GKZ approach to Feynman integrals and beyond

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Seminar series on motives and period integrals in QFT and string theory
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What Feynman integrals evaluate to?

In 1963 Regge conjectured that the convergent Feynman integrals form a new class of special functions satisfying certain differential equations. He called them generalized hypergeometric functions. His justification for this

V A Golubeva, 1976 Russ.Math.Surv.31 139

- Answer from Mellin-Barnes Boos-Davydychev, '90 techniques: \( pF_q \), Appell, Lauricella, etc. Horn-type multivariable hypergeometric functions
  Kalmikov-Bytev-Kniehl-Ward-Yost, '09

- IBPs Chetyrkin-Tkachov, '81 +Differential Equations Gehrmann-Remiddi, '99 +\( \epsilon \)-form
  Henn, '13

- Goncharov polylogarithms and related elliptic structures

Gel’fand-Kapranov-Zelevinsky (GKZ) observation: A-hypergeometric functions with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals

\[ \int \prod_{i=1}^{n} P_i(t_1, \ldots, t_n)^{a_i} \frac{dt_1 \cdots dt_n}{t_1^{b_1} \cdots t_n^{b_n}} \]

where \( P_i \) are polynomials, i.e., practically all integrals which arise in quantum field theory.

Remarks

- All the above answers expressible as infinite sums
- State of the art based on the most popular answer \( \rightarrow \) efficiency
- Precise answer to this question not only of mathematical relevance
Invitation to GKZ: Bubble

- Suppose we want to study the Feynman integral

\[
I_{\text{bubble}} = \int \left( \frac{d^d k}{\pi^{d/2}} \right) \frac{1}{[(k)^2]^\alpha_1 [(k - p)^2 + m^2]^\alpha_2}
\]

- Lee-Pomeransky (\(\beta = d/2\))

\[
I_{\text{bubble}}/\xi^{\Gamma_\alpha} := I_g(\alpha, \beta) = \int_{\mathbb{R}^2_+} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(z_1 + z_2 + (m^2 + s)z_1 z_2 + m^2 z_2^2)^\beta} \frac{dz_1}{z_1} \frac{dz_2}{z_2}
\]

- GKZ approach: consider the more general version of this integral

\[
I_g(\alpha, \beta, c) = \int_{\Omega} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1 z_1 + c_2 z_2 + c_3 z_1 z_2 + c_4 z_2^2)^\beta} \frac{dz_1}{z_1} \frac{dz_2}{z_2}
\]

- Consider the matrix of exponents of \(g(c, z)\)

\[
g(c, z) = (c_1 z_1 + c_2 z_2 + c_3 z_1 z_2 + c_4 z_2^2) \iff A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 2
\end{pmatrix}
\]
Integral satisfies the system of PDEs ($\partial_i := \partial/\partial c_i$)

$$(\partial_2 \partial_3 - \partial_1 \partial_4)I(\alpha, \beta, c) = 0,$$

$$(c_1 \partial_1 + c_2 \partial_2 + c_3 \partial_3 + c_4 \partial_4 + \beta)I(\alpha, \beta, c) = 0,$$

$$(c_1 \partial_1 + c_3 \partial_3 + \alpha_1)I(\alpha, \beta, c) = 0,$$

$$(c_2 \partial_2 + c_3 \partial_3 + 2c_4 \partial_4 + \alpha_2)I(\alpha, \beta, c) = 0.$$

Canonical series solutions (Saito-Sturmfels-Takayama) ($\beta := d/2$)

$$\phi_1 = c^{\gamma_1} _2 F_1 \left( \alpha_1, \alpha_1 + \alpha_2 - \beta; 2\alpha_1 + \alpha_2 - 2\beta + 1; \frac{c_2 c_3}{c_1 c_4} \right),$$

$$\phi_2 = c^{\gamma_2} _2 F_1 \left( 2\beta - \alpha_1 - \alpha_2, \beta - \alpha_1; 2\beta - 2\alpha_1 - \alpha_2 + 1; \frac{c_2 c_3}{c_1 c_4} \right).$$

