Feynman integrals in dimensional regularisation and extension of Calabi-Yau motives

Motives in QFT– 24-11-2021

Albrecht Klemm

with Kilian Bönisch, Claude Duhr, Fabian Fischbach, Christoph Nega, Reza Safari

based on [1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1 and [3]=arXiv:2108.05310

UNIVERSITÄT BONN

Introduction: Emerging relation between:

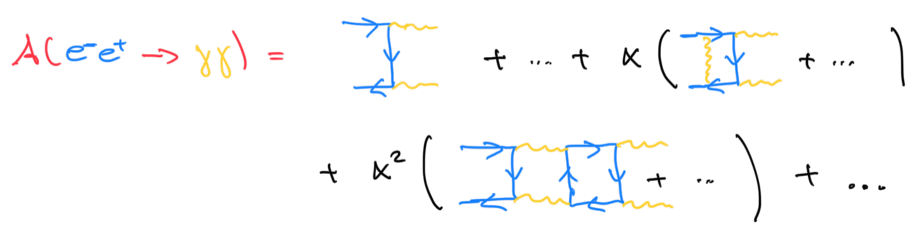
Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

For the full Feynman integral the rational functions are replaced by rational polylogarithms ✓ and the elliptic functions by elliptic polylogarithms (✓) . I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. Mc Leod, M. Hippel, M. Wilhelm, J. Broedel, L Trancredi, S. Müller-Stach, . . . + 248 cits. in [3]

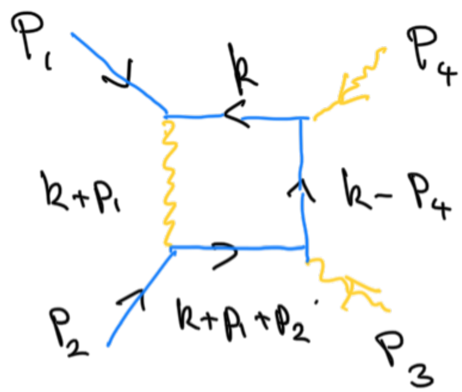
For the Banana diagrams: this dictionary is worked out to all loop orders [1,2]. The aim of this talk is to explain the applications of the dictionary relating Feynman integrals to families of Calabi-Yau motives and to extend them to include the dimensional regularization parameter [3] ϵ and comment on the extensions to other graphs as e.g. in the calculation for the probability for $e^-\ e^+$ to annihilate to two photons

$$P(e^-e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^-e^+ \to \gamma\gamma)|^2,\ \alpha \sim \frac{1}{137}$$



Decisive part e.g. for e.g. the box integral: Propagators (edge) $\rightarrow \frac{1}{q^2 - m^2 + i\cdot 0}$

$$\sum_{i=1}^{4} p_i = 0 \quad \text{momentum conservation}$$



$$\sim \int \frac{d^D k}{(k^2 - m^2)\,(k+p_1)^2\,((k+p_1+p_2)^2 - m^2)((k-p_4)^2 - m^2)}$$

Yields a function of masses and Lorentz invariant products of the external momenta that we need to know!

Master Integrals and integration by parts relations:

Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{1}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the p_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r.

$$\epsilon := \frac{D_0 - D}{2}$$

describes the deviation from a critical dimension D_0, which depends on the graph.

The Feynman integral depends besides D on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{x} = (x_1, \dots, x_N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$.

Actually, dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i. In particular, we can chose

$$z_k := x_k/x_N \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{z}$.

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. There is a finite set of integrals $I_{\underline{\nu}}(\underline{x}, D)$ so called master integrals, which yields all non-vanishing integrals in this lattice.

A generating set of master integrals can be found by integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^{\mu}} \left(q_l^{\mu} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

The IBP relations relate in particular master integrals with different exponents $\underline{\nu}$.

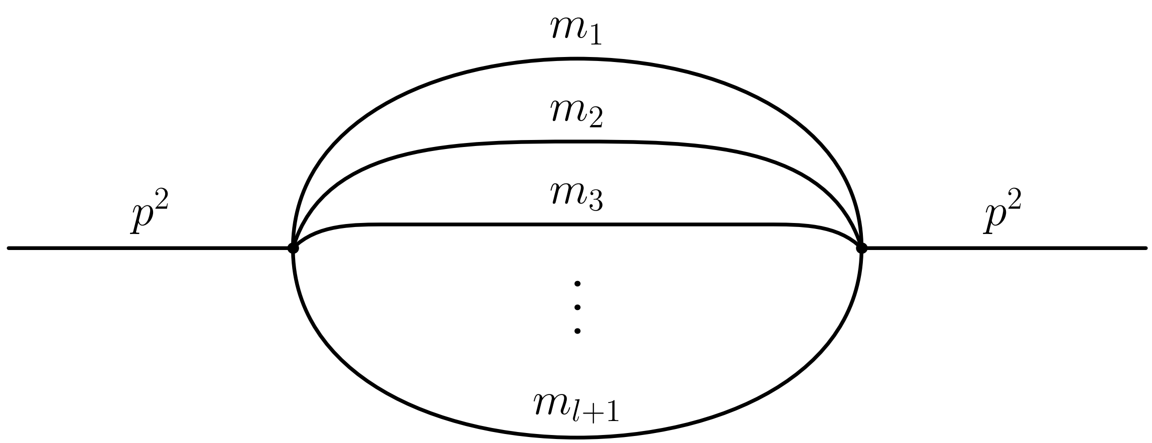
Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1 \leq j \leq p} \ .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

The important property is that there is a finite region in the lattice that contains all non-vanishing master integrals. Picking a basis one can can express derivatives of this basis w.r.t. the z_k as a linear combination of master integrals with rational coefficients.

Main Example: A very simple series of such Feynman integrals with loop order l are the banana diagrams in critical dimension $D_0 = 2$:



$$D_j = k_j^2 - m_j^2\,, \qquad 1 \leq j \leq l\,,$$

$$D_{l+1} = (k_1 + \ldots + k_l - p)^2 - m_{l+1}^2\,,$$

$$\underline{z} = (m_1^2/p^2, \ldots, m_{l+1}^2/p^2)\,.$$

Families of Calabi-Yau manifolds for the banana integral:

Using the graph polynomials $\mathcal{U}$ and $\mathcal{F}$ we can write the integral generally as:

$$I_{1,\ldots,1}(\underline{p},\underline{m},D)=\int_{\sigma_l}\left(\prod_{k=1}^{l+1}x_k^{\delta_k}\right)\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}(p^2,\underline{m}^2)^{\omega}}\mu_l\,.$$

- $\nu_i=1+\delta_i$, $\omega:=\sum_{i=1}^{l+1}\nu_i-\frac{lD}{2}-1+l\epsilon+\sum_i\delta_i$
- $\sigma_{n-1}=\{[x_1:\ldots:x_n]\in\mathbb{P}^{n-1}|x_i\in\mathbb{R}_{\geq 0}\,\forall\,1\leq i\leq n\}$ an open domain,
- $\mu_l=\sum_{k=1}^{n}(-)^{k+1}x_k\mathrm{d}x_1\wedge..\wedge\widehat{\mathrm{d}x_k}\wedge..\wedge\mathrm{d}x_n$ measure on $\mathbb{P}^l$.

