-crossing phenomenon
- Invariant under duality group $SL(2, \mathbb{Z}) \times O(6, 22, \mathbb{Z})$

$$B_{\frac{1}{4}BPS}(P, Q) = d\left(\frac{Q^2}{2}, \frac{P^2}{2}, P \cdot Q\right)$$

$$d(m,n,\ell) = (-1)^{\ell+1} \int_{\mathcal{C}} \frac{e^{-2\pi i(m\sigma+n\tau+\ell z)}}{\Phi_{10}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right)}$$

[Dijkgraaf, Verlinde[2] '96], [Dijkgraaf, Moore, Verlinde[2] '96], ...

- $\Phi_{10}(\Omega)$ is Siegel modular form of weight 10

  $$\Phi_{10}\big((A\Omega+B)(C\Omega+D)^{-1}\big) = \det(C\Omega+D)^{10}\Phi_{10}(\Omega),$$

  where $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in Sp(4,\mathbb{Z})$

- $\Phi_{10}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right)$ has double zero at $z=0$ with

  $$\lim_{z\to 0} z^{-2}\Phi_{10}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right) \sim \eta(\tau)^{24}\eta(\sigma)^{24}$$

- $d(m,n,\ell)$ 'jumps' as $\mathcal{C}$ crosses the pole (*wall-crossing*)
  [Cheng, Verlinde '07]

Φ_{10} is the multiplicative lift of elliptic genus ϕ_{K3}

Multiplicative lift

$$\Phi_{10}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right) = pqy \prod_{(n,m,l)<0} (1 - p^m q^n y^l)^{c(mn,l)}$$

where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$ and

$$\phi_{K3}(\tau, z) = \sum_{n,l} c(n, l) q^n y^l$$

Electric-magnetic duality $\sigma \leftrightarrow \tau$ manifest

Derivation by D1-D5-P system in type IIB/K3$\times T^2$

Second quantized elliptic genus

$$\frac{1}{\Phi_{10}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right)} = (\text{correction}) \times \sum_{m=0}^{\infty} e^{2\pi i m\sigma}\,\phi_{S^m K3}(\tau, z)$$

$$= (\text{correction}) \times \exp\Big(\sum_{N=0}^{\infty} e^{2\pi i N\sigma}\, T_N \phi_{K3}(\tau, z)\Big)$$

where T_N are Hecke operators

$$T_N\phi(\tau, z) := \frac{1}{N}\sum_{ad=N}\sum_{b=0}^{d-1}\phi\Big(\frac{a\tau + b}{d}, az\Big)$$

Second-quantized Mathieu Moonshine

For all commuting pairs $g, h \in M_{24}$, define [Cheng '10; Persson, Volpato '13]

$$\Phi_{g,h}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right)^{-1} = (\text{correction}) \times \sum_{m=0}^{\infty} e^{2\pi i m\sigma}\, \phi_{g,h}^{S^m K3}(\tau, z)$$

$$= (\text{correction}) \times \exp\Big(\sum_{N=0}^{\infty} e^{2\pi i N\sigma}\, \mathcal{T}_N \phi_{g,h}(\tau, z)\Big)$$

where $\mathcal{T}_N$ are *twisted equivariant* Hecke operators

$$\mathcal{T}_N\phi(\tau, z) := \frac{1}{N}\sum_{ad=N}\sum_{b=0}^{d-1}\epsilon_{g,h}\left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right)\phi_{g^d, g^{-b}h^a}\Big(\frac{a\tau+b}{d}, az\Big)$$

with $\epsilon_{g,h}$ suitable phases

Second-quantized Mathieu Moonshine

Infinite product representation

$$\Phi_{g,h}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right) = pq^{\frac{1}{N}}y \prod_{(d,m,\ell)>0} \prod_{t=0}^{M-1} \left(1 - e^{\frac{2\pi i t}{M}} q^{\frac{m}{N}} y^{\ell} p^{d}\right)^{\hat{c}_{g,h}(d,m,\ell,t)}$$