$\gamma_i$ roots of a system of polynomial equations obtained from $A$

General solution is

$$I(\alpha, c) = K_1 \phi_1 + K_2 \phi_2$$

Feynman integral is the restriction of $I(\alpha, c) \rightarrow c_1 = c_2 = 1, c_3 = (s + m^2), c_4 = m_2$
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A-philosophy I

- GKZ (Discriminants, Resultants and Multidimensional Determinants): “The study of many problems becomes more transparent if we consider not individual polynomials but polynomials with indeterminate coefficients”

- Multi-index notation ($\alpha \in \mathbb{Z}^N$)
  \[ z^\alpha := z_1^{\alpha_1} \cdots z_N^{\alpha_N} \]

- Laurent polynomials in $N$ variables of the form
  \[ b_i(z) = \sum_{j=1}^{n_i} c_{ij} z^{\alpha_{ij}}, \quad c_{ij} \in \mathbb{C}_* = \mathbb{C}\{0\}, \quad i = 1, \ldots, q \]

- For each $b_i(z)$, we have the $N \times n_i$ configuration matrix
  \[ A_i = (\alpha_{i1} \cdots \alpha_{ik} \cdots \alpha_{in_i}), \quad \alpha_{ik} \in \mathbb{Z}^N \]

- Each column of $A_i$ associated with a monomial term in $b_i(z)$

- $n = n_1 + \cdots + n_q$ is the total number of monomials
A-philosophy II

- Product of Laurent polynomials
  \[ b(z) := b_1(z) \cdots b_q(z) \]

- Define the \((N + q) \times n\) matrix
  \[
  A := \begin{pmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1 \\
  A_1 & A_2 & \ldots & A_q
  \end{pmatrix}
  \]

- Here \(0 = (0, \ldots, 0)\) and \(1 = (1, \ldots, 1)\) are row vectors of length \(|A_i|\).
  \[ \text{co}(A) := n - N - q. \]

Comment

One can associate a polytope to the matrix \(A\) and study its combinatorial properties. In this talk we will not follow this approach.
Example: \( \binom{2}{1}F_1 \)

- Single polynomial \( n_1 = 4, \ N = 2, \ q = 1 \)

\[
b(c, z) = c_1 + c_2 z + c_3 z^2 + c_4 z_1 z_2 \iff A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

- Product of polynomials \( n_1 = 2, \ n_2 = 2, \ N = 1, \ q = 2 \)

\[
b(z) = b_1^{\beta_1} b_2^{\beta_2} = (c_1 + c_2 z)^{\beta_1} (c_3 + c_4 z)^{\beta_2} \iff A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{pmatrix}
\]
Gel’fand-Kapranov-Zelevinsky systems

Defined by the following data

1. $(N + q) \times n$ matrix $A$ such that the vector $(1, \ldots, 1)$ lies in its row span

   \[ b(z) = b_1(z) \cdots b_q(z). \]

2. A vector of parameters $\kappa = (\kappa_1, \ldots, \kappa_{N+q})$, $\kappa_i \in \mathbb{C}$

3. A system of partial differential equations (PDEs) associated with $A$. Let $u, v \in \mathbb{N}^n$ and consider

\[
\left( \partial^u - \partial^v \right) F(c) = 0, \quad \text{where} \quad A u = A v,
\]

\[
\left( \sum_{j=1}^{n} a_{ij} \theta_j - \kappa_i \right) F(c) = 0, \quad i = 1, \ldots, N + q
\]

\[ \theta_j = c_j \partial / \partial c_j \quad \text{and} \quad \partial^u = \partial_{u_1} \cdots \partial_{u_n} \]

\[ A \text{-hypergeometric functions} \]

A holomorphic function $F(c)$ or formal series is called $A$-hypergeometric if it satisfies the system of PDEs
GKZ and D-modules