The two Symanzik polynomials for the banana graph are given by:

$$\mathcal{U} = \left(\prod_{i=1}^{l+1} x_i\right)\left(\sum_{i=1}^{l+1}\frac{1}{x_i}\right) = \sum_{i=1}^{l+1}\prod_{\substack{j=1\\ j\neq i}}^{l+1} x_j\,,$$

$$\mathcal{F}(p^2,\underline{m}^2) = -p^2\left(\prod_{i=1}^{l+1} x_i\right) + \left(\sum_{i=1}^{l+1} m_i^2 x_i\right)\mathcal{U}\,.$$

In the critical dimension $D_0 = 2$ one gets a maximal cut integral

$$J_{l,\underline{0}}^{\Gamma_T}(\underline{z};0) = \int_{T^l}\frac{\mu_l}{\mathcal{F}(1,\underline{z})} = \int_{T^{l-1}}\oint_{S^1}\frac{\mu_l}{\mathcal{F}(1,\underline{z})} = 2\pi i\int_{\Gamma_T=T^{l-1}}\Omega_{l-1}(\underline{z})\ ,$$

over the cycle T^l defined as

$$T^l := \{[x_1:\ldots:x_{l+1}]\in\mathbb{P}^l \mid |x_i| = 1 \text{ for all } 1\leq i\leq l+1\}\,.$$

Here we used the Griffiths residue form for the holomorphic n-form Ω for complete intersections

$$\Omega(\underline{z}) = \frac{1}{(2\pi i)^r} \oint_{S_1^1} \cdots \oint_{S_r^1} \frac{\wedge_{i=1}^m \mu_{n_i}}{P_1 \cdots P_r} ,$$

where S_k^1 encircles the constraints $P_k = 0$ in the ambient space. The crucial point is that the integral over the S^1 cycle of T^l leads to a closed period integral over $\mathbf{T} = T^{l-1}$ on

$$M_{l-1}^{\mathrm{HS}} = \{\underline{x} \in \mathbb{P}^l | \mathcal{F}(1, \underline{z}; \underline{x}) = 0\} .$$

Performing all l residua integrals one gets

$$J^{\Gamma_T}_{l,\underline{0}}(\underline{z};0) = (2\pi i)^l \sum_{n=0}^{\infty} \sum_{|k|=n} \begin{pmatrix} n \\ k_1 \ldots k_{l+1} \end{pmatrix}^2 \prod_{i=1}^{l+1} z_i^{k_i} \,,$$

with $|k| = \sum_{i=1}^{l+1} k_i$.

The hypersurface M^{HS}_{l-1} defines a singular family of Calabi-Yau motives with $l+1$ complex parameters. To get a workable smooth model one could deform $F(1,\underline{z};\underline{x})$ (toric resolution). However, one needs l^2 (complex) moduli to achieve that. This leads to a highly redundant model that is very hard to solve. We provide a better CY motive latter.

Master Integrals for the banana integrals: The banana graph has $2^{l+1}-1$ master integrals in $l+2$ sectors:

$l+1$ sectors correspond to $\vartheta(\underline{\nu}) = (1,\ldots,1,0,1\ldots1)$. These sectors correspond all to l-loop tadpole integrals

$$J_{l,i}(\underline{z};\epsilon)=\frac{(-1)^{l+1}(p^2)^{l\epsilon}\epsilon^l}{\Gamma(1+l\epsilon)}I_{1..1,0,1..1}(\underline{x};D)=-\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)}\prod_{\substack{j=1\\j\neq i}}^{l+1} z_j^{-\epsilon}.$$

$2^{l+1}-l-2$ master integrals come from the sector

$\vartheta(\underline{\nu}) = (1, \ldots, 1)$, $\underline{k} \in \{0, 1\}^{l+1}$, $1 \leq |\underline{k}| \leq l - 1$,

$$J_{l,\underline{0}}(\underline{z};\epsilon) = \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)}(p^2)^{1+l\epsilon}\ I_{1,\ldots,1}(\underline{x}; 2-2\epsilon)\,,$$
$$J_{l,\underline{k}}(\underline{z};\epsilon) = (1+2\epsilon)\cdots(1+|\underline{k}|\epsilon)\partial_{\underline{z}}^{\underline{k}} J_{l,\underline{0}}(\underline{z};\epsilon)\,.$$

Here $|\underline{k}| = \sum_{j=1}^{l+1} k_j$ and $\partial_{\underline{z}}^{\underline{k}} =: \prod_{i=1}^{l+1} \partial_{z_i}^{k_i}$.

The number of master integrals changes discontinuosly, when $\underline{x}, \epsilon$ changes:

- E.g. if $m_i^2 = m^2$, the equal-mass case yields only $l + 1$

master integrals

$$
\begin{aligned}
J_{l,0}(z;\epsilon) &= \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)}(m^2)^{l\epsilon}\epsilon^l \; I_{1,\ldots,1,0}(p^2,m^2;2-2\epsilon) = -\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)}\,,\\
J_{l,1}(z;\epsilon) &= \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)}(m^2)^{1+l\epsilon} \; I_{1,\ldots,1}(p^2,m^2;2-2\epsilon)\,,\\
J_{l,k}(z;\epsilon) &= (1+2\epsilon)\cdots(1+k\epsilon)\;\partial_z^{k-1}J_{l,1}(z;\epsilon)\,, \qquad \text{for } 2\le k\le l\,,
\end{aligned}
$$

- In the critical dimension, i.e. $\epsilon = 0$, we have only $2^{l+1} - \binom{l+2}{\lfloor \frac{l+2}{2}\rfloor} + 1$ independent master integrals. This was shown in [2] by analyzing the horizontal cohomology of a complete intersection Calabi-Yau geometry M^{CI}_{l-1}, i.e. the derivatives of the holomorphic Ω_{l-1} form, modulo the Griffith reduction relations or the vertical cohomology of its mirror W^{CI}_{l-1}

$$h^{l-1-k,k}_{\mathsf{hor}}(M^{\mathsf{CI}}_{l-1}) = \begin{cases} \binom{l+1}{k} & \text{if } k \leq \lceil \frac{l}{2} \rceil - 1 \\ \binom{l+1}{l-1-k} & \text{otherwise} \end{cases}$$

First order IBP and Gauss-Manin connection: Physics experience shows that one can recast the IBP relations as

$$\mathrm{d}\underline{I}(\underline{x};\epsilon) = \mathbf{A}(\underline{x};\epsilon)\,\underline{I}(\underline{x};\epsilon)\,,$$

where $\mathrm{d} = \sum_{k=1}^{N} \mathrm{d}x_k\,\partial_{x_k}$ and $\mathbf{A}(\underline{x};\epsilon)$ is a matrix of rational one-forms. In this first order form one can identify the master integrals $\mathrm{d}\underline{I}(\underline{x};\epsilon)$ as the Hodge bundle over $\underline{x},\epsilon$ and the first order differential form as its flat Gauss-Manin connection.

