Galois theory for periods and Lauricella hypergeometric functions

(joint work with F. Brown

+ work in progress with F. Brown, J. Fresán, M. Tapušković)

Motivation from physics

- [Abreu - Britto - Duhr - Gardi - Matthew] compute the motivic Galois theory (motivic coaction) for coefficients in ε-expansion of Feynman integrals.

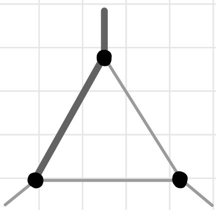
  dim reg parameter $(D = 4-2\varepsilon)$

$$I_\varepsilon(G) = \sum_{n \gg -\infty} a_n(G)\, \varepsilon^n$$

Feynman diagram   periods

$\rightsquigarrow$ get finite formulas at the level of ε-series.

$\rightsquigarrow$ conjecture a general "diagrammatic" coaction formula for $I_\varepsilon(G)$'s

$I_\varepsilon\Bigg($  $\Bigg)$

$$= \frac{e^{\gamma\varepsilon}\,\Gamma(1+\varepsilon)}{\varepsilon(1-\varepsilon)} (m^2)^{-1-\varepsilon}\; {}_2F_1\left(\begin{matrix}1\,,\,1+\varepsilon\\ 2-\varepsilon\end{matrix}\;\middle|\;\frac{p^2}{m^2}\right).$$

a_n: $\quad \zeta(n) \quad \log^n(m^2) \quad \mathrm{Li}_n\left(\frac{p^2}{m^2}\right), \ldots$

- [Schlotterer - Stieberger] similar results ("f-alphabet decomposition") for the α'-expansion of open superstring amplitudes in genus 0. (see also [Britto - Mizera - Rodriguez - Schlotterer])

Motivation from mathematics

Reconcile two geometric viewpoints on certain hypergeometric-type integrals.

(1) The "functional" viewpoint

$$B(s,t) = \frac{\Gamma(s)\,\Gamma(t)}{\Gamma(s+t)} = \int_0^1 x^s (1-x)^t \frac{dx}{x(1-x)} \qquad (s,t \in \mathbb{C})$$

"period" of a connection on $X = \mathbb{C} \setminus \{0,1\}$

$$\nabla_{s,t}(\varphi) = d\varphi + \varphi\left(s\,\frac{dx}{x} + t\,\frac{dx}{x-1}\right) \qquad d\left(x^s(1-x)^t\varphi\right) = x^s(1-x)^t\,\nabla_{s,t}(\varphi)$$

$\rightsquigarrow$ "twisted" cohomology group $H^1_{dR}(X, \nabla_{s,t})$.

dim. 1 if $s, t, s+t \notin \mathbb{Z}$

[Aomoto, Gelfand, Varchenko, Kita, Yoshida, ...]

In physics: "twisted" cohomology groups encode integration by parts;
basis = master integrands; very useful intersection pairing;
KLT formalism; difference equations; ... [..., Mizera, ...]

Warning: Twisted cohomology groups are not motives!

What this talk is about: a formal version of twisted cohomology has a motivic origin.

(2) The "series of periods" viewpoint

$$\zeta(n_1,\dots,n_r) = \sum_{1\le k_1<\dots<k_r} k_1^{-n_1}\cdots k_r^{-n_r}$$

$$B(s,t) = \frac{s+t}{st}\left(1 - \sum_{m,n\geq 1} (-s)^m(-t)^n\, \zeta(\underbrace{1,\dots,1}_{n-1},m+1)\right)$$

Better: $$B(s,t) = \frac{s+t}{st}\exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}}{n}\zeta(n)\left((s+t)^n - s^n - t^n\right)\right)$$

period of $H^n(X_n, D_n)$

Galois theory of ζ-values: for $g\in G$,

$$g\cdot\zeta(n) = \underset{\mathbb{Q}^\times}{\underset{\cap}{\lambda_g^n}}\,\zeta(n) + \underset{\mathbb{Q}}{\underset{\cap}{a_g^{(n)}}} \qquad \mathrm{Gal}(\zeta(n)) = \left\{\begin{pmatrix}1 & a^{(n)}\\ 0 & \lambda^n\end{pmatrix}\right\}$$

$$\Rightarrow\ g\cdot B(s,t) = B(s,t)\times A_g(s,t) \qquad \left(\text{needed: } g\cdot s = \lambda_g^{-1}s,\ g\cdot t = \lambda_g^{-1}t\right)$$

$\in \mathbb{Q}((s,t))^\times = GL_1(\mathbb{Q}((s,t)))$

Leitmotiv: the Galois theory of all coefficients in the expansion is controlled by (a formal version of) twisted cohomology.

[Precursors: Ihara action formula (associator); Goncharov's coaction formula for multiple polylogarithms; single-valued periods & superstring amplitudes in genus zero (Brown - D.)]

Periods

Definition: (Kontsevich-Zagier) A *period* is a complex number whose real and imaginary parts can be written as absolutely convergent integrals of the form

$$\int_\sigma f(x_1,\dots,x_n)\, dx_1 \dots dx_n$$

$f \in \mathbb{Q}(x_1,\dots,x_n)$

σ: union/intersection of $\{g(x_1,\dots,x_n) \geq 0\}$ for $g \in \mathbb{Q}[x_1,\dots,x_n]$

Examples: algebraic numbers $\left(\sqrt{2} = \int_{\substack{x \geq 0 \\ x^2 \leq 2}} 1\, dx\right)$, $\pi = \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \int_{[0,1]^n} \frac{dx_1 \dots dx_n}{1 - x_1 \dots x_n}$$, iterated integrals, ...

n: integer ≥ 2

Geometric perspective: periods arise in the comparison between de Rham and Betti cohomology $H^n(X, Z)$.

