Quadratic relations between Bessel moments

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Legendre's relation

$E\colon y^2 = 4x^3 - ax - b$ elliptic curve

$\omega = \frac{dx}{y}$ (*first kind*) $\eta = x\frac{dx}{y}$ (*second kind*)

$E(\mathbb{C}) = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$

$\gamma_1, \gamma_2\colon [0,1] \to E(\mathbb{C})$ images of paths $\gamma_1(t) = t$ and $\gamma_2(t) = t\tau$

$\omega_i = \int_{\gamma_i} \omega$ (periods), $\eta_i = \int_{\gamma_i} \eta$ (quasiperiods)

Theorem (Legendre's relation)

$$\boxed{\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i}$$

quel que soit c.

Pour abréger la notation, désignons simplement par F, E, les quantités $F^1(c)$, $E^1(c)$, et par F′, E′, les quantités $F^1(b)$, $E^1(b)$, et supposons

$$P = FE' + F'E - FF',$$

P étant une fonction de c encore inconnue.

Je différencie les deux membres par rapport à c qui est la seule variable qu'ils contiennent. Or ayant $E(\varphi) = \int \Delta d\varphi$, $F(\varphi) = \int \frac{d\varphi}{\Delta}$, $\Delta^2 = 1 - c^2 \sin^2\varphi$, la différentiation donne

$$\frac{dE}{dc} = -\int \frac{c d\varphi \sin^2\varphi}{\Delta} = \frac{1}{c}(E - F),$$
$$\frac{dF}{dc} = \int \frac{c d\varphi \sin^2\varphi}{\Delta^3} = \frac{1}{c}\int \frac{d\varphi}{\Delta^3} - \frac{1}{c}\int \frac{d\varphi}{\Delta}.$$

Mais par les formules de l'art. 9, on a $\int \frac{d\varphi}{\Delta^3} = \frac{1}{b^2}\int \Delta d\varphi - \frac{c^2 \sin\varphi \cos\varphi}{b^2 \Delta}$, et dans le cas de $\varphi = \frac{1}{2}\pi$ dont il s'agit, le second terme s'évanouit : ainsi on aura

$$\frac{dF}{dc} = \frac{1}{b^2 c}(E - b^2 F).$$

On aura semblablement $\frac{dE'}{db} = \frac{1}{b}(E' - F')$, $\frac{dF'}{db} = \frac{1}{c^2 b}(E' - c^2 F')$, et parce que $bdb + cdc = 0$, on en déduira

$$\frac{dE'}{dc} = -\frac{c}{b^2}(E' - F'),$$
$$\frac{dF'}{dc} = -\frac{1}{b^2 c}(E' - c^2 F').$$

Substituant ces valeurs dans celle de dP, on aura $dP = 0$; donc $P =$ const. Mais on a trouvé dans un cas particulier $P = \frac{1}{2}\pi$; donc l'équation (d') a lieu généralement, quel que soit c.

Cohomological interpretation

$\{[\omega],[\eta]\}$ is a basis of *algebraic de Rham cohomology*

$$\mathrm{H}^1_{\mathrm{dR}}(E) = \mathbb{H}^1(E, \mathcal{O}_E \to \Omega^1_E).$$

$\{[\gamma_1],[\gamma_2]\}$ is a *basis of Betti homology*

$$\mathrm{H}_1^{\mathrm{B}}(E) = \mathrm{H}_1(E(\mathbb{C}), \mathbb{Q}).$$

With respect to these bases, the period pairing

$$\mathrm{H}^1_{\mathrm{dR}}(E) \otimes \mathrm{H}_1^{\mathrm{B}}(E) \longrightarrow \mathbb{C}$$

is represented by the matrix $\left(\begin{smallmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{smallmatrix}\right)$.

Intersection pairings

De Rham and Betti cohomology are endowed with intersection pairings that fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{dR}}(E) \otimes \mathrm{H}^1_{\mathrm{dR}}(E) \arrow[r, "\langle\cdot,\cdot\rangle_{\mathrm{dR}}"] \arrow[d] & \mathrm{H}^2_{\mathrm{dR}}(E) \cong \mathbb{C} \arrow[d, "2\pi i"] \\
\mathrm{H}^1_{\mathrm{B}}(E)_{\mathbb{C}} \otimes \mathrm{H}^1_{\mathrm{B}}(E)_{\mathbb{C}} \arrow[r, "\langle\cdot,\cdot\rangle_{\mathrm{B}}"] & \mathrm{H}^2_{\mathrm{B}}(E)_{\mathbb{C}} \cong \mathbb{C}.
\end{tikzcd}$$

This gives the relation

$$2\pi i \cdot \langle[\omega], [\eta]\rangle_{\mathrm{dR}} = \langle \omega_1[\gamma_1]^* + \omega_2[\gamma_2]^*, \eta_1[\gamma_1]^* + \eta_2[\gamma_2]^* \rangle_{\mathrm{B}}.$$

The Betti pairing

The Betti pairing is given by *topological intersection*:

$$\langle [\gamma]^*, [\gamma']^* \rangle_{\mathrm{B}} = \sum_{P \in \gamma \cap \gamma'} i_P(\gamma, \gamma')$$

with $i_P(\gamma, \gamma') = 1$ or -1 depending on the orientation of γ and γ'.