- GKZ systems as holonomic ideals in the Weyl algebra $D$
- $D = \mathbb{C} \langle c_1, \ldots, c_n, \partial_1, \ldots, \partial_n \rangle$ modulo commutation rules between $c_i$, $\partial_j$

\[
D \ni p(z, \partial) = \sum_{\alpha, \beta} a_{\alpha \beta} c^\alpha \partial^\beta
\]

- Toric ideal of $A$

\[
I_A := \langle \partial^u - \partial^v : Au = Av, \quad u, v \in \mathbb{N}^n \rangle \subset \mathbb{C}[\partial_1, \ldots, \partial_n]
\]

- Ideal generated by $\kappa^T$ and $\theta = (\theta_1, \ldots, \theta_n)^T$

\[
\langle A\theta - \kappa^T \rangle \subset \mathbb{C}[\theta_1, \ldots, \theta_n]
\]

- $H_A(\kappa)$ is the left ideal on $D$ generated by $I_A$ and $\langle A\theta - \kappa^T \rangle$

A-hypergeometric functions

A holomorphic function $F(c)$ or formal series is $A$-hypergeometric of degree $\kappa$ if

\[
H_A(\kappa) \bullet F(c) = 0, \quad \text{rank}(H_A(\kappa)) \geq \text{vol}(A)
\]

Generic $\kappa$: \text{rank}(H_A(\kappa)) = \text{vol}(A) = \text{degree}(I_A)
Euler type solutions

- Vector of parameters $\kappa := (-\beta, -\alpha)$, $\beta \in \mathbb{C}^q$, $\alpha \in \mathbb{C}^N$

  $$I_b(\kappa) = \int_{\Omega} \frac{z^\alpha}{b(c, z)^\beta} d\eta_N,$$

  $$d\eta_N = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_N}{z_N},$$

- Cycle $\Omega \subset (\mathbb{C}_*)^N \setminus \mathcal{V}(b)$ usually assumed to be compact
- Set of zeros of $b(c, z)$ is the algebraic variety $\mathcal{V}(b)$
- GKZ, '90: If the integral $I_b(\kappa)$ converges and defines a germ of analytic functions in the variables $z$, then it represents a solution of the A-hypergeometric system $H_A(\kappa)$
- Also GKZ, '90 ... if all $b_i$ have real coefficients one can take the integral also over some connected component of $\mathbb{R}^N \setminus \mathcal{V}(b)$
- Non-compact cycles for A-hypergeometric functions (Coamoeba)
Non compact cycles: Euler-Mellin integrals

- Berkesh-Forsgård-Passare (BFP), ’13
  - Euler-Mellin integral is an integral of the form of $I_b(\kappa)$ taken over a cycle $\Omega = \text{Arg}^{-1}(\theta)$, related to coamoeba of $\mathcal{V}(b)$

Amoeba and Coamoeba of an algebraic variety

- $A_b := \text{Log}(\mathcal{V}(b))$, $\text{Log}(z) = (\log |z_1|, \ldots, \log |z_N|)$
- $A'_b := \text{Arg}(\mathcal{V}(b))$, $\text{Arg}(z) = (\text{arg}(z_1), \ldots, \text{arg}(z_N))$

- BFP consider the Euler-Mellin integral

$$I_b(\kappa) = \int_{\mathbb{R}_+^N} \frac{z^\alpha}{b(z)^\beta} d\eta_N = \int_{\mathbb{R}^N} \frac{e^{(\alpha,x)}}{b(e^x)^\beta} dx$$

- Simplest case is $\Omega \in \mathbb{R}_+^N$ which covers Feynman integrals
- The convergence of these integrals is controlled by a theorem due to Berkesh, Forsgård and Passare (BFP).
Amoebas and coamoebas

\[
b(z_1, z_2) = 1 + z_1^2 + z_2^2 + z_1z_2^2 + z_2z_1^2 = 0
\]

[Credit: Jens Forsgård mathematica package]

- Noncompact cycle is given by a representative \( \theta \in \Theta \) of a connected component of \( \mathbb{R}^N \setminus \bar{A}'_b \)
  \[
  \Omega = \text{Arg}^{-1} \theta
  \]
- For a Feynman integral we should choose \( \theta \) such that \( \Omega \in \mathbb{R}^N_+ \), in other words such that \( 0 \in \mathbb{R}^N \setminus \bar{A}'_b \)
Let us consider the weighted Newton polytopes of $b_j \tau \Delta_b = \sum_{j=1}^q \tau_j \Delta_{b_j}$.

