Iterated integrals over Riemann surfaces, flat connections and polylogarithms

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Outline

Iterated integrals

- M complex manifold.
- $\mathcal{A}^1(M)$ $\mathbb{C}$-vector-space of smooth 1-forms on M.

Definition

The **iterated integral** of $\omega_1, \cdots, \omega_n \in \mathcal{A}^1(M)$ along a smooth path $\gamma : [0,1] \to M$ is given by

$$\int_\gamma \omega_1 \cdots \omega_n := \int_{0 \le t_1 \le \cdots \le t_n \le 1} \gamma^*\omega_1(t_1) \cdots \gamma^*\omega_n(t_n) \,.$$

- Two paths γ and η are **homotopic** if $\gamma(0) = \eta(0)$, $\gamma(1) = \eta(1)$, and can "deform γ onto η continuosly".
- An iterated integral is **homotopy invariant** if its value is the same on paths which are homotopic.
- If ω is **closed**, i.e. $d\omega = 0$, then $\int_\bullet \omega$ is homotopy invariant (Stokes).
- If $\omega_1, \ldots, \omega_n$ are **holomorphic** then $\int_\bullet \omega_1 \cdots \omega_n$ homotopy invariant.

An example

- $U = \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ **simply connected**, i.e. $\pi_1(U; x, y) = 1$ $\forall x, y \in U$.
- $z \in U$, γ_z any path contained in U from 0 to z.

The dilogarithm

$$\mathrm{Li}_2(z) := \int_{\gamma_z} \frac{dt}{1-t} \cdot \frac{dt}{t}. \tag{1}$$

Rmk 1: $\mathrm{Li}_2(z)$ well-defined on U, because $\frac{dt}{1-t}$ and $\frac{dt}{t}$ are holomorphic on U, and U simply connected.

Rmk 2: If $|z| < 1$ $\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$, and $\mathrm{Li}_2(1) = \zeta(2)$.

Rmk 3: Let $M := \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ (**not** simply-connected!)
RHS of (1) defines function of z and of homot. class of γ_z in M, and induces well-defined function on **fundamental cover** $\tilde{M}$ of M.
We say it's a **multi-valued function** on M.

Rmk 4: $\mathrm{Li}_2(z)$ is a period function, $\zeta(2)$ is the period of a **mixed** motive.

Motivating examples

Multiple polylogarithms

Multi-valued fcts on $M = \mathrm{Conf}_n(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$ (conf. space of n points on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$) induced by homotopy-inv. iterated integrals.

Elliptic multiple polylogarithms

Multi-valued functions on $M = \mathrm{Conf}_n((\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}) \setminus \{0\})$ induced by homotopy-invariant iterated integrals.

String theory amplitudes

Genus-zero amplitudes $\leftrightsquigarrow$ **MZVs**
Genus-one amplitudes $\leftrightsquigarrow$ **elliptic MZVs**.

Feynman integrals

Basis of master integrals satisfies 1st-order linear diff. eq. $\rightsquigarrow$ iterated integrals as coefficients ϵ-expansion (MPLs, elliptic MPLs, iterated integrals of modular forms, much more?).

Homotopy invariance and multi-valued functions

- $\mathcal{P}$ space of all smooth paths on M.
- Iterated integration map $\int_\bullet : T\mathcal{A}^1(M) \to \mathrm{Fun}(\mathcal{P}, \mathbb{C})$ given by $\omega_1 \otimes \cdots \otimes \omega_r \mapsto (\gamma \to \int_\gamma \omega_1 \cdots \omega_r)$
- $p : \tilde{M} \to M$ universal cover, and $\Gamma := \mathrm{Aut}(\tilde{M}/M) \simeq \pi_1(M, x)$
- Group action of Γ on $C^\infty(\tilde{M}^2, \mathbb{C})$ via $\gamma \cdot f : (\tilde{x}, \tilde{y}) \mapsto f(\gamma(\tilde{x}), \gamma(\tilde{y}))$

Subalgebra $Z\mathcal{A}^1(M) \subset T\mathcal{A}^1(M)$ which gives rise to homotopy-invariant iterated integrals is the pullback

$$\begin{array}{ccc} T\mathcal{A}^1(M) & \xrightarrow{\int_\bullet} & \mathrm{Fun}(\mathcal{P}, \mathbb{C}) \\ \uparrow & & \uparrow \\ Z\mathcal{A}^1(M) & \xrightarrow{\int_\bullet} & C^\infty(\tilde{M}^2, \mathbb{C})^\Gamma \end{array}$$

Chen: Description of elements of $Z\mathcal{A}^1(M)$ (bar complex).