In our example:

$$\langle [\gamma_1]^*, [\gamma_1]^* \rangle_{\mathrm{B}} = \langle [\gamma_2]^*, [\gamma_2]^* \rangle_{\mathrm{B}} = 0,$$
$$\langle [\gamma_1]^*, [\gamma_2]^* \rangle_{\mathrm{B}} = -\langle [\gamma_2]^*, [\gamma_1]^* \rangle_{\mathrm{B}} = 1.$$

The de Rham pairing

The de Rham pairing can be computed in terms of *residues*.
Representing classes by differentials of the second kind, we get:

$$\langle [\omega], [\eta] \rangle_{\mathrm{dR}} = \sum_{P \text{ poles}} \mathrm{Res}_P \left(\int \omega \right) \eta,$$

where $\int \omega$ is any local primitive of ω around P.

In our example:

$$\langle [\omega], [\eta] \rangle_{\mathrm{dR}} = \mathrm{Res}_0 \left(\int dz \right) \mathfrak{P}(z) = 1.$$

Bessel moments

In $D = 2$ quantum field theory, moments of the Bessel functions arise as Feynman integrals of banana graphs

$$\int_{x_i \geq 0} \frac{1}{(1 + \sum_{i=1}^{\ell} x_i)(1 + \sum_{i=1}^{\ell} 1/x_i)} \prod_{i=1}^{\ell} \frac{dx_i}{x_i}$$
$$= 2^{\ell} \int_0^{\infty} I_0(t) K_0(t)^{\ell+1} t dt,$$

where $I_0(t)$ and $K_0(t)$ are the modified Bessel functions

$$I_0(t) = \frac{1}{2\pi i} \int_{|x|=1} e^{-\frac{t}{2}(x + \frac{1}{x})} \frac{dx}{x}$$

$$K_0(t) = \frac{1}{2} \int_0^{\infty} e^{-\frac{t}{2}(x + \frac{1}{x})} \frac{dx}{x} \qquad (|\arg t| < \pi/2).$$

An interesting feature of this formula is that $I_0(t)$ and $K_0(t)$ are solutions of the differential operator

$$(t\partial_t)^2 - t^2,$$

which has an irregular singularity at infinity, and hence does not come from geometry in the usual sense of encoding how periods vary in families. However, the integral

$$\int_0^\infty I_0(t) K_0(t)^{\ell+1} t dt$$

turns out to be a (classical) period itself.

The program of Broadhurst and Roberts

Broadhurst and Roberts have put forward a program to understand the motivic origin of Bessel moments

$$\int_0^\infty I_0(t)^a K_0(t)^b t^c dt$$

and relate them to special values of L-functions of Kloosterman sums. As part of their program, they found a remarkable set of quadratic relations satisfied by these numbers.

Motivic interpretation

For fixed t, the Bessel values $2\pi i I_0(t)$ and $K_0(t)$ are exponential periods of the exponential motive

$$\mathrm{H}^1(\mathbb{G}_{\mathrm{m}}, \frac{t}{2}(x+\frac{1}{x})).$$

As t varies, the exponential Picard-Fuchs differential equation they satisfy is precisely the Bessel differential equation.

In terms of connections (set $z = t^2/4$)

$$\mathrm{Kl}_2 = (\mathcal{O}_{\mathbb{G}_{\mathrm{m}}} v_0 \oplus \mathcal{O}_{\mathbb{G}_{\mathrm{m}}} v_1, \nabla),$$

$$z\nabla_{\partial_z}(v_0, v_1) = (v_0, v_1) \cdot \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}.$$

Motivic interpretation

To deal with Bessel moments, we consider symmetric powers of the Bessel differential equation $\mathrm{Sym}^k \mathrm{Kl}_2$ and their cohomology.

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \\ &= \mathrm{Im}\left(\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)\right).\end{aligned}$$

This space is endowed with a $(-1)^{k+1}$-symmetric pairing

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \to \mathbb{Q}(-k-1)$$

If $4 \nmid k$, it has basis $\omega_j = z^j v_0^k \frac{dz}{z}$ $(j = 1, \ldots, k' = \lfloor (k-1)/2 \rfloor)$.