To provide an iterative solution scheme one searches for a new basis $\underline{I}(\underline{x};\epsilon) = \mathbf{M}(\underline{x};\epsilon)\underline{J}(\underline{z};\epsilon)$

$$\begin{aligned} \mathrm{d}\underline{J}(\underline{z};\epsilon) &= \widetilde{\mathbf{A}}(\underline{z};\epsilon)\underline{J}(\underline{z};\epsilon)\,, \\ \widetilde{\mathbf{A}}(\underline{z};\epsilon) &= \mathbf{M}(\underline{x};\epsilon)^{-1}\left[\mathbf{A}(\underline{x};\epsilon)\mathbf{M}(\underline{x};\epsilon) - \mathrm{d}\mathbf{M}(\underline{x};\epsilon)\right]\,, \end{aligned}$$

so that

- $J_{i,0}(\underline{z};\epsilon) = \lim_{\epsilon\to 0} J_i(\underline{z};\epsilon)$ are finite and non-zero

- $\mathbf{A}_0(\underline{z}) := \lim_{\epsilon\to 0}\widetilde{\mathbf{A}}(\underline{z};\epsilon)$ regular and one composes the master integrals into its sectors

$$\underline{J}(\underline{z};\epsilon) = (\underline{J}_1(\underline{z};\epsilon)^T, \ldots, \underline{J}_s(\underline{z};\epsilon)^T)^T\,,$$

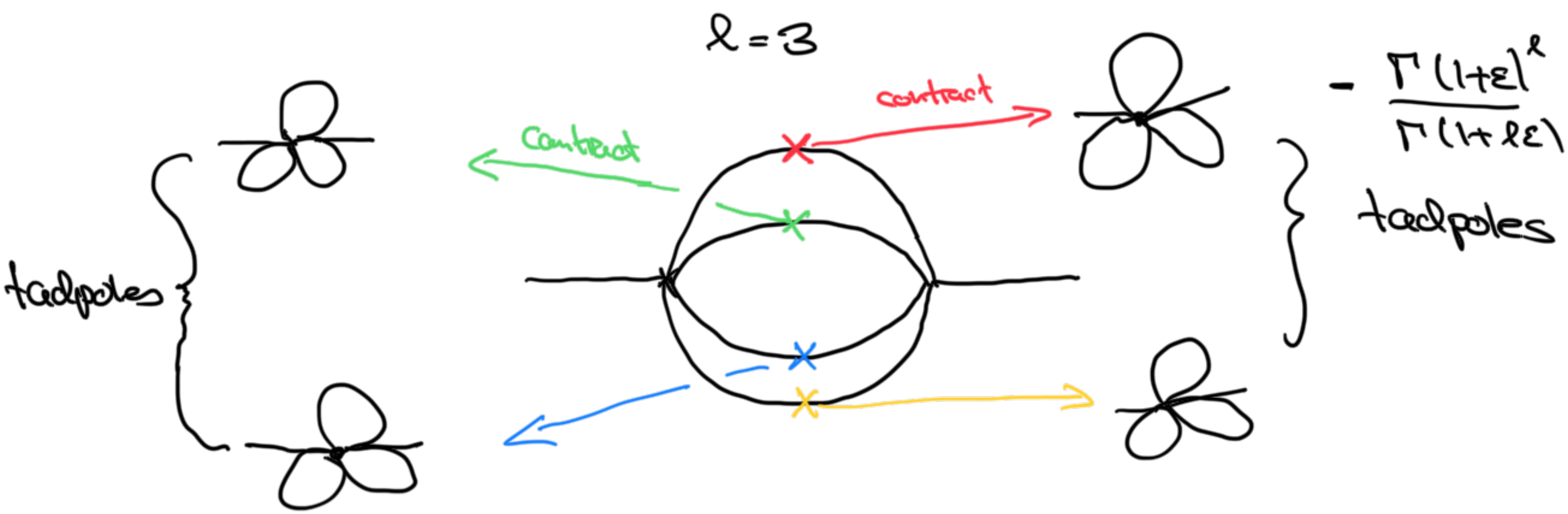
so that $\widetilde{\mathbf{A}}(\underline{z};\epsilon)$ becomes block-diagonal and the master integrals in each sector satisfy an inhomogeneous differential equation

$$\mathrm{d}\underline{J}_r(\underline{z};\epsilon) = \mathbf{B}_r(\underline{z};\epsilon)\,\underline{J}_r(\underline{z};\epsilon) + \underline{N}_r(\underline{z};\epsilon)\,, \qquad 1 \leq r \leq s\,.$$

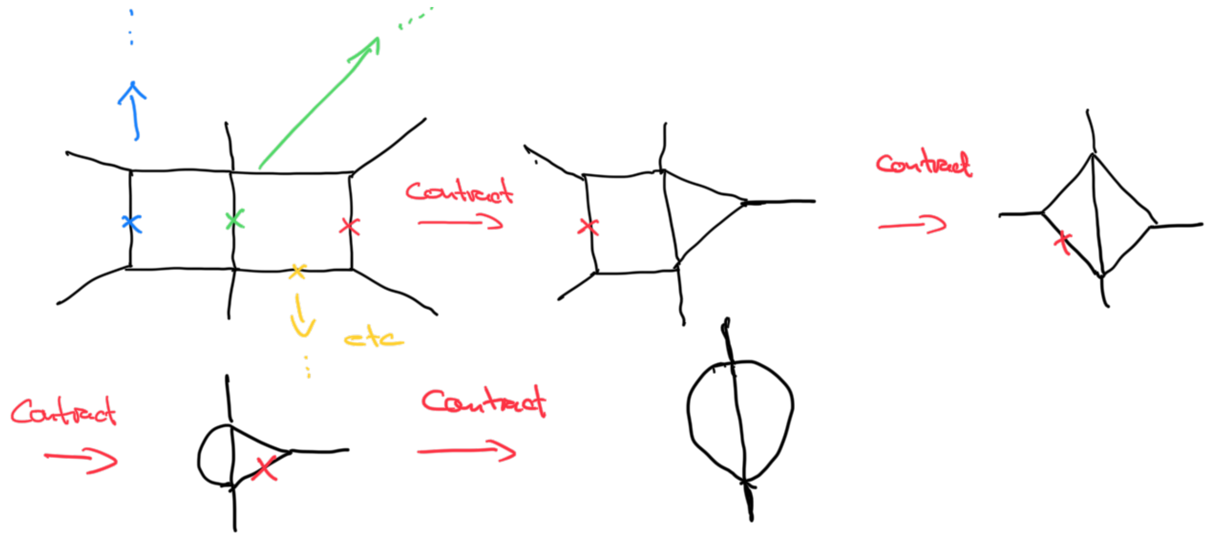
where the inhomogeneity $\underline{N}_r(\underline{z};\epsilon) = W_l(z)^{-1}\tilde{N}_l(\underline{z};\epsilon)$ contains integrals from lower sectors, which should have been characterized analytically in previous steps of the iterative scheme.

The special role of the banana integrals in this program:

- The lower sectors are all tadpoles yielding already analytic expressions.



- Banana integrals do occur in the iterative procedure within each more complicated Feynman diagram.



- The homogenous solutions or maximal cuts correspond in the critical dimension or said differently in leading order in $\epsilon \to 0$ the period integrals of families of Calabi-Yau $(n = l - 1)$-folds.

The dictionary between maximal cut integrals and families of Calabi-Yau motives:

	$l=(n+1)$-loop banana integrals in $D=2$ dimensions	Calabi-Yau (CY) geometry
1	Maximal cut integrals in $D=2$ dimensions	$(n,0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integrand-basis for maximal cuts of of master integrals in $D=2$	Middle (hyper) cohomology $H^n(M_n)$ M_n
4	Quadratic relations among maximal cut integrals	Quadratic relations from Griffiths transversality
5	Integration-by-parts (IBP) reduction	Griffiths reduction method

6	Complete set of differential operators annihilating a given maximal cut in $D = 2$ dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection
7	(Non-)maximal cut contours	(Relative) homology of CY geometry $H_n(M_n)$ $(H_{n+1}(F_{n+1}, \partial\sigma_{n+1}))$
8	Contributions from subtopologies to the differential equations	Extensions of the PFI or the GM connection
9	Full banana integrals in $D = 2$ dimensions	Chain integrals in CY geometry or extensions of Calabi-Yau motive
10	Degenerate kinematics (e.g., $m_i^2 = 0$ or $p^2/m_i^2 \to 0$)	Critical divisors of the moduli space
11	Large-momentum regime $p^2 \gg m_i^2$	Point of maximal unipotent monodromy & $\widehat{\Gamma}$-classes of W_n

12	General logarithmic degenerations	Limiting mixed Hodge structure from monodromy weight filtration
13	Analytic structure and analytic continuation	Monodromy of the CY motive and its extension
14	Special values of the integrals for special values of the z_i	Reducibility of Galois action & L-function values
15	(Generalized?) modularity of Feynman integrals	Global $\mathrm{O}(\mathbf{\Sigma}, \mathbb{Z})$-monodromy, integrality of mirror map & instantons expansion

Status of this program for the banana integrals:

- For the banana graphs the PFI is a Gel′fand-Kapranov-Zelevinskiĭ ideal and the program for $\epsilon \to 0$ has been completed [1,2].