Homotopy invariance and Chen's theorem

- $\mathcal{P}_{x,y}$ space of all smooth paths on M from x to y.
- Iterated integration map $\int_{x\bullet_y} : T\mathcal{A}^1(M) \to \mathrm{Fun}(\mathcal{P}_{x,y}, \mathbb{C})$ given by $\omega_1 \otimes \cdots \otimes \omega_r \mapsto ({}_x\gamma_y \to \int_{{}_x\gamma_y} \omega_1 \cdots \omega_r)$
- $\tilde{M}_x := p^{-1}(x)$

Subalgebra $Z_{x,y}\mathcal{A}^1(M) \subset T\mathcal{A}^1(M)$ which give rise to homotopy-invariant iterated integrals from x to y is the pullback

$$\begin{array}{ccc} T\mathcal{A}^1(M) & \xrightarrow{\int_{x\bullet_y}} & \mathrm{Fun}(\mathcal{P}_{x,y}, \mathbb{C}) \\ \uparrow & & \uparrow \\ Z_{x,y}\mathcal{A}^1(M) & \xrightarrow{\int_{x\bullet_y}} & \mathrm{Fun}(\tilde{M}_x \times \tilde{M}_y, \mathbb{C})^{\Gamma} \end{array}$$

Theorem (Chen): $Z_{x,y}\mathcal{A}^1(M)/\mathrm{Ker}(\int_{x\bullet_y}) \simeq \mathbb{C}[\pi_1^{\mathrm{un}}(M; x, y))]$.

The holomorphic case

$\Omega^1(M) \subset \mathcal{A}^1(M)$ $\mathbb{C}$-vector-space of **holomorphic** 1-forms on M

$$\begin{array}{ccc} T\Omega^1(M) & \xrightarrow{\int_\bullet} & \mathrm{Fun}(\mathcal{P},\mathbb{C}) \\ & \overset{\int_\bullet}{\searrow} & \uparrow \\ & & \mathrm{Hol}(\tilde{M}^2)^\Gamma \end{array} \qquad \begin{array}{ccc} T\Omega^1(M) & \xrightarrow{\int_{x\bullet y}} & \mathrm{Fun}(\mathcal{P}_{x,y},\mathbb{C}) \\ & \overset{\int_{x\bullet y}}{\searrow} & \uparrow \\ & & \mathrm{Fun}(\tilde{M}_x \times \tilde{M}_y,\mathbb{C})^\Gamma \end{array}$$

Rmk: $T\Omega^1(M) \hookrightarrow Z_{x,y}\mathcal{A}^1(M)$, and therefore
$T\Omega^1(M)/\mathrm{Ker}(\int_{x\bullet y}) \hookrightarrow \mathbb{C}[\pi_1^{\mathrm{un}}(M;x,y)]$.
This is not an isomorphism in general, but it is in special cases of interest:

- $M = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\} \rightsquigarrow$ MPLs and MZVs
- $M = (\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}) \setminus \{0\} \rightsquigarrow$ elliptic MPLs and elliptic MZVs

Question: describe $\mathrm{Im}(\int_\bullet) \subset \mathrm{Hol}(\tilde{M}^2)^\Gamma$ (space of multi-valued fcts induced by homotopy inv. iterated integrals of holomorphic 1-forms).

A space of multi-valued functions

- $M := \mathbb{P}^1(\mathbb{C}) \setminus \{\sigma_0, \sigma_1, \ldots, \sigma_N, \infty\}$ punctured Riemann sphere.
- $\mathcal{O}(M) = \mathbb{C}\left[z, \left(\frac{1}{z-\sigma_i}\right)_i\right]$ ring of regular fcts.
- Abstract alphabet $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$, Σ^* non-commut. words.
- Fix $p : \tilde{M} \to M$, and pre-image $\tilde{0}$ of 0.
- **Hyperlogarithms** $\{L_w\}_{w\in\Sigma^*}$ are family of fcts in $\mathrm{Hol}(\tilde{M})$ def. by

$$L_{\sigma_{i_1}\cdots\sigma_{i_n}}(z) := \int_{\tilde{0}\gamma_z} \frac{dt}{t-\sigma_{i_1}} \cdots \frac{dt}{t-\sigma_{i_n}}.$$

- $\mathcal{H} := \mathcal{O}(M)[\{L_w\}_{w\in\Sigma^*}]$ differential algebra.

