Symbol Alphabets from the Landau Singular Locus

Georgios Papathanasiou

City, University of London & DESY

Seminar on Motives in Quantum Field and String Theory
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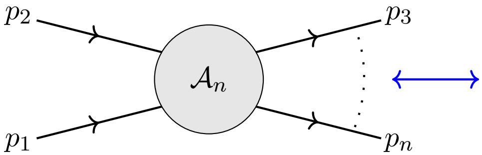
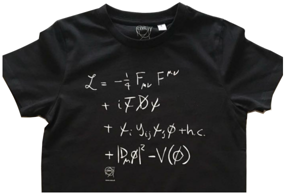
Motivation: Scattering Amplitudes $\mathcal{A}_n$ in Quantum Field Theory





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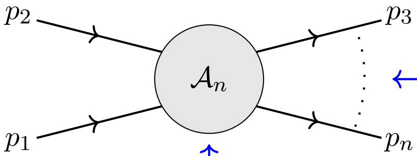
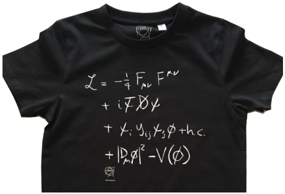
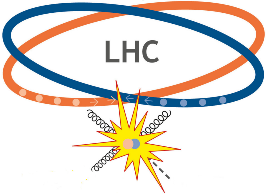




Collider Experiments



- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC & High-Luminosity upgrade

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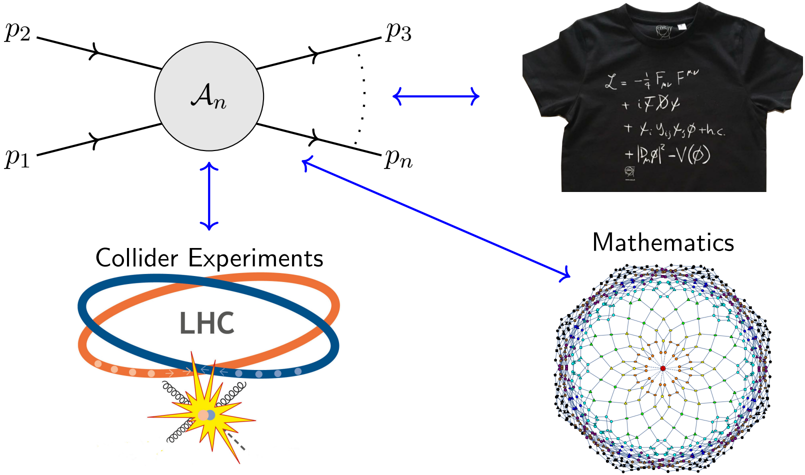

- Theoretical predictions for outcome of elementary particle collisions, central for experiments such as the LHC & High-Luminosity upgrade
- Exhibit remarkably deep mathematical structures

Motivation: Feynman Graphs

Building blocks of perturbative calculations in coupling g,

$$\mathcal{A}_n = g^{n-2} \sum_{L=0,1\ldots} g^{2L} \mathcal{A}_n^{(L)} \,.$$

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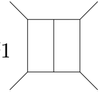
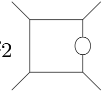
$\mathcal{A}_4^{(2)} = c_1$  $+ c_2$  $+ \ldots$

where each graph $G \to$ integral $I_G = \int \prod_{l=1}^{L} \frac{d^D k_l}{i\pi^{D/2}} \prod_{i=1}^{E} \frac{1}{(-q_i^2 + m_i^2)^{\nu_i}}$,

for each loop l, internal edge i, in $D = D_0 - 2\epsilon$ dimensions.