Motivic interpretation

The Betti counterpart of these cohomology groups is

$$\begin{aligned}&\mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k\mathrm{Kl}_2)\\&\qquad = \mathrm{Im}\left(\mathrm{H}_1^{\mathrm{rd}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k\mathrm{Kl}_2) \to \mathrm{H}_1^{\mathrm{mod}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k\mathrm{Kl}_2)\right),\end{aligned}$$

where ‘rd’ and ‘mod’ stand for rapid decay and moderate growth.

Elements are represented by twisted cycles $c \otimes e$, where c is a path and e a horizontal section of $\mathrm{Sym}^k\mathrm{Kl}_2$ with rapid decay/moderate growth along c.

If $4 \nmid k$, basis given by $\alpha_i = [0, \infty] \otimes e_0^i e_1^{k-i} \quad (i = 1, \ldots, k')$

Motivic interpretation

Intersection pairing:

$$\mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes \mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \to \mathbb{Q}(-k-1)$$

Period pairing:

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \otimes \mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathbb{C}$$

Quadratic relations between Bessel moments

Theorem (with Claude Sabbah and Jeng-Daw Yu)

1. *With respect to the bases $\{\alpha_i\}$ and $\{\omega_j\}$, the period matrix is given by Bessel moments:*

$$\mathrm{P}_k^{\mathrm{mid}} = \left((-1)^{k-i}\,2^{k+1-2j}(\pi i)^i \int_0^\infty I_0(t)^i K_0(t)^{k-i} t^{2j-1}\,dt\right)_{i,j}.$$

2. *There are quadratic relations*

$$\boxed{\mathrm{P}_k^{\mathrm{mid}} \cdot (\mathrm{S}_k^{\mathrm{mid}})^{-1} \cdot {}^t\mathrm{P}_k^{\mathrm{mid}} = (-2\pi i)^{k+1}\,\mathrm{B}_k^{\mathrm{mid}}}$$

where $\mathrm{B}_k^{\mathrm{mid}}$ and $\mathrm{S}_k^{\mathrm{mid}}$ are the matrices of the Betti and de Rham intersection pairings in the bases $\{\alpha_i\}$ and $\{\omega_j\}$.

Closed-form expression for the Betti matrix

$$\mathrm{B}_k^{\mathrm{mid}} = \left((-1)^{k-i} \, \frac{(k-i)!(k-j)!}{k!} \, \frac{B_{k-i-j+1}}{(k-i-j+1)!} \right)_{i,j},$$

where B_n stands for the n-th Bernoulli number.

The de Rham matrix is lower-right triangular with anti-diagonal

$$\begin{cases} (-2)^{k'} \, \dfrac{k'!}{k!!} & \text{if } k \text{ is odd,} \\ \dfrac{(-1)^{k'+1}}{2^{k'}(k'+1-2i)} \cdot \dfrac{(k-1)!!}{(k'+1)!} & \text{if } k \text{ is even.} \end{cases}$$

Algorithm to compute the full matrix gives for example:

$$S_5^{\mathrm{mid}} = \begin{pmatrix} 0 & \frac{2^3}{3\cdot 5} \\ \frac{2^3}{3\cdot 5} & \frac{2^4\cdot 13}{3^3\cdot 5^3} \end{pmatrix}, \qquad S_6^{\mathrm{mid}} = \begin{pmatrix} 0 & -\frac{5}{2^3} \\ \frac{5}{2^3} & 0 \end{pmatrix}$$

$$S_7^{\mathrm{mid}} = \begin{pmatrix} 0 & 0 & -\frac{2^4}{5\cdot 7} \\ 0 & -\frac{2^4}{5\cdot 7} & -\frac{2^5\cdot 863}{3\cdot 5^3\cdot 7^3} \\ -\frac{2^4}{5\cdot 7} & -\frac{2^5\cdot 863}{3\cdot 5^3\cdot 7^3} & -\frac{2^4\cdot 79\cdot 36919}{3^2\cdot 5^5\cdot 7^5} \end{pmatrix}.$$

[Broadhurst and Roberts conjectured relations of the form

$$\boxed{\mathrm{P}_k^{\mathrm{BR}} \cdot \mathrm{D}_k^{\mathrm{BR}} \cdot {}^t\mathrm{P}_k^{\mathrm{BR}} = \mathrm{B}_k^{\mathrm{BR}}}$$

Up to normalisation, $(\mathrm{S}_k^{\mathrm{mid}})^{-1}$ and $\mathrm{D}_k^{\mathrm{BR}}$ agree for $k \leq 22$.
Spoiler alert: talk by Y. Zhou in this seminar]