Brown: $\mathcal{H}$ is the smallest diff. algebra of $\mathrm{Hol}(\tilde{M})$ which contains $\mathcal{O}(M)$ and is closed under taking primitives.

Fact: $\mathcal{H} = \mathrm{Im}\left(\int_{\tilde{0}\bullet} : T\Omega^1(M) \to \mathrm{Hol}(\tilde{M})\right)$.

Outline

1. Iterated integrals
2. Iterated integrals on Riemann surfaces
3. Higher-genus KZB-connections

Riemann surfaces

- C compact Riemann surface of genus $g \geq 1$ (donuts).
- $S \subset C$ finite non-empty set of pts $\rightsquigarrow C \setminus S$ affine.
- $H^1_{\text{sing}}(C, \mathbb{Q}) \simeq \mathbb{Q}^{2g}$, $H^1_{\text{sing}}(C \setminus S, \mathbb{Q}) \simeq \mathbb{Q}^{2g+|S|-1}$.
- $\omega \in \Omega^1(C)$ called "1st kind" differential ($\rightsquigarrow H^{1,0}(C) \simeq \mathbb{C}^g$).
- $\Omega^1_{2\text{nd}}(C, S)$ "2nd kind" differentials (with poles in S), consists of $\omega \in \Omega^1(C \setminus S)$ s.t. $\text{Res}_P \omega = 0 \; \forall P \in S$.
- **Fact 1:** $\frac{\Omega^1(C \setminus S)}{d\mathcal{O}(C \setminus S)} \tilde{\to} H^1_{\text{Sing}}(C \setminus S, \mathbb{Q}) \otimes \mathbb{C}$ via integration.
- **Fact 2:** $\forall S$ $\frac{\Omega^1_{2\text{nd}}(C,S)}{d\mathcal{O}(C \setminus S)} \tilde{\to} H^1_{\text{Sing}}(C, \mathbb{Q}) \otimes \mathbb{C}$ via integration
  ($\int \omega : \sigma \to \int_\sigma \omega$ well-defined by residue theorem!)
- **Fact 3:** $T\Omega^1(C \setminus S)/\text{Ker}(\int_{x \bullet_y}) \tilde{\to} \mathbb{C}[\pi_1^{\text{un}}(C \setminus S; x, y)]$.
- **Fact 4:** $T\Omega^1(C)/\text{Ker}(\int_{x \bullet_y}) \hookrightarrow \mathbb{C}[\pi_1^{\text{un}}(C; x, y)]$.

2nd-kind iterated integrals

Idea: Let $\gamma, \eta : [0,1] \to C \setminus S$ be homotopic as paths on C, but not as paths on $C \setminus S$. In general, if $\omega_1, \ldots, \omega_n \in \Omega^1(C \setminus S)$ then $\int_\gamma \omega_1 \cdots \omega_n \neq \int_\eta \omega_1 \cdots \omega_n$.

Definition (Hain)

An iterated integral of meromorphic differential forms with poles in S is a **2nd-kind iterated integral** if its value is the same on paths on $C \setminus S$ which are homotopic as paths in C.

Length 1: $\int_\bullet \omega$ 2nd-kind $\Leftrightarrow \omega \in \Omega^1_{\mathrm{2nd}}(C, S)$.

Length 2 (Hain)

Suppose that $\omega_1, \omega_2 \in \Omega^1_{\mathrm{2nd}}(C, S) \rightsquigarrow$ locally at any $P \in S$ $\exists f_1^{(P)}$ s.t. $\omega_1 = df_1^{(P)}$, and $\int_P \omega_1\omega_2 = \int_P f_1^{(P)}\omega_2 = \mathrm{Res}_P(f_1^{(P)}\omega_2) \neq 0$ in general. **But** if $\sum_{P \in S} \mathrm{Res}_P(f_1^{(P)}\omega_2) = 0$ then $\exists \alpha \in \Omega^1(C \setminus S)$ s.t. $\mathrm{Res}_P(f_1^{(P)}\omega_2) = -\mathrm{Res}_P(\alpha)$, and therefore $\int_\bullet \omega_1\omega_2 + \alpha$ is 2nd-kind.

Hain's theorem

- Fix $x, y \in C \setminus S$.
- Fix $p : \tilde{C} \to C$ and $\pi : \widetilde{C \setminus S} \to C \setminus S$ universal covers.
- $\tilde{C}_x := p^{-1}(x)$, $\widetilde{C \setminus S}_x := \pi^{-1}(x)$.
- $\Gamma_C := \mathrm{Aut}(\tilde{C}/C)$, $\Gamma_{C\setminus S} := \mathrm{Aut}(\widetilde{C \setminus S}/C \setminus S)$.