- In this work [3] we further generalize that to include the general ϵ dependence.

Definition: A Calabi -Yau n-fold M is a Kähler manifold of complex dimension n with the following additional equivalent properties

- a.) The Ricci curvature vanishes $R_{i\bar{\jmath}} = 0$
- b.) The canonical class is trivial $K_M = c_1(T_M) = 0$
- c.) It has a no-where vanishing holomorphic $(n, 0)$-form Ω (and a type $(1, 1)$ Kähler form ω)
- d.) It has $SU(n)$ holonomy
- e.) It has two covariant constants spinors . . .

Constructions: Trick make $c_1(M) = 0$ by adjunction formula: Take any Fano variety F, i.e. a Kähler variety with $c_1(F) > 0$ and

- a). take a section of K_F (compact)
- b). take the complement of a section of K_F in F (non-compact)
- c) take the total space of the anti-canonical line bundle $\mathcal{O}(-K_F) \to F$ (non-compact)

Examples:

- a.) $F = \mathbb{P}^2$, $K_F = 3H$, i.e. a section of K_F is a homogenous cubic, often written in Weierstrass form

$$wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$$

  Using the Weierstrass $\mathcal{P}(\tau(z), \zeta)$-function this is identified with a 2-torus with complex structure $\tau(z) \sim_{loc} z$, which is (Ricci) flat. Note $\Omega = dx/y$

- b.) $F = \mathbb{P}^1 \sim S^2$, $K_F = 2H$. The section $x^2 + y^2 = 0$ defines two Pts. A sphere minus 2 Pts is 2-cylinder which is (Ricci) flat

- c.) $F = \mathbb{P}^1$: $\mathcal{O}(-2) \to \mathbb{P}^1$ is the local neighbourhood of a two sphere in $K3$, which is flat by the adjunction formula ....

Main properties:

- Their complex moduli spaces $\mathcal{M}_{cs}$ are unobstructed. This leads to families of CY n-folds over $\mathcal{M}_{cs}$.

- Periods $\Pi(z) = \int_\Gamma \Omega(z)$ parametrize locally faithfully the $\mathcal{M}_{cs}$ (local) Torelli Theorem

- Periods are completly characterized by homogenous linear differential equations called Gauss-Manin– or Picard-Fuchs system.

- CY manifolds come generically in mirror pairs (M,W) for which the complex deformations and the complexified Kähler deformations are exchanged

- Periods fulfill Griffths transversality conditions that leads to a special geometric structure on $\mathcal{M}_{cs}$

Periods and Calabi-Yau motives:

- Period integrals for geometric families of Calabi-Yau n-folds M_n

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z}) \,,$$

are pairings

$$\Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C} \,.$$

- Here Γ_i is a fixed basis of homology $H_n(M_n, \mathbb{Z})$ and $\gamma^j(\underline{z})$ a basis of cohomology $H^n(M_n, \mathbb{C})$ varying with the complex structure $\underline{z}$.

- One can made a fix choice $\tilde{\gamma}^j \in H^n(M_n, \mathbb{Z})$ so that

$\int_{\Gamma_i} \tilde{\gamma}^j = \delta_i^j$ and $\int_{M_n} \tilde{\gamma}^i \wedge \tilde{\gamma}^j = \Sigma^{ij}$ yields the intersection form Σ.

- Σ is an even lattice form or an odd integer symplectic form if n is even or odd,respectively.

- The periods are solution to a homogenous Gauss-Manin-System and correspond for fixed Γ_i to the maximal cut integrals.

- Maximal cut integrals are characterized either by being homogeneous solutions of the GM-System or that the corresponding contours enclose all propagator poles.

- For families of Calabi-Yau manifolds (motives) the first order Gauss-Manin system is equivalent to the Picard-Fuchs differential ideal (PFI).

- For $l = 1$ the periods are rational — for $l = 2$ elliptic functions.

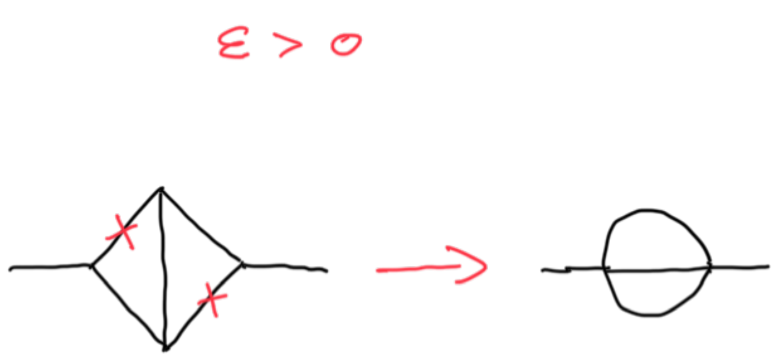
- For higher l they generalise to periods of (weight or) dimension $n = l - 1$ families of Calabi-Yau manifold (motives).

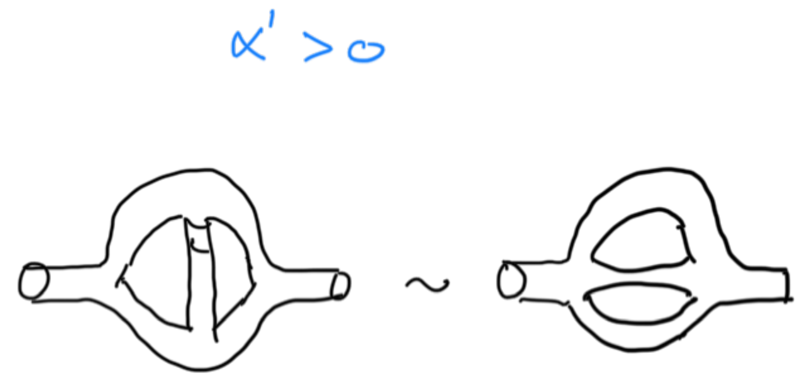
- Calabi-Yau motives are characterized by their Picard-Fuchs differential ideal, the intersection form Σ and monodromies in $\mathcal{O}(\Sigma, \mathbb{Z})$.

- The inhomogenous solutions for $\epsilon \to 0$ correspond to the extension of these Calabi-Yau motives by chain integrals.

- Both structures can be analytically solved everywhere in the parameter space using the PFI, the $\widehat{\Gamma}$ class and its extension.

- Connecting graphs of different topologies using the symmetry preseriving ϵ regularisation resembles strongly the unification of topologies using the α' regularisation and string/QFT correspondence principle.