$I(\kappa)$ converges and defines an analytic function with parameters $\kappa = (-\beta, -\alpha)$ on the tube domain

$$\{ (\alpha, \beta) \in \mathbb{C}^{N+q} | \tau := \text{Re } \beta \in \mathbb{R}_+, \quad \sigma := \text{Re } \alpha \in \text{int}(\tau \Delta_b) \}.$$ 

If the polynomials $b(z)$ vanish on the positive orthant we can take a connected component $\Theta$ of $\mathbb{R}^N \setminus \overline{A}_b'$, where $\overline{A}_b'$ denotes the closure of the coamoeba of $b$ and consider the integral

$$I_b(\kappa) = \int_{\text{Arg}^{-1} \theta} \frac{z^\alpha}{b(z)^\beta} d\eta_N = \int_{\mathbb{R}^N} \frac{e^{\alpha \cdot (x+i\theta)}}{b(e^x+i\theta)^\beta} d\eta_N,$$

Promoting the coefficients of $b(z)$ to indeterminates

$$I_b(c, \kappa) = \int_{\text{Arg}^{-1} \theta} \frac{z^\alpha}{b(c, z)^\beta} d\eta_N$$

represents an A-hypergeometric function (Theorem 4.2 in BFP)

For generic parameters $\kappa$ provides a basis of solutions of $H_A(\kappa)$

Each integral is evaluated on a representative of $\Theta$ for each connected component of $\mathbb{R}^N \setminus \overline{A}_b'$.
Formal series Solutions

\[ \mathcal{L} := \{ u \in \mathbb{Z}^n : Au = 0 \} \]

- For \( u \in \mathcal{L} \), we can write \( u = u_+ - u_- \), where \( u_\pm \in \mathbb{N}^n \) have disjoint support.
- For \( \gamma \in \mathbb{C}^n \) we define (falling factorials)

\[
[\gamma]_{u_-} := \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (\gamma_i - j + 1) = \prod_{i:u_i<0} (-1)^{-u_i} (\gamma_i)_{-u_i},
\]

\[
[\gamma + u]_{u_+} := \prod_{i:u_i>0} \prod_{j=1}^{u_i} (\gamma_i + u_i - j + 1) = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (\gamma_i + j) = \prod_{i:u_i>0} (\gamma_i + 1)^{u_i}
\]

\((a)_x\) are Pochhammer symbols

- Series solution

\[
\phi_{\gamma} := \sum_{u \in \mathcal{L}} \frac{[\gamma]_{u_-}}{[\gamma + u]_{u_+}} c^{(\gamma+u)}
\]
Canonical series algorithm (Saito-Sturmfels-Takayama)

Frobenious method

\[
\frac{d^2 y}{dx^2} + \omega^2 y = 0
\]

1. Try series solution

\[
y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}, \ a_0 \neq 0
\]

2. Indicial equation

\[
k(k - 1) = 0
\]

Canonical Series

\[
H_A(\kappa) \bullet F(c) = 0
\]

1. Formal solution

\[
\phi_\gamma := \sum_{u \in \mathcal{L}} \frac{[\gamma] u_-}{[\gamma + u] u_+} c^{(\gamma + u)}
\]

2. Fake indicial ideal

\[
\text{find}_w(H_A(\kappa))
\]

Convergence

Weight vector \( w \) chooses a region of convergence: representative of the Gröbner fan of \( I_A \)
Canonical series algorithm (Saito-Sturmfels-Takayama)