The subalgebra $Z_{x,y}\Omega^1(C \setminus S) \subset T\Omega^1(C \setminus S)$ giving rise to 2nd-kind iterated integrals from x to y is the pullback

$$\begin{array}{ccc}
T\Omega^1(C \setminus S) & \xrightarrow{\int_{x \bullet y}} & \mathrm{Hol}(\widetilde{C \setminus S}_x \times \widetilde{C \setminus S}_y)^{\Gamma_{C\setminus S}} \\
\uparrow & & \uparrow \\
Z_{x,y}\Omega^1(C \setminus S) & \xrightarrow{\int_{x \bullet y}} & \mathrm{Hol}(\tilde{C}_x \times \tilde{C}_y)^{\Gamma_C}
\end{array}$$

Theorem (Hain): $Z_{x,y}\Omega^1(C \setminus S)/\mathrm{Ker}(\int_{x \bullet y}) \simeq \mathbb{C}[\pi_1^{\mathrm{un}}(C; x, y)]$.

Multi-valued functions

- Fix $p : \tilde{C} \to C$ and $\pi : \widetilde{C \setminus S} \to C \setminus S$ universal covers.
- $\Gamma_C := \mathrm{Aut}(\tilde{C}/C)$, $\Gamma_{C\setminus S} := \mathrm{Aut}(\widetilde{C \setminus S}/C)$.

The subalgebra $Z\Omega^1(C \setminus S) \subset T\Omega^1(C \setminus S)$ giving rise to 2nd-kind iterated integrals is the pullback

$$\begin{array}{ccc} T\Omega^1(C \setminus S) & \xrightarrow{\int_\bullet} & \mathrm{Hol}(\widetilde{C \setminus S}^2)^{\Gamma_{C\setminus S}} \\ \uparrow & & \uparrow \\ Z\Omega^1(C \setminus S) & \xrightarrow{\int_\bullet} & \mathrm{Hol}((\tilde{C} \setminus p^{-1}(S))^2)^{\Gamma_C} \end{array}$$

- Description of elements of $Z\Omega^1(C \setminus S)$ and of $\mathrm{Im}\big(\int_\bullet : Z\Omega^1(C \setminus S) \to \mathrm{Hol}((\tilde{C} \setminus p^{-1}(S))^2)\big)$?
- Configuration space analogues?
- Relation with string amplitudes?
- Smooth case: use flat connections! (Brown-Levin, elliptic MPLs).

Outline

Maurer-Cartan elements

- $\mathfrak{g}$ (pro-)nilpotent Lie algebra.
- $J \in \Omega^1(C \setminus S) \otimes \mathfrak{g}$ is a Maurer-Cartan element if $dJ + [J, J]/2 = 0$.
- A Maurer-Cartan element rise to a (flat) connection $d + J$ on the trivial $\exp(\mathfrak{g})$-principal bundle over $C \setminus S$.
- Let $\mathrm{MC}(C, S, \mathfrak{g})$ = set of Maurer-Cartan elements s.t. $d + J$ has no monodromy at points of S.

Fact 1: The "coefficients" of $\sum_{n\geq 0} \underbrace{J \otimes \cdots \otimes J}_{n} \in T\Omega^1(C \setminus S) \otimes \exp(\mathfrak{g})$ belong to $Z\Omega^1(C \setminus S)$, and so the "coefficients" of $\sum_{n\geq 0} \int_\bullet \underbrace{J \cdots J}_{n}$ are 2nd-kind iterated integrals.

Fact 2: If $\mathfrak{g}$ is the (pro-nilpotent) Lie algebra of $\pi_1^{\mathrm{un}}(C, x)$, then the existence of an element $J \in \mathrm{MC}(C, S, \mathfrak{g})$ implies Hain's theorem.

Bezrukavnikov: constructs explicit pro-nilpotent Lie algebra $\mathfrak{t}_{g,n}$ s.t. $\pi_1^{\mathrm{un}}(\mathrm{Conf}_n(C), x) \simeq \exp(\mathfrak{t}_{g,n})$.