where M, N orders of g, h in *central extension* of $C_{M_{24}}(g)$

$$\hat{c}_{g,h}(d,m,\ell,t) := \sum_{k=0}^{M-1} \sum_{b=0}^{N-1} \frac{e^{-\frac{2\pi i t k}{M}}}{M} \frac{e^{\frac{2\pi i b m}{N}}}{N} \epsilon_{g,h}\left(\begin{smallmatrix}k & b\\ 0 & d\end{smallmatrix}\right) c_{g^d, g^{-b}h^k}\left(\frac{md}{N}, \ell\right)$$

with

$$\phi_{g,h}(\tau, z) = \sum_{r,l} c_{g,h}(r,l) q^r y^l \qquad r \in \mathbb{Q},\ l \in \mathbb{Z}$$

Second-quantized Mathieu Moonshine

Properties:

- All $\Phi_{g,h}$ invariant under finite index subgroups of paramodular groups $\Gamma_t(N) \subset Sp(4,\mathbb{R})$
  [Gritsenko, Nikulin] [Gritsenko, Cléry]
- 'Electric-magnetic duality'

$$\Phi_{g,h}\begin{pmatrix} \sigma & z \\ z & \tau \end{pmatrix} = \Phi_{g,h'}\begin{pmatrix} \tau/N & z \\ z & N\sigma \end{pmatrix}$$

  where h' is *not necessarily* the same as h

- 'Wall-crossing':

$$\lim_{z\to 0} z^{-2}\,\Phi_{g,h}\begin{pmatrix} \sigma & z \\ z & \tau \end{pmatrix} \sim \eta_{g,h}(\tau)\eta_{g,h'}(\sigma)$$

  where $\eta_{g,h}$ are η-products defining 'old Mathieu Moonshine'
  [Mason '89-'90]

A physical meaning?

Suppose g, h are commuting discrete symmetries of internal CFT (non-linear sigma model on K3$\times T^2$)

- We compactify on the g-orbifold of the internal CFT
  $\rightarrow$ new 4-dim $\mathcal{N} = 4$ model (CHL model)
  [Chauduri, Hockney, Lykken '95]

- Consider the h-twisted indices in the g-orbifold

  $\frac{1}{2}$BPS: $B_{\frac{1}{2};g,h}(Q) := \mathrm{Tr}_{g;Q}(h\ldots) = d_{g,h}(\frac{Q^2}{2})$

  $\frac{1}{4}$BPS: $B_{\frac{1}{4};g,h}(P,Q) := \mathrm{Tr}_{g;P,Q}(h\ldots) = d_{g,h}(\frac{Q^2}{2}, \frac{P^2}{2}, P\cdot Q)$

A physical meaning?

Computed for various pairs of symmetries g, h

[Cheng, Dabholkar, David, Gaiotto, Govindarajan, Jatkar, Murthy, Pioline, Sen, Verlinde, ...]

$$\frac{1}{2}\text{-BPS:} \quad d_{g,h}(n) = \oint \frac{e^{-2\pi i n\tau}}{\eta_{g,h}(\tau)}$$

$$\frac{1}{4}\text{-BPS:} \quad d_{g,h}(m,n,\ell) = (-1)^{\ell+1} \int_{\mathcal{C}} \frac{e^{-2\pi i(m\sigma+n\tau+\ell z)}}{\Phi_{g,h}\left(\begin{smallmatrix}\sigma & z\\ z & \tau\end{smallmatrix}\right)}$$

$\eta_{g,h}$ and $\Phi_{g,h}$ coincide with some functions from second quantized Mathieu Moonshine

Summary and outlook

- Generalized MM suggests structure similar to CFT
- Problems in building a consistent CFT with right properties
- Second-quantized twisted-twining genera satisfy ‘electric-magnetic duality’ and ‘wall-crossing’

Some open questions:

- Precise modular properties of $\Phi_{g,h}$?
- Can they be described as additive lifts?
- Denominators of BKM algebras?
- Relation with Umbral Moonshine?
- Physical interpretation of what we observe?