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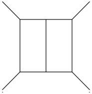
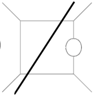
$\mathcal{A}_4^{(2)} = c_1'$  $+\, 0$  $+ \ldots$

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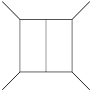
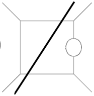
$\mathcal{A}_4^{(2)} = c_1'$  $+\, 0$  $+ \ldots$

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Serious bottlenecks

1. Eliminate huge number of linear (IBP) relations
2. Evaluate basis $\vec{f}$ of Feynman integrals (FI)

Evaluation of Feynman Integrals

State of the art: Canonical differential equations

For *polylogarithmic* FI, find basis transformation $\vec{g} = T \cdot \vec{f}$ such that
[Gehrmann,Remiddi'99] [Henn'13]

$$d\vec{g} = \epsilon \, d\widetilde{M} \, \vec{g}, \qquad \widetilde{M} \equiv \sum_i \overbrace{\tilde{a}_i}^{\text{constant matrices}} \log \underbrace{W_i}_{\text{letters}} .$$

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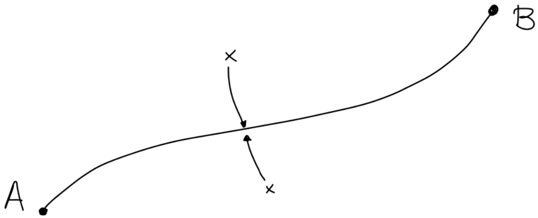
This strategy in line with e.g. [Abreu,Ita,Moriello,Page,Tschernow,Zeng'20]

The Role of the Landau Equations

Yield specific values of (kinematic) parameters of any (Feynman) integral, for which it may become singular. [Landau'59]



Formulated as conditions for the contour of integration ($A \to B$) to become trapped between two poles of integrand ($\times$). Recent revival of their study, e.g. [Berghoff,Brown,Collins,Hannesdottir,Klausen,McLeod,Mizera,Panzer] [Schwartz,Spradlin,Telen,Vergu,Volovich. . .]

Believed for long to only provide information on where $W_i = 0$.

This work

Evidence through two loops: Rational letters of polylogarithmic FI captured by Landau equations, when recast as polynomial of the kinematic variables of integral, known as the *principal* A*-determinant* E_A!

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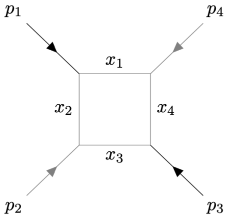
Example: 'Two-mass easy' box with $p_2^2 = p_4^2 = 0$, $p_1^2, p_3^2 \neq 0$:



E_A equipped with natural factorization, $(s = (p_1 + p_2)^2,\ t = (p_1 + p_4)^2)$

$$E_A = (p_1^2 p_3^2 - st) p_1^2 p_3^2 st (p_1^2 + p_3^2 - s - t)(p_3^2 - t)(p_3^2 - s)(p_1^2 - t)(p_1^2 - s).$$

where each factor is indeed a letter of the integral!

Outline

Introduction and Motivation

Feynman integrals, Landau singularities & GKZ systems

One-loop principal A-determinants and symbol letters

Conclusions and Outlook

Outline

Feynman Integrals in the Lee-Pomeransky Representation:

$$I_G = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2 - \sum_i \nu_i)} \int_0^\infty \prod_{i=1}^{E} \left(\frac{x^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \frac{1}{\mathcal{G}^{D/2}}$$

where $\mathcal{G} = \mathcal{U} + \mathcal{F}$ is the sum of the 1^{st} and 2^{nd} Symanzik polynomials,

- Of degree $L, L+1$ in the x_i, respectively.
- Coefficients of $\mathcal{U}$ are numbers, of $\mathcal{F}$ depend on kinematic parameters
- Obtained easily from data of graph G.

.

In this form, I_G is special case[1] of $\mathcal{A}$-hypergeometric function as defined by Gelfand, Graev, Kapranov & Zelevinsky (GKZ). [de la Cruz'19][Klausen'19]

Very active field of research, e.g.
[Ananthanarayan,Banik,Bera,Chang,Chen,Datta, Feng,Klemm,Nega,Safari,Vanhove,Walther,Zhang]

[1] Generic case: All $\mathcal{G}$ polynomial coefficients are variables, different from each other.