X: alg. variety/$\mathbb{Q}$; Z: closed subvariety

$\rightsquigarrow$ natural framework = motives [Grothendieck]

Galois theory for (motivic) periods : [André]

$$G_{dR} \circlearrowright \{\text{motivic periods}\} = \mathcal{P}^{mot}$$

(matrix coefficients in a Tannakian category of motives)

$$\Big\downarrow \; \cong \,? \quad (\text{period conjecture})$$

$$\{\text{periods}\} = \mathcal{P}$$

algebra of functions on G_{dR}

Action $\xleftrightarrow{\text{dual}}$ **Coaction**

$$G_{dR} \times \mathcal{P}^{mot} \to \mathcal{P}^{mot}$$

$$g \cdot \alpha = \sum_i \underbrace{\varphi_i(g)}_{\in\, \mathbb{Q}} \alpha_i$$

$$\mathcal{P}^{mot} \xrightarrow{\rho} \mathcal{P}^{mot} \otimes \mathcal{O}(G_{dR})$$

$$\rho(\alpha) = \sum_i \alpha_i \otimes \varphi_i$$

"de Rham periods"

Warning: In the "mixed Tate case" (multiple polylogarithms), de Rham periods are "motivic periods modulo $2\pi i$" $\rightsquigarrow$ "de Rham projection"

No interaction in general between $\mathcal{P}^{mot}$ and $\mathcal{O}(G^{dR})$.

The motivic coaction can be effectively computed for iterated integrals on $\mathbb{A}^1 \setminus \Sigma$. [Ihara, Drinfeld, Goncharov, Brown]

Lauricella hypergeometric functions

$$\Sigma = \{\underset{\substack{\parallel \\ 0}}{\sigma_0}, \sigma_1, \dots, \sigma_n\} \subset \mathbb{C} \quad (\text{or } \overline{\mathbb{Q}})$$

<u>Def</u>: Lauricella matrix $(n \times n)$

$$L_{i,j}(s_0, s_1, \dots, s_n) = -s_j \int_0^{\sigma_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j}$$

$(1 \leq i, j \leq n)$

<u>$n = 1$</u>: $\Sigma = \{0, 1\}$: $L_{1,1}(s,t) = \dfrac{st}{s+t} B(s,t)$

<u>$n = 2$</u>: $\Sigma = \{0, 1, \frac{1}{y}\}$:

$$B(b, c-b)\, {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| y\right) = \int_0^1 x^{b} (1-x)^{c-b} (1-yx)^{-a} \frac{dx}{x(1-x)}$$

$$= \frac{c}{b(c-b)} L_{1,1}(b, c-b, -a) + \frac{1}{b} L_{1,2}(b, c-b, -a)$$

"Functional" viewpoint: L is a "period" matrix of twisted cohomology $H^1(\mathbb{A}^1 \setminus \Sigma, \nabla_{s_0, s_1, \dots, s_n})$.

"Series of periods" viewpoint: expand and lift to

$$L^{mot} \in \mathcal{M}_{n\times n}\left(\mathcal{P}^{mot}[\![s_0, s_1, \dots, s_n]\!]\right)$$

via $\pi_1^{mot}(\mathbb{A}^1\setminus\Sigma, 0, \sigma_i)$

tangential basepoints!

<u>Theorem</u>: [Brown–D] There exists a matrix

$$L^{dR} \in \mathcal{M}_{n\times n}\left(\mathcal{O}(G_{dR})[\![s_0, s_1, \dots, s_n]\!]\right)$$

such that the motivic Galois theory / coaction of L^{mot} reads:

$$g \cdot L^{mot} = L^{mot} \times L^{dR}(g) \quad / \quad \rho(L^{mot}) = L^{mot} \otimes L^{dR}$$

Coincides with a motivic Galois theory / coaction that would exist at the level of (formal) twisted cohomology:

$$G_{dR}(\mathbb{Q}) \xrightarrow{L^{dR}} GL_n(\mathbb{Q}((s_0, \dots, s_n)))$$

$$\| $$

$$\mathrm{Aut}_{\mathbb{Q}((s_0,\dots,s_n))}\left(\begin{array}{c}\text{formal twisted}\\ \text{cohomology group}\end{array}\right)$$

Work in progress (with F. Brown, J. Fresán, M. Tapušković)

- General result

"motivic Galois theory controlled by formal twisted cohomology"

$$f : X \to T \simeq \mathbb{G}_m^N \rightsquigarrow F(\underline{s}) = \int_\sigma f_1^{s_1} \cdots f_N^{s_N} \omega \qquad \text{"Mellin transforms"}$$

Unprecise theorem : G_{dR} acts on the coefficients of a motivic lift of $F(\underline{s})$ via an algebraic group $/ \mathbb{Q}((\underline{s}))$:

$$G_{dR} \longrightarrow \mathrm{Res}_{\mathbb{Q}((\underline{s}))/\mathbb{Q}} GL\left(H^n_{dR}(X, f^*\nabla_{\underline{s}}) / \mathbb{Q}((\underline{s}))\right)$$

(Need to interpret twisted cohomology motivically!)

- Tannakian interpretation of this algebraic group $/ \mathbb{Q}((\underline{s}))$?
- Global theory ? (replace $\mathbb{Q}((\underline{s}))$ with $\mathbb{Q}(\underline{s})$ or $\mathbb{Q}[\underline{s}]$) ?
- Application to Feynman integrals in dim. reg. ?
- Application to classical motives / motivic Galois theory ?

Quote from A. Varchenko : (Proceedings of Kyoto ICM 1990)

"[...] it might be possible to pass from Conformal Field Theory to Algebraic K-Theory by analytic continuation on parameters."