Poincaré series for Modular Graph Forms and Iterated integrals

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Joint work w/ A. Kleinschmidt and O. Schlotterer [2109.05017] [2109.05018]

Previous work with A.K. [1903.09250]

Online seminar on motives and period integrals

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My Fantastic Collaborators

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Max Planck - AEI

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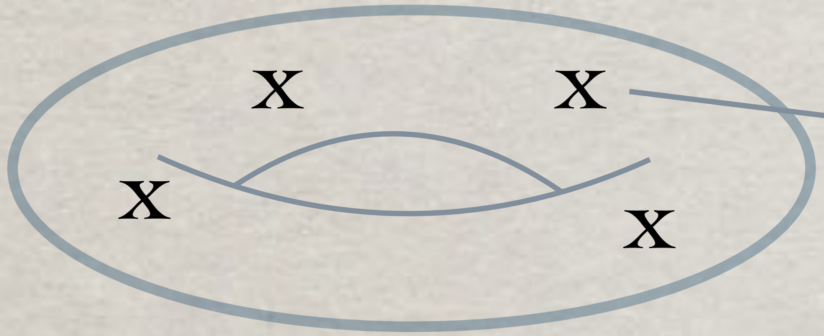
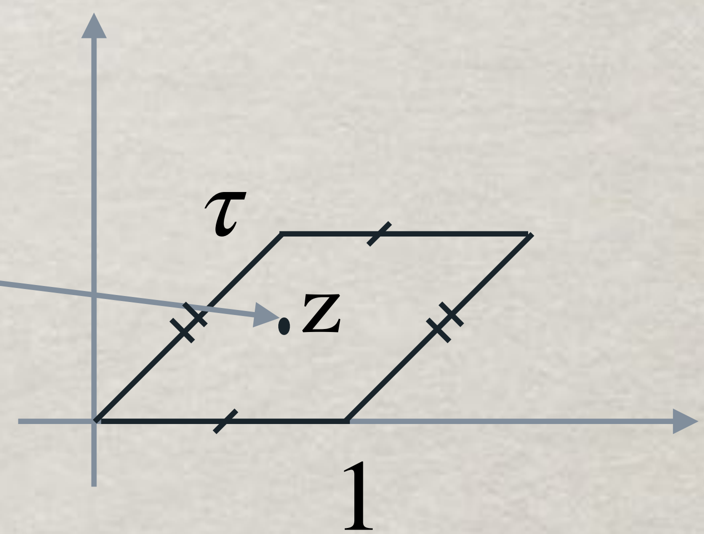
Setup:

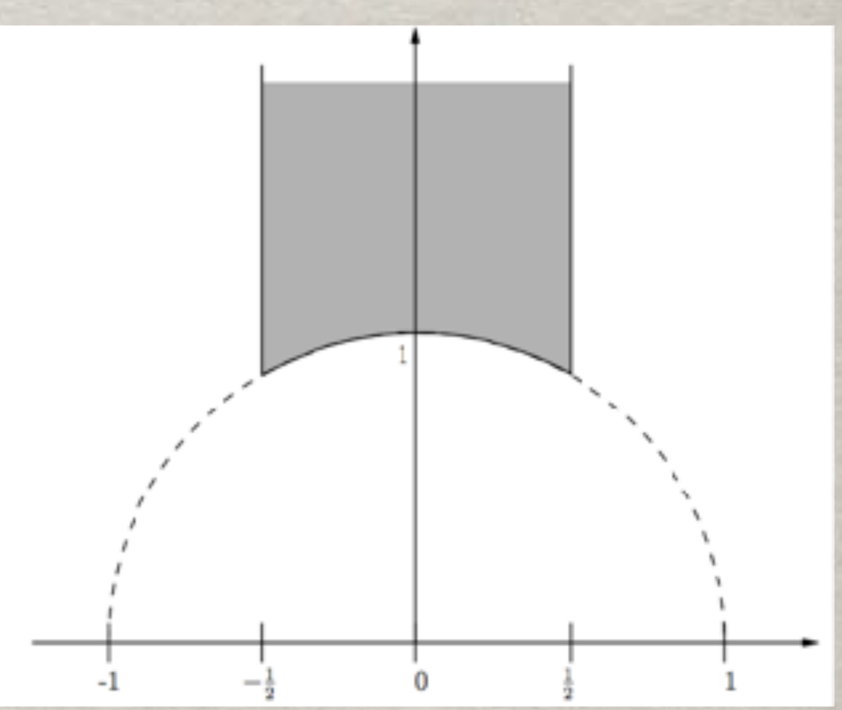
Closed-string scattering amplitudes @ genus-one:

Worldsheet = $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and $\tau = \tau_1 + i\tau_2$





$$\mathcal{F}_\tau = \{\tau \in \mathbb{H} \,\text{s.t.}\, |\tau_1| \leq \frac{1}{2}\,;\, |\tau| \geq 1\}$$



Setup:

The n-pt genus-one scattering amplitude

$$\int_{\mathcal{F}_\tau} \frac{d^2\tau}{\tau_2^2} \int_{\Sigma_\tau} \prod_{i=2}^{n} \frac{d^2 z_i}{\tau_2} \langle V_1(z_1) \cdots V_n(z_n) \rangle$$

Specifies the type of n-particles scattered and The string theory considered

Setup:

The n-pt genus-one scattering amplitude

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We will NOT be interested in

Setup:

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Non-holomorphic modular invariant function under $SL(2,\mathbb{Z})$. Main focus of this talk.

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Non-holomorphic modular invariant function under *SL*(2,$\mathbb{Z}$). Main focus of this talk.

Example: 4-graviton scattering in IIB

$$I(s_{ij};\tau,\bar{\tau}) = \int_{\Sigma_\tau} \prod_{i=2}^{3} \frac{d^2 z_i}{\tau_2} \exp\Big[\sum_{i\leq i<j\leq 4} s_{ij} G(z_i - z_j|\tau) \Big]$$

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Non-holomorphic modular invariant function under $SL(2,\mathbb{Z})$. Main focus of this talk.

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Mandelstam Variables

$s_{ij} = -\alpha' k_i \cdot k_j$

Torus Green's function

Example: 4-graviton scattering in IIB

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Not known exactly but can be computed in $\alpha' \to 0$
aka low energy expansion:
Mandelstam Variables $s_{ij} = -\alpha' k_i \cdot k_j$

The coefficient of the α'^n term is a non-holmorphic modular invariant function:

Diagrammatic graph expansion -> Modular Graph Functions (Forms)

[Green, Russo, Vanhove - D'Hoker, Green, Vanhove]

MGFs:

$$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|n\tau + m|^2} e^{2\pi i(nu-mv)}$$

$$z = u + v\tau; \qquad p = m + n\tau; \qquad < z, p >= nu - mv$$

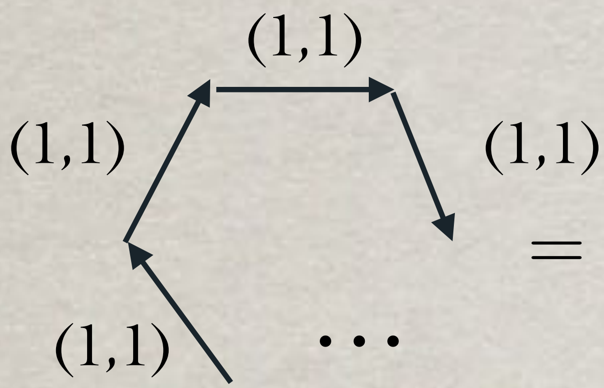
Building Block:

$$z_i \xrightarrow{(a,b)} z_j = \sum_{p\in\Lambda'} \frac{e^{2\pi i<p,z_i-z_j>}}{p^a \bar{p}^b} \qquad \Lambda' = \mathbb{Z} + \tau\mathbb{Z} \setminus \{0\}$$

$$exp(2\pi i < p, z_i - z_j >)$$

Very easy z integrals:
momentum conservation at each graph vertex

Examples:



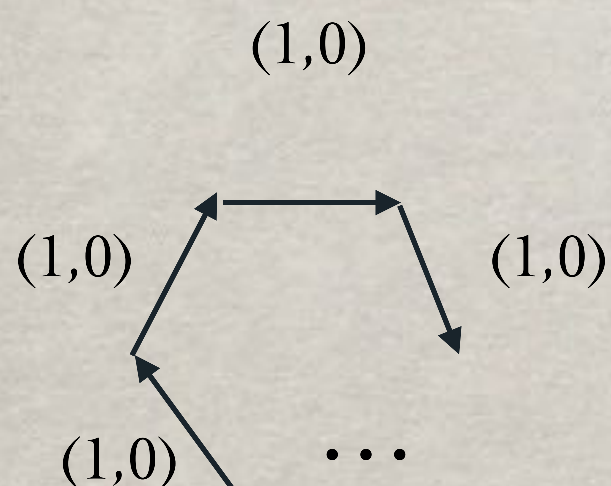
k vertices

$$= \sum_{p\in\Lambda'} \frac{1}{|p|^{2k}} = \left(\frac{\pi}{\tau_2}\right)^k E_k(\tau,\bar\tau)$$

non-holo,
modular invariant Eisenstein

$y = \pi\tau_2$

$$E_k(\tau,\bar\tau) = \mathbb{Q}y^k + \mathbb{Q}\zeta_{2k-1}y^{1-k} + O(q,\bar q)$$



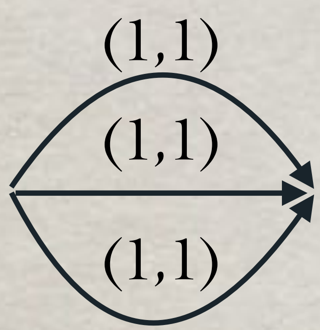
k vertices

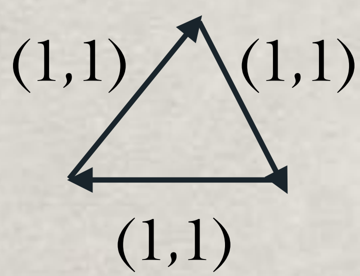
$$= \sum_{p\in\Lambda'} \frac{1}{p^k} = G_k(\tau) = 2\zeta_{2k} + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n$$

holomorphic,
Eisenstein modular form
of weight k

Examples:

$$C_{a,b,c} = \sum_{\substack{p1,p_2\in\Lambda' \\ p_1+p_2\neq 0}} \frac{(\tau_2/\pi)^{a+b+c}}{|p_1|^{2a}|p_2|^{2b}|p_1+p_2|^{2c}}$$

$C_{1,1,1} =$ 

$=$ 

$+\zeta_3(\pi/\tau_2)^3$ [Zagier]

$$\sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1+p_2+p_3)(\tau_2/pi)^3}{|p_1|^2|p_2|^2|p_3|^2} = \sum_{p\in\Lambda'} \frac{(\tau_2/\pi)^3}{|p|^6} + \zeta_3$$

Examples:

$$C_{a,b,c} = \sum_{\substack{p1,p_2\in\Lambda' \\ p_1+p_2\neq 0}} \frac{(\tau_2/\pi)^{a+b+c}}{|p_1|^{2a}|p_2|^{2b}|p_1+p_2|^{2c}}$$

Various techniques to study these objects:
In particular generating series [Gerken,Kleinschmidt,Schlotterer]
expressed in terms of certain objects:

$$\beta^{\mathrm{sv}}\left[\begin{matrix} j_1 & j_2 & \dots \\ k_1 & k_2 & \dots \end{matrix};\tau\right] \qquad 0\leq j_i\leq k_i-2\,, \qquad k_i=4,6,8,\dots$$

non-holomorphic, non SL invariant but with nice cocycles and nice Cauchy-Riemann equations

Some Properties:

Cauchy-Riemann's eq:

$$2\pi i(\tau-\bar\tau)^2\partial_\tau \beta^{\rm sv}\left[\begin{smallmatrix} j \\ k\end{smallmatrix};\tau\right] = (k-2-j)\beta^{\rm sv}\left[\begin{smallmatrix} j+1 \\ k\end{smallmatrix};\tau\right] - \delta_{j,k-2}(\tau-\bar\tau)^k G_k(\tau)$$

"Modular Weight"

$$\beta^{\rm sv}\left[\begin{matrix} j_1 & \dots \\ k_1 & \dots\end{matrix};-\frac{1}{\tau}\right] = \bar\tau^{\sum_i k_i-2-2j_i}\beta^{\rm sv}\left[\begin{matrix} j_1 & \dots \\ k_1 & \dots\end{matrix};\tau\right] + (\text{lower}-\text{depth})$$