Singularities of GKZ-systems

Let $\mathcal{G} = \sum_{j=1}^{m} c_j \prod_{i=1}^{E} x_i^{a_{ij}}$, c_j all independent.
GKZ-system singular for c_i values solving

$$E_A(\mathcal{G}) = 0$$

Principal A-determinant of $\mathcal{G}$: Polynomial in c_j with integer coefficients, that vanishes whenever equations

$$\mathcal{G} = x_1 \frac{\partial \mathcal{G}}{\partial x_1} = \ldots = x_E \frac{\partial \mathcal{G}}{\partial x_E} = 0 \text{ have solution}.$$

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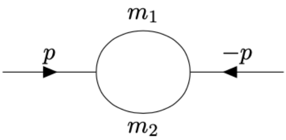
In practice, compute via theorem factorizing it into contributions from each face Γ of polytope with vertices $(a_{1j}, \ldots, a_{Ej})$,

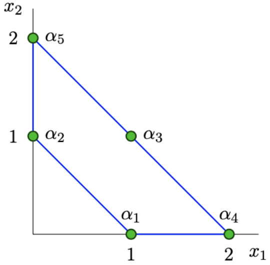
$$E_A(\mathcal{G}) = \prod_{\Gamma} \Delta_\Gamma(\mathcal{G}_\Gamma)$$

A-discriminant: Polynomial in c_i, that vanishes when $\mathcal{G}_\Gamma = \mathcal{G}|_{x_{m_j}=0, m_j \notin \Gamma}$

$$\mathcal{G}_\Gamma = \frac{\partial \mathcal{G}_\Gamma}{\partial x_{m_1}} = \ldots = \frac{\partial \mathcal{G}_\Gamma}{\partial x_{m_k}} = 0 \text{ have solution}\,.$$

Example: Principal A-determinant of bubble





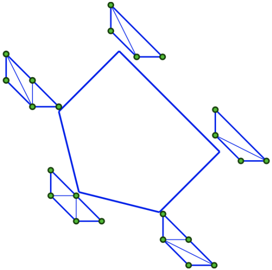
(Newton) polytope of $\mathcal{G}$ polynomial exponents, Newt($\mathcal{G}$)

$$\mathcal{G} = x_1 + x_2 + (m_1^2 + m_2^2 - p^2)x_1 x_2 + m_1^2 x_1^2 + m_2^2 x_2^2 \,,$$

$$\begin{aligned} E_A(\mathcal{G}) &= \Delta_{\alpha_4}\Delta_{\alpha_5}\Delta_{\alpha_4\alpha_5}\Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} \\ &= m_1^2 m_2^2 (p^4 + m_1^4 + m_2^4 - 2p^2 m_1^2 - 2p^2 m_2^2 - 2m_1^2 m_2^2) p^2 \,, \end{aligned}$$

Interpretation of $E_A(\mathcal{G})$ polytope

Newt($E_A(\mathcal{G})$), built out of exponents of $E_A(\mathcal{G})$ polynomial: Keeps track of *triangulations* of Newt($\mathcal{G}$).



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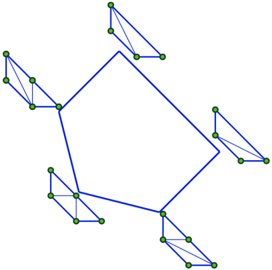
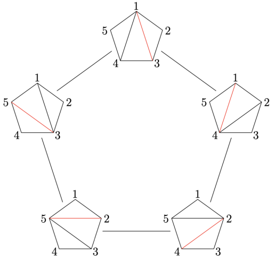




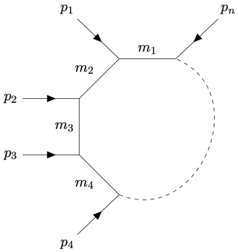
Cluster algebras also describe triangulations of geometric spaces [Fomin,Zelevinsky'01] [Felikson,Shapiro,Tumarkin'11]

First-principle derivation of observed cluster-algebraic structure of Feynman integrals? [Chicherin,Henn,Papathanasiou'20]... [He,Liu,Tang,Yang'22]