The genus-one KZB-connection

Let $\tau \in \mathbb{H}$, $\mathbb{T}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ (genus-1 compact RS), $\mathbb{T}_\tau^* := \mathbb{T}_\tau \setminus \{0\}$

The Kronecker function

For $z \in \mathbb{T}_\tau^*$, α formal variable, define $F(z,\alpha;\tau) := \frac{\theta'(0,\tau)\theta(z+\alpha,\tau)}{\theta(z,\tau)\theta(\alpha,\tau)}$

- $F(z,\alpha;\tau) - \frac{1}{z} - \frac{1}{\alpha} \in \mathbb{C}[[z,\alpha]]$
- $F(z+1,\alpha;\tau) = F(z,\alpha;\tau)$, $F(z+\tau,\alpha;\tau) = \exp(-2\pi i\alpha)F(z,\alpha;\tau)$

Fact: $\mathrm{Lie}(\pi_1^{\mathrm{un}}(\mathbb{T}_\tau^*, x)) \simeq \mathrm{Lie}(a,b)^\wedge$.

The Knizhnik-Zamolodchikov-Bernard connection

$K(z) := \mathrm{ad}_b F(z, \mathrm{ad}_b;\tau)(a)\,dz$ $\mathrm{Lie}(a,b)^\wedge$-valued 1-form (multi-valued) on $\mathbb{T}_\tau^*$ $\rightsquigarrow$ $d+K$ (flat) connection on principal $\exp(\mathrm{Lie}(a,b)^\wedge)$-bundle over $\mathbb{T}_\tau^*$, regular singularity at 0.

Define 1-forms ω_n by $K =: \sum_{n\geq 0} \omega_n\, \mathrm{ad}_b^n(a)$ ($\mathrm{ad}_b^n(a) := [b,\ldots,[b,a]]$).

Examples: $\omega_0(z,\tau) = dz$, $\omega_1(z,\tau) = (\zeta(z,\tau) - G_2(\tau)z)dz$.

Iterated integrals $\int_0^z \omega_{i_1}\cdots\omega_{i_n}$ (over $\mathbb{C}$) $\rightsquigarrow \sum_{n\in\mathbb{Z}} \mathrm{Li}_k(e^{2\pi i(z+n\tau)})$.

Higher-genus KZB-connections

Fact: $\mathfrak{t}_{g,1} = (\mathrm{Lie}(a_1, \dots, a_g, b_1, \dots, b_g)/(\sum_i [a_i, b_i]))^\wedge$

Enriquez

For C genus g compact RS, for any $P \in C$, there exists a $\mathrm{Lie}(a_1, \dots, a_g, b_1, \dots, b_g)^\wedge$-valued holomorphic 1-form K_P (multi-valued) on $C \setminus P$, uniquely determined by:

- K_P has a simple pole at P with residue $\sum_i [b_i, a_i]$.
- $A_i^* K(z) = K(z)$, $B_i^* K(z) = \exp(-2\pi i \mathrm{ad}_{b_i}) K(z)$.

$\rightsquigarrow$ K_P holomorphic at P as a $\mathfrak{t}_{g,1}$-valued form, induces (flat) connection $d + K$ on principal $\exp(\mathfrak{t}_{g,1})$-bundle $\mathcal{P}$ over C, independent of P.

Rmk 1: Analogous construction for $\mathrm{Conf}_n(C)$.

Rmk 2: $K_P(z) =: \sum_n \sum_{i_1, \dots, i_n, j} \omega^P_{i_1 \cdots i_n j} [b_{i_1} \cdots [b_{i_n}, a_j]]$ is a generating function of multi-valued 1-forms $\omega^P_{i_1 \cdots i_n j}$ which are higher-genus analogues of ω_n, but their construction is not explicit.

Trivialisation

- Fix arbitrary point $\infty \in C$.

Fact: A *trivialisation* of the principal bundle $\mathcal{P}$ over $C \setminus \infty$ (i.e. an isomorphism with the trivial $\exp(\mathfrak{t}_{g,1})$-bundle) is given by a holomorphic multi-valued function $g : C \setminus \infty \to \exp(\mathfrak{t}_{g,1})$ s.t. $\mathrm{Ad}_g(K)$ is single-valued on $C \setminus \infty$.

Enriquez, FZ (2021)

Explicit recursive construction of trivialisation g (in terms of 2nd kind iterated integrals over A and B cycles!).

Rmk: Analogous result for $\mathrm{Conf}_n(C)$.

Main consequences

- $J := gd(g^{-1}) + \mathrm{Ad}_g(K) \in \mathrm{MC}(C, S, \mathfrak{g})$: single-valued, induces 2nd-kind iterated integrals (similar statement holds on $\mathrm{Conf}_n(C)$).
- Recursive explicit formulas for the multi-valued 1-forms $\omega^{\infty}_{i_1 \cdots i_n j}$ (in terms of coeffs of g and fundamental form of 3rd kind).

THE END

Thanks!