A better Calabi-Yau motive: A very elegant way to circumvent this problem was proposed in [2]. Consider the complete intersection of two polynomials of degree

$(1, \ldots, 1)$ in

$$\mathbb{P}_{l+1} := \bigotimes_{i=1}^{l+1} \mathbb{P}^1_{(i)} \,,$$

i.e., we have after an étale map to mirror coordinates

$$\begin{aligned} M^{\mathsf{CI}}_{l-1} \quad &= \left\{ \left(w_1^{(i)} : w_2^{(i)} \right) \in \widehat{\mathbb{P}^1_{(i)}}, \forall i \right| \\ & P_1 := \textstyle\sum_{i=1}^{l+1} a^{(i)} w_1^{(i)} + b^{(i)} w_2^{(i)} = \sum_{i=1}^{l+1} c^{(i)} w_1^{(i)} + d^{(i)} w_2^{(i)} =: P_2 = 0 \Bigg\} \,. \end{aligned} \tag{2}$$

This is transversal $\mathrm{d}P_1 \wedge \mathrm{d}P_2 \neq 0$ when $P_1 = P_2 = 0$ iff

$$\det \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \neq 0$$

for all $i = 1, \ldots, l+1$. On every $\mathbb{P}^1_{(i)}$ there is a natural $\mathrm{SL}_i(2, \mathbb{C})$ action which allows to make the choices

$$\begin{aligned} & a^{(i)} = -\frac{m_i^2}{p^2} = -z_i, \quad d^{(i)} = x, \quad i = 1, \ldots, l+1, \\ & b^{(1)} = \frac{x}{w_2^{(1)}}, \quad c^{(1)} = \frac{1}{w_1^{(1)}}, \\ & b^{(i)} = c^{(i)} = 0, \quad i = 2, \ldots, l+1, \end{aligned}$$

One can construct a birational map from the smooth complete intersection geometry to the singular hypersurface geometry by solving for $P_1 = 0$ such that one gets $x = \sum_{i=1}^{l+1} \frac{m_i^2}{p^2} w_1^{(i)}$. P_2 becomes $P_2 = 1 - x \sum_{i=1}^{l+1} w_2^{(i)}$.

Passing to $\mathbb{C}^*$ coordinates $w_1^{(i)} = W_i$ and $w_2^{(i)} = 1/W_i$, for $i = 1, \ldots, l+1$, we find

$$P_2 = p^2 - \left(\sum_{i=1}^{l+1} m_i^2 W_i \right) \left(\sum_{i=1}^{l+1} \frac{1}{W_i} \right) .$$

This is the hypersurface written in $\mathbb{C}^*$ coordinates!

Remarks on Calabi-Yau motives:

- The smooth hypersurface geometry M_n^{HS} and M_n^{CI} have not the same topology. Indeed it is easy to check that

$$\chi(M_3^{\mathsf{HS}}) = 20\,, \qquad \chi(M_4^{\mathsf{HS}}) = 540\,, \qquad \ldots\,,$$
$$\chi(M_3^{\mathsf{CI}}) = -80\,, \qquad \chi(M_4^{\mathsf{CI}}) = 720\,, \qquad \ldots\,.$$

- They are related by a singular transition. E.g. for $n=3$ we have

$$X_u^{16,26} \to X_a^{\mathsf{sing}} \to \hat{X}^{5,45}\,. \tag{3}$$

Here $X_u^{16,26}$ is our deformed space M_3^{HS} with $\chi(M_3^{\mathsf{HS}}) =$

20. The superscripts are h_{21} and h_{11}, respectively. X_a^{sing} is the singular five-parameter space in the physical slice written in torus variables, and $\hat{X}^{5,45}$ is a small resolution of X_a^{sing}. X_a^{sing} has 30 nodes, where S^3-spheres are shrinking. Replacing each singular loci by $\mathbb{P}^1$ resolved them small. Each resolution adds $\chi(\mathbb{P}^1) = 2$ to the Euler number leading the topology $\hat{X}^{5,45}$, which mirror to M_3^{CI}!

- Indeed, M_n^{CI} is self mirror in the following sense: Consider the $l+1$ deformations of (2). By taking deriviatives of Ω_n w.r.t. these parameters modulo the

Griffiths partial integration relation

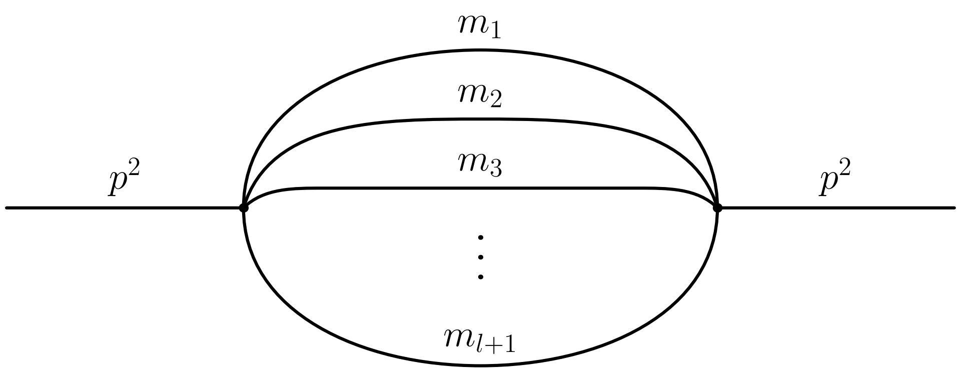
$$\sum_{k\neq j}\frac{m_k}{m_j-1}\frac{P_j}{P_k}\frac{q\partial_{x_i}P_k}{\prod_{l=1}^r P_l^{m_l}}\mu \;=\frac{1}{m_j-1}\frac{P_j\partial_{x_i}q}{\prod_{l=1}^r P_l^{m_l}}\mu-\frac{q\partial_{x_i}P_j}{\prod_{l=1}^r P_l^{m_l}}\mu\ ,$$

we can define $H_n^{\text{hor}}(M_{l-1}^{\text{CI}})$. The latter is mirror to the vertical cohomology $H_{\text{vert}}^{k,k}(W_n^{\text{CI}})$ that is inherited from $\mathbb{P}_{l+1}$. Similar remarks hold for the associated homology groups. If we restrict ourselves to these vertical– and horizontal subspaces, then the geometries for the banana graphs are self mirrors

$$M_{l-1}^{\text{CI, res}}=W_{l-1}^{\text{CI, res}}\ .$$

The mirror picture:

The vertical quantum cohomology of W^{CI}_{l-1} relates natural to the banana graph



$$\longleftrightarrow W^{\mathsf{CI}}_{l-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array} \right) \subset \left(\begin{array}{c||c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 \end{array} \right) = F_l \,.$$

In particular, in the high energy regime we get a one-to-one identification of the complexified (large volume) Kähler parameters t^k of the $l+1$ rational curves $\mathbb{P}^1_k$ with the physical parameters m_i^2/p^2

$$t^k \simeq \frac{1}{2\pi i}\int_{\mathbb{P}^1_k}(i\omega - b) + \mathcal{O}(e^{-t^k}) = \frac{\log}{2\pi i}\left(\frac{m_k^2}{p^2}\right) = \frac{\log(z_k)}{2\pi i}$$

for $k = 1, \ldots, l+1$. Away from the limit the mirror symmetry for complete intersections and, in particular, the associated GKZ system provides the exact answer, including the exponentially suppressed $\mathcal{O}(e^{-t^k})$ corrections.

The fiberation Structure: $E = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \mathbb{P}_2 & 1 & 1 \\ \mathbb{P}^1_3 & 1 & 1 \end{array} \right)$ is the elliptic curve associated to the two-loop graph. The K3 associated to the three-loop graph $K_3 = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \mathbb{P}^1_2 & 1 & 1 \\ \mathbb{P}^1_3 & 1 & 1 \\ \mathbb{P}^1_4 & 1 & 1 \end{array} \right)$ is fibered in four ways by E over each of its $\mathbb{P}^1_k$. The K3 fibres the Calabi-Yau three-fold W_3^{CI} in five ways and so on.

<u>The $\widehat{\Gamma}$-classes:</u> A powerful application of the geometric realization W_{l-1}^{CI} is the $\widehat{\Gamma}$-class formalism. It relates the Frobenius $\mathbb{Q}$-basis of solutions at the point of maximal unipotent monodromy (MUM) to an integral $\mathbb{Z}$-basis of solutions to the PFI.