Outline

Generic n-point 1-loop integrals

All $m_i, p_i^2 \neq 0$ and different from each other



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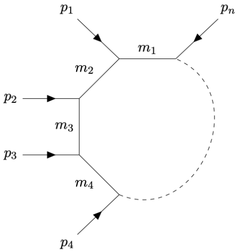

A-discriminants reduce to usual determinants

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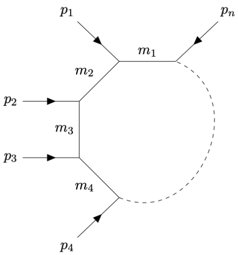

A-discriminants reduce to usual determinants $\Rightarrow$ *Modified Cayley matrix* $\mathcal{Y}$, [Melrose'65]

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \qquad \begin{aligned} Y_{ii} &= 2m_i^2 \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij-1} \\ s_{ij} &= (p_i + \ldots + p_j)^2 \end{aligned}$$

captures all Landau singularity information.

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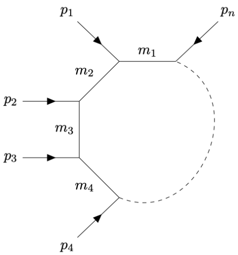

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- $\Delta(\mathcal{F}) = \det Y$: Leading[1] Landau singularity of type I[2]

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[1]Where all $x_i \neq 0$

[2]Type I (II): Integration contour pinched at finite (∞) values of loop momentum k.

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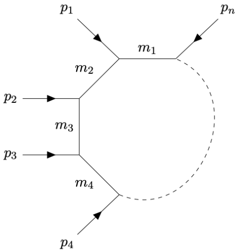

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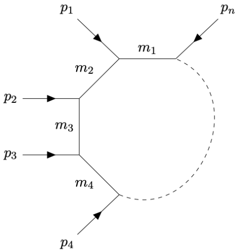

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- $\Delta(\mathcal{F}) = \det Y$: Leading[1] Landau singularity of type I[2]
- $\Delta(\mathcal{G}) = \det \mathcal{Y}$: Leading[1] Landau singularity of type II[2]
- Subleading Landau singularity where $x_{i_1}, \ldots, x_{i_m} = 0 \sim$ Leading singularity of subgraph where internal edges $i_1, \ldots, i_m$ removed [Klausen'21]

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[1] Where all $x_i \neq 0$

[2] Type I (II): Integration contour pinched at finite (∞) values of loop momentum k.

1-loop Subleading Landau Singularities=Subdeterminants

For any matrix A with elements a_{mn}, let (j,k)-th minor of A be

$$
A\begin{bmatrix} j \\ k \end{bmatrix} \equiv \begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & k & a_{1,k+1} & \cdots & a_{1,N} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k-1} & & a_{2,k+1} & \cdots & a_{2,N} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{j-1,1} & a_{j-1,2} & a_{j-1,3} & \cdots & a_{j-1,k-1} & & a_{j-1,k+1} & \cdots & a_{j-1,N} \\
j & & & & & & & & \\
a_{j+1,1} & a_{j+1,2} & a_{j+1,3} & \cdots & a_{j+1,k-1} & & a_{j+1,k+1} & \cdots & a_{j+1,N} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,k-1} & & a_{N,k+1} & \cdots & a_{N,N}
\end{vmatrix},
$$

where shading indicates removal of row and column. Similarly $A\begin{bmatrix} i_1 \dots i_k \\ j_1 \dots j_k \end{bmatrix}$,

$A\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \det A$.