2F1 two integral

\[
\begin{align*}
\text{In}[54] &= \{1, 1, 1, 1\}, \{0, 1, 0, 1\}, \{0, 0, 1, 1\} \\
& \quad \text{kerfromM2}[A] \\
& \quad \text{finwIA}[A, \{0, 1, 1, 1\}, \{-\beta, -\alpha[1], -\alpha[2]\}] \\
& \quad \text{fakeExp}[^%, \text{Table}[\theta[i], \{i, 1, 4\}]] \\
& \quad \text{co1res}[A, \{0, 1, 1, 1\}, \{-\beta, -\alpha[1], -\alpha[2]\}] \\
\text{Out}[55] &= \{(1, \{-1\}, \{-1\}, \{1\}\}\} \\
\text{Out}[57] &= \{-\beta + \alpha[1], -\alpha[1] + \alpha[2], \theta, -\alpha[2]\}, \{-\beta + \alpha[2], \theta, \alpha[1] - \alpha[2]\}, \\
\text{Out}[58] &= \{-n, n, n, -n\}, \{-n\}, \{n\}, \{\theta[2] \times \theta[3], \beta + \theta[1] + \theta[2] + \theta[3] + \theta[4], \beta - \alpha[1], -\alpha[1] + \alpha[2], \theta, -\alpha[2]\}, \\
& \quad \left\{\begin{array}{c}
(\beta - \alpha[1])_n \\
(\alpha[2])_n \\
(-\alpha[1] + \alpha[2] + 1)_n \\
(1)_n \\
(\alpha[1] - \alpha[2] + 1)_n
\end{array}\right\}
\end{align*}
\]

Interface: Macaulay2+Mathematica
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Lee-Pomeransky representation of Feynman integrals

\[ I_F(\alpha) = \xi \Gamma_\alpha \int_{\mathbb{R}_+^N} \left( \prod_{i=1}^{N} \frac{dz_i}{z_i^{\alpha_i}} \right) \frac{1}{(U + F)^{d/2}} = \xi \Gamma_\alpha \int_{\mathbb{R}_+^N} \frac{z^{\alpha}}{(U + F)^{d/2}} d\eta_N \]

- Factor independent of the kinematics

\[ \xi \Gamma_\alpha := \frac{\Gamma(d/2)}{\Gamma((L + 1)d/2 - \sum_{i=1}^{N} \alpha_i) \prod_{i=1}^{N} \Gamma(\alpha_i)} \]

Symanzik polynomials

- Dimensionless (scaling in \( F \) assumed)
- Euclidean kinematics: invariants \(-(p_i + p_j)^2 > 0\)
- \( U \) homogeneous polynomial of degree \( L \)
- \( F \) homogeneous polynomial of degree \( L + 1 \)
- \( U, F \) positive functions of their parameters
- kinematic dependence is in \( F \)
- \( U \) and \( F \) can only vanish on the boundaries of the integration region
Proposal based on canonical series

Idea

Consider Feynman integrals as special points of A-hypergeometric functions. A Feynman integral is A-hypergeometric whenever we can compute its canonical series $\phi_i$.

$$I_F(\kappa) = K_1 \phi_1 + \cdots + K_r \phi_r$$

Feynman integrals

- Consider the coefficients in $g(z) = U + F$ as indeterminate

$$g(z) \to g(z, c) = U(c) + F(c) \iff A$$

- In general we add a deformation to $g(z, c)$ to ensure canonical series when $\text{co}(A) = 0$, $g(z, c) \to g_r(z, c) := r(z) + g(z, c)$

- Feynman integrals are obtained from the restriction of canonical series to kinematics values.

- Designed for algorithmic evaluation of Feynman integrals
Proposal based on canonical series

**Theorem**

Let

\[ g_r(c, z) = \sum_{i=1}^{n} c_i z^{a_i} \iff A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}. \]

The Euler-Mellin integral

\[ I_{g_r}(\kappa) = \int_{\Omega} \frac{z^\alpha}{g_r(c, z)^{d/2}} \, d\eta_N \]

is a solution of the A-hypergeometric system \( H_A(\kappa) \) of degree \( \kappa = (-d/2, -\alpha) \).