Invariant under T: $\tau \to \tau+1$

sv Iterated Eisenstein Integrals:

$$\beta^{\rm sv}\left[\begin{smallmatrix} j \\ k\end{smallmatrix};\tau\right] = \frac{(2\pi i)^{-1}}{(4y)^{k-2-j}}\int_\tau^{i\infty} d\tau'(\tau-\tau')^{k-2-j}(\bar\tau-\tau')^j G_k(\tau') - \text{c.c.}$$

Similarly at higher depth. Possibly "single-valued" versions of Brown's iterated Integrals

Key Point: in the Generating Series NOT all betasv appear independently for MGFs. Conjectural matrix representation of Tsunogai's derivation algebra! I.e. there will be some holo cusp forms lurking behind!

Key Point: Study betasv depth by depth and weight by weight, $w = \text{sum}\, k_i$.

Depth one: just derivatives of non-holo Eisenstein series.

Depth two: Contain ALL $C_{a,b,c}$ but go beyond this space.

Depth-Two β^{sv}

With Axel and Oli we used differential + c.c. properties for depth two betasv to boil ALL of them (modular invariant case here) down to:

$$(\Delta - s(s-1))\mathrm{F}^{+(s)}_{m,k} = \mathrm{E}_m\mathrm{E}_k\,, \qquad s \in \{k-m+2, k-m+4, \ldots, k+m-4, k+m-2\}\ ,$$

$$(\Delta - s(s-1))\mathrm{F}^{-(s)}_{m,k} = \frac{(\nabla\mathrm{E}_m)(\overline{\nabla}\mathrm{E}_k) - (\nabla\mathrm{E}_k)(\overline{\nabla}\mathrm{E}_m)}{2(\mathrm{Im}\,\tau)^2}$$

$$s \in \{k-m+1, k-m+3, \ldots, k+m-3, k+m-1\}\ ,$$

$$\Delta = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$$

$$\nabla = 2i\tau_2^2\partial_\tau$$

$$\bar\nabla = -2i\tau_2^2\partial_{\bar\tau}$$

Odd/Even under:

$$\tau \leftrightarrow \bar\tau$$

Depth-Two β^{sv}

With Axel and Oli we used differential + c.c. properties for depth two betasv to boil ALL of them (modular invariant case here) down to:

Focus on:

$$(\Delta - s(s-1))\mathrm{F}^{+(s)}_{m,k} = \mathrm{E}_m\mathrm{E}_k\,, \qquad s \in \{k-m+2, k-m+4, \ldots, k+m-4, k+m-2\}\ ,$$

Appeared in other contexts!
For example with half-integer Eisenstein sources.
[Green,Miller,Vanhove]

Spectral decomposition in terms of L^2 Eisensteins
[Klinger-Logan]

Strategy: Poincaré series to simplify depth two-source term
[DD, Kleinschmidt]

Poincaré Series:

Write a modular invariant function as a sum over SL orbits of an easier non-modular seed function:

$$F(\tau) = \sum_{\gamma \in B(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \varphi(\gamma \cdot \tau)$$

$$B(\mathbb{Z}) = \{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}, n \in \mathbb{Z} \}$$

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Write a modular invariant function as a sum over SL orbits of an easier non-modular seed function.

Poincaré Series:

Write a modular invariant function as a sum over SL orbits of an easier non-modular <u>seed</u> function:

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E.g.

$$E_k(\tau) = \left(\frac{2\zeta_{2k}}{\pi^k}\right) \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \mathrm{Im}\Big(\gamma \cdot \tau\Big)^k$$

We "fold" the Eisenstein $E_k \to y^k$.

Poincaré Series:

We "fold" the Eisenstein $E_k \to y^k$

$$(\Delta - s(s-1))F^{+(s)}_{m,k} = E_m E_k$$

wlog we assume $k \geq m$

$$(\Delta - s(s-1))f^{+(s)}_{m,k} = \frac{2\zeta_{2k}}{\pi^{2k}} y^k E_m$$

Solve the easier Laplace equation for the seed function!