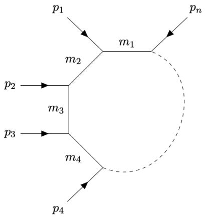
1-loop Subleading Landau Singularities=Subdeterminants

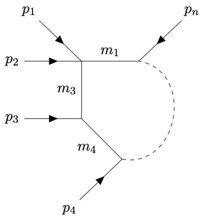
For any matrix A with elements a_{mn}, let (j,k)-th minor of A be

$$A\begin{bmatrix}j\\k\end{bmatrix} \equiv \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,k-1} & k & a_{1,k+1} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k-1} & & a_{2,k+1} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{j-1,1} & a_{j-1,2} & a_{j-1,3} & \cdots & a_{j-1,k-1} & & a_{j-1,k+1} & \cdots & a_{j-1,N} \\ j & & & & & & & & \\ a_{j+1,1} & a_{j+1,2} & a_{j+1,3} & \cdots & a_{j+1,k-1} & & a_{j+1,k+1} & \cdots & a_{j+1,N} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \cdots & a_{N,k-1} & & a_{N,k+1} & \cdots & a_{N,N} \end{vmatrix},$$

where shading indicates removal of row and column. Similarly $A\begin{bmatrix}i_1 \ldots i_k\\ j_1 \ldots j_k\end{bmatrix}$, $A\begin{bmatrix}\cdot\\ \cdot\end{bmatrix} = \det A$.

$$\mathcal{Y}\begin{bmatrix}3\\3\end{bmatrix}\left(\quad \right) = \mathcal{Y}\begin{bmatrix}\cdot\\ \cdot\end{bmatrix}\left(\quad \right)$$





Principal A-determinant of generic 1-loop graphs

Gathering previous bits of information, arrive at

$$E_A(\mathcal{G}) = \mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \prod_{i=1}^{n+1} \mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix} \dots \prod_{i_{n-1} > \dots > i_1 = 1}^{n+1} \mathcal{Y}\begin{bmatrix} i_1 \dots i_{n-1} \\ i_1 \dots i_{n-1} \end{bmatrix} \prod_{i=2}^{n+1} \mathcal{Y}_{ii} \,.$$

Contains all diagonal k-dimensional minors of $\mathcal{Y}$, $1 \leq k \leq n+1$, but $\mathcal{Y}_{11} = 0$.

$2^{n+1} - n - 2$ factors, e.g. $1, 4, 11, 26, 57, 120$ factors for $n = 1, \dots, 6$.

Each factor = polynomial symbol letter W_i!

Polylogarithmic integral singular for $W_i = 0 \Rightarrow E_A(\mathcal{G}) = 0$

From 1-loop polynomial to square-root letters

Square-root letters often present. How to obtain them?

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Re-factorize E_A with *Jacobi determinant identities* of the form

$$p \cdot q = f^2 - g = (f - \sqrt{g})(f + \sqrt{g}),$$

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1. where p, q factors of E_A, i.e. polynomial letters.
2. $f \pm \sqrt{g}$ contain leading singularity of FI in 2^{nd} term. [Cachazo'08]

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Motivation: 1-loop integrals = volumes of spherical simplices. [Davydychev,Delbourgo'99]

Crucial for their computation are the *Jacobi identities*,

$$A\begin{bmatrix}\cdot\\\cdot\end{bmatrix} A\begin{bmatrix}i & j\\ i & j\end{bmatrix} = A\begin{bmatrix}i\\ i\end{bmatrix} A\begin{bmatrix}j\\ j\end{bmatrix} - A\begin{bmatrix}i\\ j\end{bmatrix} A\begin{bmatrix}j\\ i\end{bmatrix} \overset{A=A^T}{=} A\begin{bmatrix}i\\ i\end{bmatrix} A\begin{bmatrix}j\\ j\end{bmatrix} - A\begin{bmatrix}i\\ j\end{bmatrix}^2.$$

Point 2 adopts widely observed pattern in 1- and 2-loop computations.