Noncompact cycles \( \Omega \) considering the coamoeba of \( \mathcal{V}(g_r(c, z)) \) and choosing representatives \( \theta \in \Theta \) in \( \mathbb{R}^N \setminus \overline{\mathcal{A}}_{g_r} \).

**Proof.**

Show that \( I_{g_r}(\kappa) \) satisfies the GKZ system. Validity for non-compact cycles demonstrated by as we discussed before [BFP, ’13](#).
Relation to other proposals and methods

- Maximal cuts and \( n \)-loop bananas, fixed dimensions, compact cycles:
  Vanhove, '18 Klemm-Nega-Safari, '19 Bönish-Fischbach-Klemm-Nega-Safari, '20

- Full massive sunset with emphasis on triangulations of polytopes: Klausen, '19

- Feynman Integrals satisfying GKZ differential equations also in
  Nasrollahpoursamami, '16

Remark

- Above approaches emphasize triangulations of Convex Polytopes \( \rightarrow \) Gamma series representations

- Canonical series through Gröbner bases for some \( w \) and triangulations of polytopes are intimately connected
  Sturmfels, Gröbner bases and convex polytopes, '95
More recent developments

- Cohen-Macaulay property \textit{Tellander-Helmer'21}
  \[
  \text{in}_{(w,-w)}(H_A(\kappa)) = \langle A\theta - \kappa^T \rangle + \text{in}_w(I_A)
  \]

- Choice of $w$ can simplify sum representation

- Kinematic singularities of Feynman integrals through A-determinants
  \textit{Klausen '21, Mizera-Telen'21, Fevola-Mizera-Tellen, '23}
  \[
  g(z) = \mathcal{U} + \mathcal{F}
  \]

- Banana Feynman integrals from series representation (Frobenius method) \textit{Bönisch-Duhr-Fischbach-Klemm-Nega'21}

- Analytic continuation tool \textit{Ananthanarayan-Bera-Friot-Pathak'21}

\textbf{FeynGKZ}

\textit{Ananthanarayan-Banik, Souvik Bera-Datta, '22} A Mathematica package for solving Feynman integrals using GKZ hypergeometric systems
Example: Triangle

\[ s_1 = -p_1^2, \quad s_2 = -p_2^2, \quad s_3 = -(p_1 + p_2)^2 \]

- **Polynomial**

\[ g(z) = z_1 + z_2 + z_3 + s_3 z_1 z_2 + s_1 z_1 z_3 + s_2 z_2 z_3 \iff A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \]

- **co(A) = 2. Two variable hypergeometric function**
Example: Triangle

\[ I_g(\kappa) = \int_{\Omega} d\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_1 z_2 + c_5 z_1 z_3 + c_6 z_2 z_3)^\beta} \]

Macaulay 2 + Mathematica

- **Input:** \( w = (0, 0, 1, 0, 0, 0) \), \( A \)
- \( \text{fin}_w(H_A(\kappa)) = \langle \theta_2 \theta_5, \theta_3 \theta_4 \rangle + \langle A\theta - \kappa^T \rangle \)
- **Output:**

\[
\{\gamma_i\} = \{(-\alpha_1, C - \beta, B - \beta, 0, 0, \beta - A), (\alpha_2 - \beta, C - \beta, 0, \beta - B, 0, -\alpha_3), \\
\quad (\alpha_3 - \beta, 0, B - \beta, 0, \beta - C, -\alpha_2), (A - 2\beta, 0, 0, \beta - B, \beta - C, \alpha_1 - \beta)\}
\]

\[ A = \alpha_1 + \alpha_2 + \alpha_3, \quad B = \alpha_1 + \alpha_2, \quad \text{and} \quad C = \alpha_1 + \alpha_3. \]
Example: Triangle