Poincaré Series:

$$(\Delta - s(s-1)) f^{+(s)}_{m,k} = \frac{2\zeta_{2k}}{\pi^{2k}} y^k E_m$$

<u>Key Point:</u> ALL these $f^{+(s)}_{m,k}$ can be written as finite linear combinations with rational coefficients of depth-one iterated integrals seeds:

$$y^\ell \mathrm{Re}\, \mathcal{E}_0(2m, 0^{k+m-\ell-1}; \tau)$$

Depth-one iterated integrals of holo Eisensteins:

$$\mathcal{E}_0(2m, 0^p; \tau) \sim \int_\tau^{i\infty} d\tau_1 \int_{\tau_1}^{i\infty} d\tau_2 \cdots \int_{\tau_{p-1}}^{i\infty} d\tau_p G^0_{2m}(\tau_p)$$

[Brown - Brödel, Schlotterer,Zerbini]

Cuspidal part only

Poincaré Series:

Standard Method:

$$F(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n \tau_1} a_n(\tau_2) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \varphi(\gamma\cdot\tau)$$

?

$$\varphi(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n \tau_1} b_n(\tau_2)$$

$$a_n(\tau_2) =$$

$$b_n(\tau_2) + \sum_{m\in\mathbb{Z}}\sum_{c>0}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i\frac{md+nd^{-1}}{c}} \int d\omega\, e^{-2\pi i n\omega - 2\pi i m \frac{\omega}{c^2(\tau_2^2+\omega^2)}} b_n\left(\frac{\tau_2}{c^2(\tau_2^2+\omega^2)}\right)$$

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Kloosterman' sum

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Convergent in our case, in particular we found the asymptotic expansion @ the cusp $\tau_2 \to \infty$

Laurent Polynomials

$$\mathrm{F}^{+(s)}_{m,k} = \frac{(-4)^{k+m}\mathrm{B}_{2m}\mathrm{B}_{2k}}{(k+m-s)(k+m+s-1)(2m)!(2k)!}y^{k+m} - \frac{2(-1)^m 4^{1+m-k}\mathrm{B}_{2m}\Gamma(2k-1)\zeta_{2k-1}}{\Gamma(k)\Gamma(k)(m-k+s)(m-k-s+1)(2m)!}y^{1+m-k}$$
$$- \frac{2(-1)^k 4^{1+k-m}\mathrm{B}_{2k}\Gamma(2m-1)\zeta_{2m-1}}{\Gamma(m)\Gamma(m)(k-m+s)(k-m-s+1)(2k)!}y^{1+k-m}$$
$$+ \frac{4^{3-m-k}\Gamma(2m-1)\Gamma(2k-1)\zeta_{2m-1}\zeta_{2k-1}}{[\Gamma(m)\Gamma(k)]^2(k+m-s-1)(k+m+s-2)}y^{2-k-m} + c^{(s)}_{m,k}\zeta_{k+m+s-1}y^{1-s} + O(q,\bar{q})$$

See also [D'Hoker, Duke]

$$F^{+(s)}_{m,k} = \mathbb{Q}y^{k+m} + \mathbb{Q}\zeta_{2k-1}y^{1+m-k} + \mathbb{Q}\zeta_{2m-1}y^{1+k-m} + \mathbb{Q}\zeta_{2k-1}\zeta_{2m-1}y^{2-k-m} + \mathbb{Q}\zeta_{k+m+s-1}y^{1-s} + O(q,\bar{q})$$

weight w = k+m = uniform transcendental weight of Laurent polynomials, [y]=1, [ζ_k]=k

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$$+ \frac{4^{3-m-k}\Gamma(2m-1)\Gamma(2k-1)\zeta_{2m-1}\zeta_{2k-1}}{[\Gamma(m)\Gamma(k)]^2(k+m-s-1)(k+m+s-2)}y^{2-k-m} + c^{(s)}_{m,k}\zeta_{k+m+s-1}y^{1-s} + O(q,\bar{q})$$

The only "new" (homogenous soln.) term

$$\beta^{(s)}_{(s_1,s_2)} = \frac{4\pi^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)}\,\frac{\zeta^*(s-s_1-s_2+1)\zeta^*(s+s_1-s_2)\zeta^*(s-s_1+s_2)\zeta^*(s+s_1+s_2-1)}{(2s-1)\,\zeta^*(2s)}, \qquad \text{(D.12)}$$

with $\zeta^*(s) = \zeta(s)\Gamma(s/2)/\pi^{s/2}$.