All 1-loop letters I

Need only ratio $\frac{f-\sqrt{g}}{f+\sqrt{g}}$, as product already contained in polynomial alphabet. Letting $D = D_0 - 2\epsilon$, obtain N letters of type,

$$W_{1,\dots,(i-1),\dots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}}}, & D_0 + n \text{ odd}, \\ \dfrac{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ 1 \end{bmatrix} + \sqrt{\mathcal{Y}\begin{bmatrix} i \\ i \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}, & D_0 + n \text{ even}. \end{cases}$$

All 1-loop letters II

In addition, $n(n-1)/2$ letters of type,

$$W_{1,\ldots,(i-1),\ldots,(j-1),\ldots,n} = \begin{cases} \dfrac{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} i \\ j \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\mathcal{Y}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}}, & D_0 + n \text{ odd}, \\[2em] \dfrac{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} - \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}{\mathcal{Y}\begin{bmatrix} 1 & j \\ 1 & i \end{bmatrix} + \sqrt{-\mathcal{Y}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\mathcal{Y}\begin{bmatrix} 1 & i & j \\ 1 & i & j \end{bmatrix}}}, & D_0 + n \text{ even}, \end{cases}$$

All 1-loop letters III

Our procedure also predicts $\mathcal{Y}\left[\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}\right]$ and $\mathcal{Y}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ as individual rational letters, but in fact only the ratio

$$W_{1,2,\dots,n} = \frac{\mathcal{Y}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}}{\mathcal{Y}\begin{bmatrix}1\\1\end{bmatrix}},$$

appears, as we'll get back to in next slide.

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Total letter count: Assuming $n \leq d+1$ for external kinematics dimension d,

$$|W| = 2^{n-3}\left(n^2+3n+8\right) - \frac{1}{6}\left(n^3+5n+6\right),$$

e.g. $|W| = 1, 5, 18, 57, 166$ for $n = 1, \ldots, 5$ and D_0 even.

Verification through differential equations & comparison with literature

From letter prediction, derived canonical differential equations through numeric IBP relations ⇒ confirmation.

By explicit computation up to $n = 10$, infer general form, e.g. $n + D_0$ even:

$$
\begin{aligned}
d\mathcal{J}_{1\ldots n} =& \epsilon \; d\log W_{1\ldots n}\; \mathcal{J}_{1\ldots n} \\
&+ \epsilon \sum_{1\le i\le n} (-1)^{i+\left\lfloor \frac{n}{2}\right\rfloor} d\log W_{1\ldots(i)\ldots n}\; \mathcal{J}_{1\ldots\widehat{i}\ldots n} \\
&+ \epsilon \sum_{1\le i<j\le n} (-1)^{i+j+\left\lfloor \frac{n}{2}\right\rfloor} d\log W_{1\ldots(i)\ldots(j)\ldots n}\; \mathcal{J}_{1\ldots\widehat{i}\ldots\widehat{j}\ldots n}.
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\end{aligned}
$$

Furthermore, compared to previous results for D_0 even based on

1. the diagrammatic coaction [Abreu,Britto,Duhr,Gardi'17]
2. the Baikov representation [Chen,Ma,Yang'22]

Agreement in form of CDE, as well as in letters for orientations presented in 2, see also. [Jiang,Yang'23]

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Proved that E_A has well-defined limit when any $m_i^2, p_j^2 \to 0$ (unique regardless of order with which we send them to zero).

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Multivariate generalization straightforward, but highly nontrivial that limit does not depend on order. E.g. triangle Cayley in limit $p_i^2 \to 0$:

$$\det Y = 0 + 2\sum_{i=1}^{3} p_i^2 (m_i^2 - m_{i-1}^2)(m_{i+1}^2 - m_{i-1}^2) + \mathcal{O}(p_j^2 p_k^2)\,,$$

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Strong evidence that non-generic FI alphabet obtained as limit.