The latter contains the maximal cut integral that corresponds to the unique period $\Pi_{\mathbf{S}}$ over a $S^{l-1} =: \mathbf{S}$ that vanishes at the nearest conifold and describes the imaginary part of the banana integral above threshold. An extension of the $\widehat{\Gamma}$-class also yields the full Feynman integral in the critical dimension. Note that $\mathbf{S} \cap \mathbf{T} = 1$

and both cycles play a crucial role in homological mirror symmetry.

Let I_p an index set of order $|I_p| = p$ and define the Frobenius basis at the MUM point:

$$S_{(p),k}(\underline{z}) = \frac{1}{(2\pi i)^p p!} \sum_{I_p} \kappa_{(p),k}^{i_1,\ldots,i_p} \varpi_0(\underline{z}) \log(z_{i_1}) \cdots \log(z_{i_p}) + \mathcal{O}(\underline{z}^{1+\alpha})\,.$$

Here $|S_{(p)}(\underline{z})|$ denotes the total number of solutions which are of leading order p in $\log(z_i)$ and $\kappa_{(p),k}^{i_1,\ldots,i_p}$ are intersection numbers of the mirror W_{l-1}^{CI}.

In particular, the Kähler parameters t^k are given by the mirror map

$$t^k(\underline{z}) = \frac{S_{(1),k}(\underline{z})}{S_{(0),0}(\underline{z})} = \frac{1}{2\pi i}\left(\log(z_k) + \frac{\Sigma_k(\underline{z})}{\varpi_0(\underline{z})}\right),$$

for $k = 1, \ldots, h^{11}(W_n) = h^{n-1,1}(M_n)$.

Homological mirror symmetry predicts then a maximal cut integral

$$\Pi_{\mathbf{S}}(\underline{t}(\underline{z})) = \int_{W_{l-1}} e^{\underline{\omega}\cdot\underline{t}}\,\widehat{\Gamma}(TW_{l-1}) + \mathcal{O}(e^{-\underline{t}}) \qquad (4)$$

and an extension also yields the full Feynman integral

$$J_{l,\underline{0}}(\underline{z},0) = \int_{F_l} e^{\underline{\omega}\cdot\underline{t}}\,\widehat{\Gamma}_{F_l}(TF_l) + \mathcal{O}(e^{-\underline{t}}) \ . \qquad (5)$$

Here the extended $\widehat{\Gamma}$-class is given by

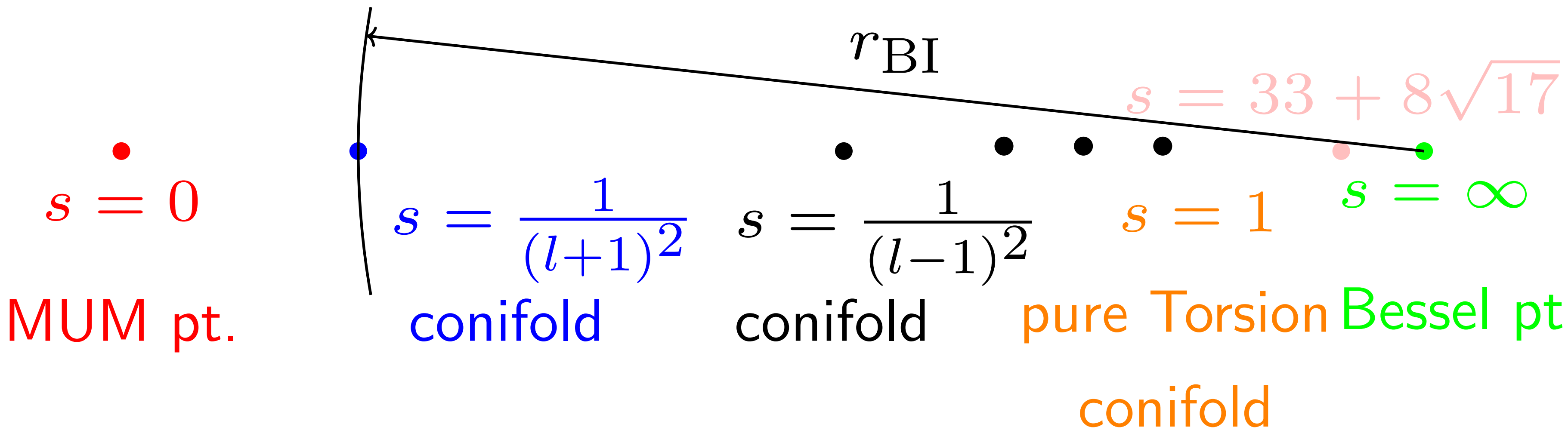
$$\widehat{\Gamma}_F(TF) = \frac{\widehat{A}(TF)}{\widehat{\Gamma}^2(TF)} = \frac{\Gamma(1-c_1)}{\Gamma(1+c_1)}\cos(\pi c_1) \ .$$

By comparing the powers of $t^k \sim \log(z_k)$ on both sides of (4),(5) using the mirror map these formulas determine uniquely the exact boundary conditions for the integrals in terms of topological intersection calculations on W^{CI}_{l-1} or the Fano variety F_l and the Frobenius basis for the banana graph [2]. Let us give an example for $I_{l,1}(\underline{p},\underline{m},D=2)$ up to five loops

l	$S_{(0),1}$	$S_{(1),1}$	$S_{(2),1}$	$S_{(3),1}$	$S_{(4),1}$
1	$-2\pi i$				
2	$18\zeta(2)$	$6\pi i$			
3	$-16\zeta(3)+24i\pi\zeta(2)$	$-72\zeta(2)$	$-12\pi i$		
4	$-450\zeta(4)-80i\pi\zeta(3)$	$80\zeta(3)-120\pi i\zeta(2)$	$180\zeta(2)$	$20\pi i$	
5	$-288\zeta(5)+1440\zeta(2)\zeta(3)-$ $540i\pi\zeta(4)$	$2700\zeta(4)+$ $480i\pi\zeta(3)$	$-240\zeta(3)+$ $360\pi i\zeta(2)$	$-360\zeta(2)$	$-30\pi i$

Roadmap to the physical moduli space:

$$s = 1/t \in \mathcal{M}_{cs}(M_{l-1}) = \mathbb{P}^1 \setminus \left(\bigcup_{j=0}^{\lfloor \frac{l+1}{2} \rfloor} \left\{ \frac{1}{(l+1-2j)^2} \right\} \cup \{0\} \right)$$



Comparison with a Barnes integral representations:

Using the indentity

$$\frac{1}{(A+B)^{\lambda}} = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}\xi}{2\pi i}\; A^{\xi} B^{-\xi-\lambda} \frac{\Gamma(-\xi)\Gamma(\xi+\lambda)}{\Gamma(\lambda)}\,,$$

one can rewrite $\mathcal{F}(p^2, \underline{m}^2)^{-\omega}$ as

$$\tilde{I} = \int \frac{\mathrm{d}\xi_0}{2\pi i} \frac{\Gamma(-\xi_0)\Gamma(\xi_0+\omega)}{\Gamma(\omega)} (-p^2)^{\xi_0} \times \int_{[0,\infty)^l} \mathrm{d}x_1 \ldots \mathrm{d}x_l \left(\prod_{i=1}^{l+1} x_i^{\xi_0+\delta_i}\right) \left(\sum_{i=1}^{l+1} \prod_{\substack{j=1\\ j\neq i}}^{l+1} x_j\right)^{-\xi_0-\frac{d}{2}} \left(\sum_{i=1}^{l+1} m_i^2 x_i\right)^{-\xi_0-\omega} . \tag{6}$$