[Green, Russo, Vanhove]

Laurent Polynomials

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Only single-valued zeta but not sv-mzv, we need to go to depth ≥ 3 [Zerbini]

$(q\bar{q})^n$ can be reconstructed from deformed, asymptotic expansion at the cusp via resurgent analysis.

non-zero Fourier modes difficult due to Kloosterman'sums

Back to β^{sv}

$$F^{+(3)}_{2,3} = 30\Big(\beta^{\rm sv}\big[\begin{smallmatrix}2 & 1\\ 4 & 6\end{smallmatrix}\big] + \beta^{\rm sv}\big[\begin{smallmatrix}3 & 0\\ 6 & 4\end{smallmatrix}\big]\Big) + \text{lower depth}$$

Modular Invariant
From Poincaré sum

Not Modular invariant

needed lower depth terms

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$$F^{+(3)}_{2,3} = -\frac{1}{4}\Big(C_{3,1,1} - \frac{43}{35}E_5 + \frac{\zeta_5}{60}\Big)$$

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However in general #F > # C

$$\dim \mathcal{V}_{\mathrm{F}^+}(w,s) - \dim \mathcal{V}_C(w,s) = \dim \mathcal{S}_{2s} = \begin{cases} \left\lfloor \frac{2s}{12} \right\rfloor - 1 & : \ 2s = 2 \mod 12\,, \\ \left\lfloor \frac{2s}{12} \right\rfloor & : \ \text{otherwise} \end{cases}$$

weight w = k+m = a+b+c = transcendental weight of Laurent polynomials

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We expect holomorphic cusp forms to play some role!

Back to β^{sv}

$$\beta^{\text{sv}}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] \sim \int_{\tau}^{i\infty} (\tau-\tau_2)^{k_2-j_2-2}(\bar{\tau}-\tau_2)^{j_2} G_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} d\tau_1 (\tau-\tau_1)^{k_1-j_1-2}(\bar{\tau}-\tau_1)^{j_1} G_{k_1}(\tau_1) + \ldots$$

Under S transformation $\tau \to -1/\tau$ we produce multiple modular values of the form:

$$\mathfrak{m}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_1\, \tau_1^{j}\, \mathrm{G}_k(\tau_1)\,,$$

$$\mathfrak{m}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_2\, \tau_2^{j_2}\, \mathrm{G}_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} \mathrm{d}\tau_1\, \tau_1^{j_1}\, \mathrm{G}_{k_1}(\tau_1)\,,$$

Back to β^{sv}

$$\beta^{\rm sv}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] \sim \int_{\tau}^{i\infty} (\tau-\tau_2)^{k_2-j_2-2}(\bar\tau-\tau_2)^{j_2} G_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} d\tau_1 (\tau-\tau_1)^{k_1-j_1-2}(\bar\tau-\tau_1)^{j_1} G_{k_1}(\tau_1) + \ldots$$

Under S transformation $\tau \to -1/\tau$ we produce multiple modular values of the form:

$$\mathfrak{m}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_1\, \tau_1^j\, \mathrm{G}_k(\tau_1)\,,$$

$$\mathfrak{m}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] = \int_0^{i\infty} \mathrm{d}\tau_2\, \tau_2^{j_2}\, \mathrm{G}_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} \mathrm{d}\tau_1\, \tau_1^{j_1}\, \mathrm{G}_{k_1}(\tau_1)\,,$$

Depth-one are easy:

$$\mathfrak{m}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right] = \begin{cases} -\dfrac{2(2\pi i)^{k-j-1} j!}{(k-1)!}\zeta_{j+1}\zeta_{j+2-k} & \text{for } j>0\,, \\ -\dfrac{2\pi i \zeta_{k-1}}{k-1} & \text{for } j=0 \end{cases}$$

Back to β^{sv}

$$\beta^{\mathrm{sv}}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] \sim \int_{\tau}^{i\infty} (\tau-\tau_2)^{k_2-j_2-2}(\bar{\tau}-\tau_2)^{j_2} G_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} d\tau_1 (\tau-\tau_1)^{k_1-j_1-2}(\bar{\tau}-\tau_1)^{j_1} G_{k_1}(\tau_1) + \ldots$$

Under S transformation $\tau \to -1/\tau$ we produce multiple modular values of the form:

Higher depth appear only in "reduced" combinations

$$\mathcal{M}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right] = \mathfrak{m}\left[\begin{smallmatrix} j_2 & j_1 \\ k_2 & k_1 \end{smallmatrix}\right] - \mathfrak{m}\left[\begin{smallmatrix} j_2 \\ k_2 \end{smallmatrix}\right] \overline{\mathfrak{m}\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix}\right]} + \overline{\mathfrak{m}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix}\right]}$$

but do contain interesting new numbers:

Back to β^{sv}

$$\beta^{\mathrm{sv}}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix} \sim \int_{\tau}^{i\infty} (\tau-\tau_2)^{k_2-j_2-2}(\bar{\tau}-\tau_2)^{j_2} G_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} d\tau_1 (\tau-\tau_1)^{k_1-j_1-2}(\bar{\tau}-\tau_1)^{j_1} G_{k_1}(\tau_1) + \ldots$$

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but do contain interesting new numbers:

$$\mathcal{M}\begin{bmatrix} 0 & 1 \\ 4 & 10 \end{bmatrix} = -\frac{43 i\pi^{11}\zeta_3}{25259850} + \frac{11 i\pi^3\zeta_{11}}{540} + \frac{256 i\pi^{13}\Lambda(\Delta_{12},12)}{1913625},$$

$$\mathcal{M}\begin{bmatrix} 0 & 3 \\ 4 & 10 \end{bmatrix} = \frac{17 i\pi^{11}\zeta_3}{265228425} - \frac{i\pi^5\zeta_9}{17010} - \frac{16 i\pi^{13}\Lambda(\Delta_{12},12)}{229635},$$

$$\mathcal{M}\begin{bmatrix} 1 & 0 \\ 4 & 10 \end{bmatrix} = \frac{8 i\pi^{11}\zeta_3}{2525985} + \frac{2 i\pi^5\zeta_9}{243} - \frac{11 i\pi^3\zeta_{11}}{135} + \frac{256 i\pi^{13}\Lambda(\Delta_{12},12)}{1913625},$$

$$\mathcal{M}\begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix} = -\frac{4 i\pi^{11}\zeta_3}{10609137} + \frac{i\pi^5\zeta_9}{5670} - \frac{16 i\pi^{13}\Lambda(\Delta_{12},12)}{229635},$$

$$\mathcal{M}\begin{bmatrix} 1 & 4 \\ 4 & 10 \end{bmatrix} = -\frac{4 i\pi^{11}\zeta_3}{37889775} + \frac{64 i\pi^{13}\Lambda(\Delta_{12},12)}{1148175},$$

Ramanujan $\Delta_{12} \in \mathcal{S}_{12}$

Back to β^{sv}

$$\beta^{\rm sv}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix} \sim \int_{\tau}^{i\infty} (\tau-\tau_2)^{k_2-j_2-2}(\bar\tau-\tau_2)^{j_2} G_{k_2}(\tau_2) \int_{\tau_2}^{i\infty} d\tau_1 (\tau-\tau_1)^{k_1-j_1-2}(\bar\tau-\tau_1)^{j_1} G_{k_1}(\tau_1) + \ldots$$

Under S transformation $\tau \to -1/\tau$ we produce multiple modular values of the form:

Higher depth appear only in "reduced" combinations

$$\mathcal{M}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix} = \mathfrak{m}\begin{bmatrix} j_2 & j_1 \\ k_2 & k_1 \end{bmatrix} - \mathfrak{m}\begin{bmatrix} j_2 \\ k_2 \end{bmatrix} \overline{\mathfrak{m}\begin{bmatrix} j_1 \\ k_1 \end{bmatrix}} + \overline{\mathfrak{m}\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix}}$$

but do contain interesting new numbers:

L-values of holomorhic cusp forms OUTSIDE the critical strip!

$$\Delta(\tau) = \sum_{n>0} a(n) q^n \in \mathcal{S}_{2s}$$

$$\Lambda(\Delta, t) = (2\pi)^{-t}\Gamma(t) \sum_{n>0} \frac{a(n)}{n^t} = (-1)^s \Lambda(\Delta, 2s-t)$$

β^{sv} and iterated integrals of cusp Forms

In General:

$$F^{\pm(s)}_{m,k} = \beta^{\rm sv} + \sum_{\Delta_{2s}\in\mathcal{S}_{2s}} a^{\pm}_{\Delta_{2s},m,k} H^{\pm}_{\Delta_{2s}}$$