Mathematica Notebook

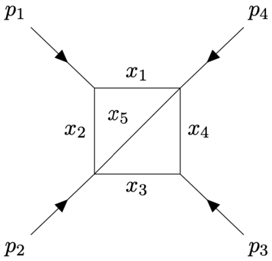
LandauAlphabetDE.nb

```
(* Symbol alphabets*)
(* Generic Box in even dimension *)
D0 = 4;
EvaluateLetter[AllLettersList[4]] // Short
(* Two-mass easy box limit *)
Factor[PLimit[%, ms[1] → 0, ms[2] → 0, ms[3] → 0, ms[4] → 0, s[1, 3] → 0, ps[2] → 0]];
(* In the limit the letters become multiplicatively dependent. Since all of them are rational, a basis may be found as follows *)
dlExpand[dl /@ %];
dlCollect /@ DeleteCases[RowReduce[CoefficientArrays[%, Variables[%]][[2]]].Variables[%], 0] /. dl[x_] → x
Length[%]
(* Product indeed yields corresponding limit of the principal A-determinant *)
LS2meBox - Times @@ %%

(* Differential equations *)
(*Box basis*)
basis @@ Range[4]
(*Box canonical differential equations*)
CDEs[%] // MatrixForm
```

```
Short=
ms[1] ≪24≫ (ms[3]^2 ps[1]^2 - 2 ms[3] × ms[4] ps[1]^2 + ms[4]^2 ps[1]^2 - 2 ms[1] × ms[3] × ps[1] × ps[2] +
  2 ms[1] × ms[4] × ps[1] × ps[2] + ≪172≫ + ms[3]^2 s[2, 3]^2 - 2 ms[1] × s[1, 2] s[2, 3]^2 - 2 ms[3] × s[1, 2] s[2, 3]^2 + s[1, 2]^2 s[2, 3]^2)

Out[180]= ps[1] × ps[3] (ps[1] - s[1, 2]) (ps[3] - s[1, 2]) s[1, 2] (ps[1] - s[2, 3]) (ps[3] - s[2, 3]) (ps[1] + ps[3] - s[1, 2] - s[2, 3]) s[2, 3] (ps[1] × ps[3] - s[1, 2] × s[2, 3])

Short=
{-1/(2 ms[1]), -1/(2 ms[2]), ≪53≫, (-ms[1] × ms[3] × ps[1] + ≪73≫)/(-ms[1] × ms[3] × ps[1] + ≪71≫ + √≪1≫),
 (-2 ms[1] × ms[2] × ps[1] + ≪65≫)/(-2 ms[1] × ms[2] × ps[1] + ≪63≫ + √((-ms[1]^2 + 2 ms[1] × ms[2] - ≪1≫^2 + ≪1≫ + 2 ms[2] × ps[1] - ps[1]^2) (≪1≫)))}

Out[185]= {ps[1], ps[3], ps[1] - s[1, 2], ps[3] - s[1, 2], s[1, 2], ps[1] - s[2, 3], ps[3] - s[2, 3], ps[1] + ps[3] - s[1, 2] - s[2, 3], s[2, 3], ps[1] × ps[3] - s[1, 2] × s[2, 3]}

Out[186]= 10

Out[187]= 0

Out[188]= {IG[2][1], IG[2][2], IG[2][3], IG[2][4], IG[2][1, 2], IG[2][1, 3], IG[2][1, 4], IG[2][2, 3], IG[2][2, 4], IG[2][3, 4], IG[4][1, 2, 3], IG[4][1, 2, 4], IG[4][1, 3, 4], IG[4][2, 3, 4], IG[4][1, 2, 3, 4]}

MatrixForm=
```

w[1]	0	0	0	0	0	0	0	0	0
0	w[2]	0	0	0	0	0	0	0	0
0	0	w[3]	0	0	0	0	0	0	0
0	0	0	w[4]	0	0	0	0	0	0
-w[1, {2}]	w[{1}, 2]	0	0	w[1, 2]	0	0	0	0	0
-w[1, {3}]	0	w[{1}, 3]	0	0	w[1, 3]	0	0	0	0
-w[1, {4}]	0	0	w[{1}, 4]	0	0	w[1, 4]	0	0	0
0	-w[2, {3}]	w[{2}, 3]	0	0	0	0	w[2, 3]	0	0
0	-w[2, {4}]	0	w[{2}, 4]	0	0	0	0	w[2, 4]	0
0	0	-w[3, {4}]	w[{3}, 4]	0	0	0	0	0	w[3, 4]
-w[1, {2}, {3}]	w[1, {2}, {3}] + w[{1}, {2}, 3]	-w[{1}, {2}, 3]	0	-w[1, 2, {3}]	w[1, {2}, 3]	0	-w[{1}, 2, 3]	0	0