The actual specifications of the contours in this integral is very complicated. But one can correctly close in the large momentum region and here also with $\delta_i = 0$ for all i the contours to find

$$I_{1,\ldots,1}(p^2, \underline{m}^2; 2-2\epsilon) = \frac{1}{\Gamma(1+l\epsilon)} \left(\frac{1}{-p^2 - i0}\right)^{1+l\epsilon} \sum_{\underline{j}\in\{0,1\}^{l+1}} \frac{\Gamma(-\epsilon)^j\,\Gamma(\epsilon)^{l+1-j}\Gamma(1+(j-1)\epsilon)}{\Gamma(-j\epsilon)}$$

$$\times \left[\prod_{i=1}^{l+1}\left(\frac{m_i^2}{-p^2 - i0}\right)^{(j_i-1)\epsilon} \sum_{\underline{n}\in\mathbb{N}_0^{l+1}} \frac{(1+j\epsilon)_n\,(1+(j-1)\epsilon)_n}{\prod_{i=1}^{l+1}(1+(-1)^{j_i+1}\epsilon)_{n_i}} \prod_{i=1}^{l+1}\frac{1}{n_i!}\left(\frac{m_i^2}{p^2}\right)^{n_i}\right] \tag{7}$$

and infer the leading asymptotic behavior at large momentum.

Letting $\underline{n} = (0, \ldots, 0)$ in eq. (7), we can extract the leading behavior of the banana integrals at large momentum, e.g. for the generic-mass case

$$I_{1,\ldots,1}(p^2, \underline{m}^2; 2-2\epsilon) = -\frac{1}{\Gamma(1+l\epsilon)} e^{i\pi l\epsilon} \left(\frac{1}{p^2}\right)^{1+l\epsilon}$$
$$\times \sum_{\underline{j}\in\{0,1\}^{l+1}} e^{i\pi(j-1)\epsilon} \frac{\Gamma(-\epsilon)^j\,\Gamma(\epsilon)^{l+1-j}\;\Gamma(1+(j-1)\epsilon)}{\Gamma(-j\epsilon)} \prod_{i=1}^{l+1} z_i^{(j_i-1)\epsilon} + \mathcal{O}\left(z_i^2\right) . \tag{8}$$

This gives the leading asymptotics of $I_{1,\ldots,1}(p^2, \underline{m}^2; 2-2\epsilon)$ and can be used as a boundary condition to solve the differential equations for the banana graphs. In particular, in the equal-mass case the expression further simplifies to

$$J_{l,1}(z;\epsilon) = -\sum_{k=1}^{l+1} \binom{l+1}{k} \frac{\Gamma(-\epsilon)^k\ \Gamma(\epsilon)^{l+1-k}}{\Gamma(-k\epsilon)} \frac{\Gamma(1+(k-1)\epsilon)}{\Gamma(1+l\epsilon)} e^{(k-1)i\pi\epsilon}\, z^{1+(k-1)\epsilon} + \mathcal{O}\left(z^2\right) .$$

Expanding this around $\epsilon = 0$, one obtains

$$J_{l,1}(z;\epsilon) = \sum_{n=0}^{\infty} J_{l,1}^{(n)}(z)\ \epsilon^n\ . \tag{9}$$

The leading order in ϵ, i.e. $J_{l,1}^{(0)}$, precisely reproduces the logarithmic structure of the l-loop banana Feynman integral in $D = 2$ spacetime dimensions!

The homogeneous differential operators $\mathcal{L}_{l,\epsilon}$ that annihilate the maximal cuts of the banana integrals in $D = 2 - 2\epsilon$ dimensions (the inhomogeniety is $-\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)}$):

Loop order l	Differential operator $\mathcal{L}_{l,\epsilon}$
1	$1 + \epsilon - 2z - (1 - 4z)\theta$
2	$(1 + 2\epsilon)(1 + \epsilon - 3z + z\epsilon) + \left(-2 - 3\epsilon + 10z + 10z\epsilon + 9z^2\epsilon\right)\theta + (1 - z)(1 - 9z)\theta^2$
3	$(1 + 2\epsilon)(1 + 3\epsilon)(1 + \epsilon - 4z + 2z\epsilon) + (-3 - 12\epsilon + 18z + 60z\epsilon - 11\epsilon^2 + 28z\epsilon^2$ $+ 64z^2\epsilon^2)\theta - 3(-1 + 10z)(1 + 2\epsilon)\theta^2 - (1 - 4z)(1 - 16z)\theta^3$
4	$(1 + 2\epsilon)(1 + 3\epsilon)(1 + 4\epsilon)(1 + \epsilon - 5z + 3z\epsilon) + (-4 - 30\epsilon + 28z + 189z\epsilon$ $+ 26z^2\epsilon - 225z^3\epsilon - 70\epsilon^2 + 343z\epsilon^2 - 225z^3\epsilon^2 - 50\epsilon^3 + 84z\epsilon^3 + 414z^2\epsilon^3)\theta$ $+ (6 - 63z + 26z^2 - 225z^3 + 30\epsilon - 315z\epsilon - 675z^3\epsilon + 35\epsilon^2 - 343z\epsilon^2 - 363z^2\epsilon^2$ $- 225z^3\epsilon^2)\theta^2 - 2\left(2 - 35z + 225z^3 + 5\epsilon - 105z\epsilon + 259z^2\epsilon + 225z^3\epsilon\right)\theta^3$ $+ (1 - z)(1 - 9z)(1 - 25z)\theta^4$

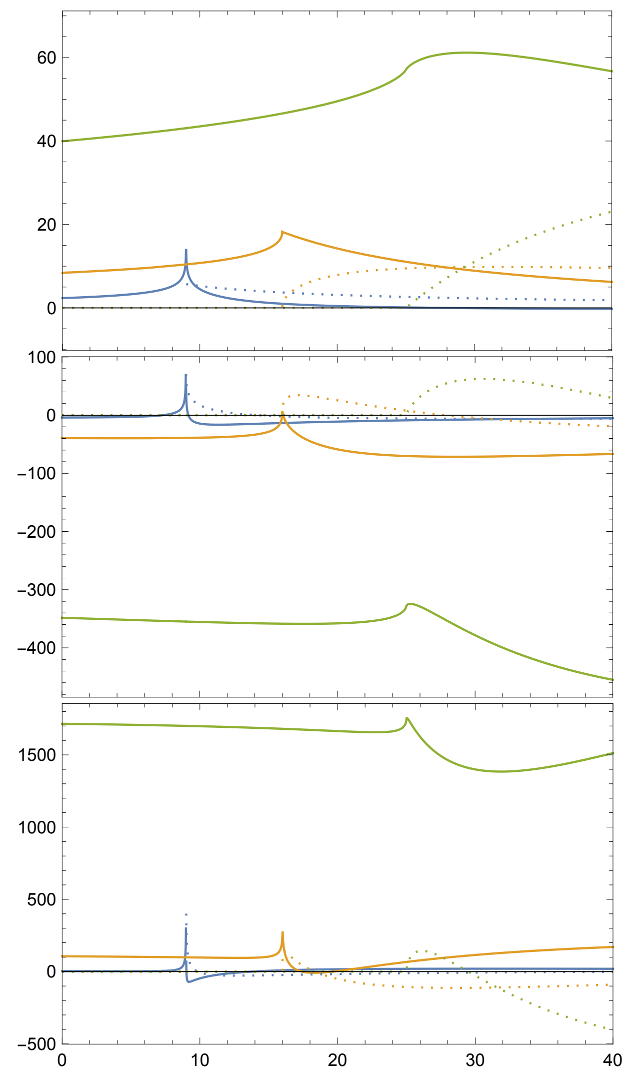


Figure 1: The banana integrals $J_{l,1}^{(n)}$ for $l = 2, 3, 4$ (blue, orange, green) and $n = 0, 1, 2$ (upper, middle and lower panels) against $1/z$.The solid: real part, dashed lines: imaginary part.