Iterated Eisenstein integrals

Iterated Cusp integrals

Neither of the two is modular invariant on its own.

$$\mathrm{H}^{\pm}_{\Delta_{2s}}(\tau) = (-1)^s \frac{\pi^{2s-1} i}{\Gamma(s)} y^{1-s} \int_{\tau}^{i\infty} \mathrm{d}\tau_1\, (\tau-\tau_1)^{s-1} (\bar\tau-\tau_1)^{s-1} \Delta_{2s}(\tau_1) \pm \mathrm{c.c.}\,,$$

β^{sv} and iterated integrals of cusp Forms

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$$(\Delta - s(s-1)) H^{\pm}_{\Delta_{2s}} = 0$$

$$\delta_S H^{\pm}_{\Delta_{2s}} = H^{\pm}_{\Delta_{2s}}(\tau) - H^{\pm}_{\Delta_{2s}}(-1/\tau) = \begin{cases} y^{1-s}\Lambda(\Delta_{2s}, 2s-1) p^{+}_{\Delta_{2s}}(\tau,\bar\tau) \\ y^{1-s}\Lambda(\Delta_{2s}, 2s-2) p^{-}_{\Delta_{2s}}(\tau,\bar\tau) \end{cases}$$

Generalisation of period polynomials in 2 variables (use Manin)

β^{sv} and iterated integrals of cusp Forms

In General:

$$F^{\pm(s)}_{m,k} = \beta^{\rm sv} + \sum_{\Delta_{2s}\in\mathcal{S}_{2s}} a^{\pm}_{\Delta_{2s},m,k} H^{\pm}_{\Delta_{2s}}$$

$$\delta_S \beta^{\rm sv} = -\delta_S H^{\pm}_{\Delta_{2s}}$$

$$a^{\pm}_{\Delta_{2s},m,k} = \mathbb{Q}_{\Delta_{2s}} \times \frac{\Lambda(\Delta_{2s}, m+k+s-1)}{\Lambda(\Delta_{2s}, 2s-1 \,\text{or}\, -2)}$$ [Brown - DD,AK,OS]

↑

Number field associated with cusp (e.g. $\mathbb{Q}[\sqrt{144169}]$ for 2s=24)

β^{sv} and iterated integrals of cusp Forms

In General:

$$F^{\pm(s)}_{m,k} = \beta^{\rm sv} + \sum_{\Delta_{2s}\in\mathcal{S}_{2s}} a^{\pm}_{\Delta_{2s},m,k} H^{\pm}_{\Delta_{2s}}$$

$$\delta_S \beta^{\rm sv} = -\delta_S H^{\pm}_{\Delta_{2s}}$$

For example:

$$F^{+(6)}_{2,6} = \beta^{sv}\text{part} + \frac{2}{17275}\frac{\Lambda(\Delta_{12},13)}{\Lambda(\Delta_{12},11}H^{+}_{\Delta_{12}}$$

$$F^{+(6)}_{4,4} = \beta^{sv}\text{part} + \frac{7}{10365}\frac{\Lambda(\Delta_{12},13)}{\Lambda(\Delta_{12},11}H^{+}_{\Delta_{12}}$$

Neither of the two is a MGFs but:

$$35F^{+(6)}_{2,6} - 6F^{+(6)}_{4,4} = \sum_{\substack{a+b+c=8\\ a,b,c\geq 1}} \mathbb{Q}C_{a,b,c} + \mathbb{Q}E_8$$

β^{sv} and iterated integrals of cusp Forms

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In MGFs generating series there is a conjectural representation of Tsunogai's derivation algebra. Pollack's algebraic relations selects only $\beta^{\rm sv}$ combinations where H_Δ drops out!

Conclusions:

- Study of all depth-two $F^{\pm(s)}_{m,k}$ non-holo, modular invariant functions via inhomogeneous Laplace equations.
- Asymptotic expansion @ cusp only in terms of mzv via Poincaré series.
- Broader class compared to $C_{a,b,c}$, these functions $F^{\pm(s)}_{m,k}$ contain iterated integrals of Eisenstein G @ depth-two and iterated integrals of holomorphic cusp forms @ depth-one.
- Higher depth generalisation? Appearance of sv-mzv?
- Can we see the appearance of these completed L-values from Kloosterman'sums?

Thanks!

Back-up Slides