Two-loop example of principal A-determinant-alphabet relation



1-mass slashed box,

$$p_1^2 \neq 0, p_2^2 = p_3^2 = p_4^2 = 0$$

$$E_A(\mathcal{G}) = (p_1^2 - t)(p_1^2 - s)(p_1^2 - s - t)(s + t)stp_1^2.$$

Agrees precisely with (2dHPL) alphabet known to describe 2-loop master integrals with these kinematics! [Gehrmann,Remiddi'00]

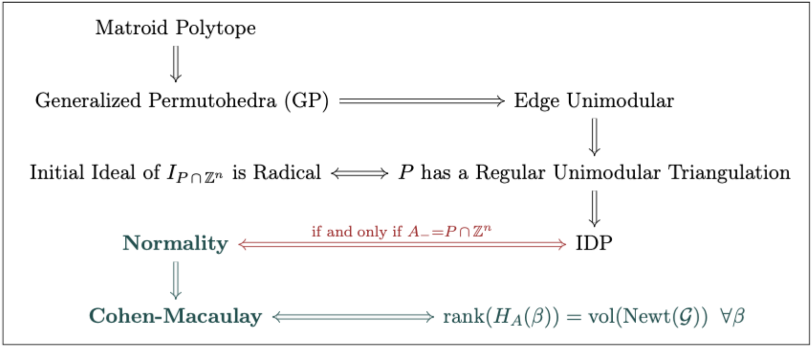
Further mathematical properties of Feynman integrals: Cohen-Macauley

Guarantees that

$$\#\ \text{master integrals} = \text{volume of Newt}(\mathcal{G})$$

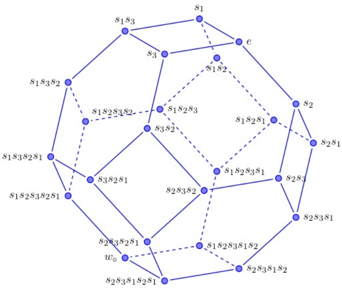
Proved it for currently largest known class of 1-loop integrals, including completely on-shell/massless. For earlier work, see [Tellander,Helmer'21] [Walther'22]

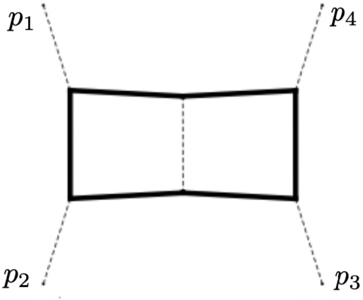
Relation to other properties:



Further mathematical properties of Feynman integrals
:Generalized permutohedron (GP) property

A polytope $P \subset \mathbb{R}^n$ is GP if and only if every edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$, where $\mathbf{e}_i$ is unit vector on coordinate axis, for some $i, j \in \{1, \ldots, n\}$. E.g.





Practical utility: This property facilitates new methods for fast Monte Carlo evaluation of Feynman integrals. [Borinsky'20] [Borinsky,Munch,Tellander'23]

Previously proven for generic kinematics. [Schultka'18] Here: Generalized to any graph where all external vertices joined by massive path.

Conclusions and Outlook

Evidence that rational letters of polylogarithmic FI captured by polynomial form of Landau equations in terms of *principal A-determinant* E_A!

- Through 2 loops
- 1 loop: Also obtain square-root letters from Jacobi identities + CDE
- Strong evidence for well-defined limits to non-generic kinematics
- Easy-to-use Mathematica file with our results

Next Stage

1. More efficient evaluation of E_A + more 2-loop checks
   [Helmer, GP,Tellander'24]
2. New predictions for pheno, e.g. letters for $2 \to 3$ with 2 massive legs
   [Les Houches Standard Model Precision Wishlist'21]
3. Explore implications for beyond-polylogarithmic case