Properties of Calabi-Yau motives and their significances for Feynman integrals:

- Griffiths transversality: Let $\Pi(\underline{z}) = \left(\int_{\Gamma_1} \Omega, \ldots, \int_{\Gamma_r} \Omega\right)^T$ the period vector. Then one gets as a generalization of the observations of Bryant and Griffiths for Calabi-Yau n-folds:

$$\underline{\Pi}(\underline{z})^T \, \mathbf{\Sigma} \, \partial_{\underline{z}}^{\underline{k}} \underline{\Pi}(\underline{z}) = \int_{M_n} \Omega \wedge \partial_{\underline{z}}^{\underline{k}} \Omega = \begin{cases} 0 & \text{for } 0 \leq r < n \\ C_{\underline{k}}(\underline{z}) & \text{for } |k| = n \end{cases} \quad ,$$

where the $C_{\underline{k}}(\underline{z})$ are rational functions in the complex structure parameters. For the first equality, expand

Ω in an integer symplectic basis of cohomology. The second equality follows from Griffiths transversality and consideration of the Hodge type. Note an arbitrary local basis $\underline{\tilde{\Pi}}(\underline{z})$ corresponding to an (implicit) choice of a basis of cycles $\tilde{\Gamma}^i \in H_n(M_n, \mathbb{C})$, obtained as independent local solutions of the Picard-Fuchs differential ideal, one can find a $\tilde{\mathbf{\Sigma}}$ and write down the corresponding relations $\underline{\tilde{\Pi}}(\underline{z})^T \, \tilde{\mathbf{\Sigma}} \, \partial^k_{\underline{z}} \underline{\tilde{\Pi}}(\underline{z})$ among the solutions very explicitly. It implies that there are quadratic relations among the maximal cut integrals. For the banana graphs we checked explicitly that these are the only ones.

In particular $W_l(z)^{-1} = \Sigma W_l(z) C(z)^{-1}$.

- Self-adjointness: Let the Picard-Fuchs differential ideal be generated by a single (normalized) differential operator (as it is the case for one-parameter families), i.e.

$$\mathcal{L}^{(n+1)} = \partial_z^{n+1} + \sum_{i=0}^{n} a_i(z)\, \partial_z^i \,.$$

Then the Yukawa coupling C_n fulfills the differential equation

$$\frac{\partial_z C_n(z)}{C_n(z)} = \frac{2}{n+1} a_n(z)\,. \tag{10}$$

One can define the adjoint differential operator

$$\mathcal{L}^{*(n+1)} = \sum_{i=0}^{n+1} (-\partial_z)^i \, a_i(z) \, .$$

An operator is called essentially self-adjoint if

$$\mathcal{L}^{*(n+1)} A(z) = (-1)^{n+1} A(z) \mathcal{L}^{(n+1)} \, ,$$

where $A(z)$ satisfies the differential relation $\frac{\partial_z A(z)}{A(z)} = \frac{2}{n+1} a_n(z)$. Note that $A(z)$ is up to a multiplicative constant given by the Yukawa coupling $C_n(z)$.

A one-paramter maximal cut Feynman integral has to be annihilated by a self-adjoint linear differential operator if it comes from a CY geometry!

- Landman's theorem: It states that all possible monodromy matrices of an algebraic n-fold have to obey

$$(\mathbf{T}^k - \mathbb{1})^{n+1} = 0 \ . \tag{11}$$

Here $k \in \mathbb{N}_0$, implying that the indicial α has to be a rational number. A monodromy matrix $\mathbf{T}$ can be unipotent of lower order $m < n$, i.e., $(\mathbf{T}^k - \mathbb{1})^{m+1} = 0$. It is clear that m is the size of the biggest Jordan block in $\mathbf{T}$. The maximal n that can appear is $n = \dim(M)$. It is not too hard to see that the unipotency of order $m \leq n$ implies that a period on an n-fold cannot degenerate worse than with a logarithmic singularity of type $\log(\Delta)^n$. This has an important consequence for Feynman integrals. Assume that we have a maximal cut of a Feynman integral in integer dimensions that degenerates in a dimensionless physical parameter Δ (or, more generally, some polynomial combination thereof) as $\log(\Delta)^m$. Then it follows from Landman's theorem that the geometry associated to this maximal cut integral cannot be an algebraic manifold of dimension less than m, or a Calabi-Yau motive of weight less than m!

- The $\mathrm{SL}(2,\mathbb{C})$ theorem: This uses the limiting mixed Hodge Structure to restrict the structure of the Jordan Blocks further. For example, for a Calabi-Yau three-fold the classification of one-parameter operators (known under der name AESZ list) uses the fact that the only possible degenerations are of the following types:
  - The generic point F is characterized by generic local exponents.
  - The conifold point C has local exponents (a, b, b, c) and a single 2×2 Jordan block.

– The K point has local exponents (a, a, b, b) and two 2×2 Jordan blocks.

– Finally, the MUM-point M has local exponents (a, a, a, a) and a 4×4 Jordan block.

Here different characters stand for different rational numbers and the *limiting mixed Hodge diamond* $H^{p,q}_{\lim}$ for the different degenerations at $F-, C-, K-, M-$ points are depicted below:

$$
F \; : \; \begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
\qquad
C : \; \begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
$$

$$
K \quad : \begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 1 & & 0 & & 1 & \\
0 & & 0 & & 0 & & 0 \\
 & 1 & & 0 & & 1 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
\quad M : \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} .
$$

For example, the previous considerations allow us to completely classify the singular points of the Calabi-Yau three-fold associated to the four-loop banana integral. The complex moduli space is $\mathcal{M}_{\text{4-loop}} = \mathbb{P}^1 \setminus \{z = 0, 1/25, 1/9, 1, \infty\}$. The local exponents of the singular points are summarized in the Riemann $\mathcal{P}_l$-symbol in eq. (12) for $\epsilon \to 0$. In particular, using the list of local exponents above, we see that $z = 0$ is a M-point (MUM-point), $z = 1/25, 1/9, 1$ are C-points (conifolds), and $z = \infty$ is a K-point.

$$\mathcal{P}_2 \left\{ \begin{array}{llll} 0 & \frac{1}{9} & 1 & \infty \\ \hline 1+\epsilon & -2\epsilon & -2\epsilon & 0 \\ 1+2\epsilon & 0 & 0 & \epsilon \end{array} \right\}, \quad \mathcal{P}_3 \left\{ \begin{array}{llll} 0 & \frac{1}{16} & \frac{1}{4} & \infty \\ \hline 1+\epsilon & 0 & 0 & -\epsilon \\ 1+2\epsilon & \frac{1}{2}-3\epsilon & \frac{1}{2}-3\epsilon & 0 \\ 1+3\epsilon & 1 & 1 & \epsilon \end{array} \right\},$$

$$\mathcal{P}_4 \left\{ \begin{array}{lllll} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 1+\epsilon & 0 & 0 & 0 & 0 \\ 1+2\epsilon & 1-4\epsilon & 1-4\epsilon & 1-4\epsilon & \epsilon \\ 1+3\epsilon & 1 & 1 & 1 & 1 \\ 1+4\epsilon & 2 & 2 & 2 & 1+\epsilon \end{array} \right\}. \tag{12}$$