Mathematica

\[ \phi_1 = c^{\gamma_1} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\alpha_1)_{m+n} (A - \beta)_{m+n}}{(-\beta + C + 1)_n (-\beta + B + 1)_m (1)_m (1)_n} x^m y^n, \]

\[ \phi_2 = c^{\gamma_2} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\beta - \alpha_2)_{m+n} (\alpha_3)_{m+n}}{(-\beta + C + 1)_n (1)_m (\beta - B + 1)_m (1)_n} x^m y^n, \]

\[ \phi_3 = c^{\gamma_3} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\beta - \alpha_3)_{m+n} (\alpha_2)_{m+n}}{(1)_n (-\beta + B + 1)_m (\beta - C + 1)_n} x^m y^n, \]

\[ \phi_4 = c^{\gamma_4} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(2\beta - A)_{m+n} (\beta - \alpha_1)_{m+n}}{(1)_n (1)_m (\beta - B + 1)_m (\beta - C + 1)_n} x^m y^n, \]

\[ x = \frac{c_3 c_4}{c_1 c_6} \ \text{and} \ y = \frac{c_2 c_5}{c_1 c_6}. \]
Example: Triangle

Integration constants from positions of zero's in roots

\[ K_r = \frac{1}{\Gamma(\beta)} \prod_{i \neq 0} \Gamma(-\gamma^i) \]

Restriction to physical values

\[ c_1 = c_2 = c_3 = 1 \text{ and } c_4 = s_3, c_5 = s_1, c_6 = s_2 \]

\[ I(\alpha, \beta) = \]

\[ K_1 s_2^{\beta - A} F_4(\alpha_1, A - \beta; -\beta + \alpha_{13} + 1, -\beta + \alpha_{12} + 1; s_3/s_2, s_1/s_2) \]

\[ + K_2 s_2^{-\alpha_3} s_3^{\beta - B} F_4(\beta - \alpha_2, \alpha_3; C - \beta + 1, -B + \beta + 1; s_3/s_2, s_1/s_2) \]

\[ + K_3 s_2^{-\alpha_2} s_1^{\beta - C} F_4(\beta - \alpha_3, \alpha_2; B - \beta + 1, -C + \beta + 1; s_3/s_2, s_1/s_2) \]

\[ + K_4 s_1^{\beta - C} s_2^{\alpha_1 - \beta} s_3^{\beta - B} F_4(2\beta - A, \beta - \alpha_1; -B + \beta + 1, -C + \beta + 1; s_3/s_2, s_1/s_2) \]

- [Mellin-Barnes](Boos, Davydychev, '91) and negative dimension approach
- Anastasiou-Glover-Oleari, '00
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Holonomic properties of scattering amplitudes

- GKZ ideal $H_A(\kappa)$ is a holonomic $D$-ideal

$$H_A(\kappa) = I_A \cup \langle \text{A} \theta - \kappa \rangle$$

- Key property of Feynman integrals: **holonomicity**
  Kashiwara-Kawai, '77, Bitoun-Bogner-Klausen-Panzer '17

- Basic equation of **generalized unitarity**
  Bern-Dixon-Dunbar-Kosower, '94

$$A_n = \sum_i c_i (A^{\text{trees}}) I^\text{basis}_i + \text{rational}$$

- Coefficients $c_i$ can be computed from tree-level amplitudes (rational functions of spinor variables)

- More generally $c_i$ are algebraic functions

- Algebraic functions are also holonomic hence amplitudes! Elementary consequence of holonomic D-modules (See Chapter 20 of Coutinho’s book)

- Let us start with trees ...
Biadjoint scalars

- Biadjoint scalar amplitudes

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \varphi_{a\alpha} \partial^\mu \varphi_{a\alpha} - \frac{\lambda}{3!} f^{abc} \tilde{f}^{\alpha\beta\gamma} \varphi_{a\alpha} \varphi_{b\beta} \varphi_{c\gamma} \]

- Admit a recursive formula [Mafra, ’16]

\[ \phi_{w_1,w_2} = \frac{1}{s_{w_1}} \sum_{x,y=w_1} \sum_{a,b=w_2} \left[ \phi_{x,a} \phi_{y,b} - (x \leftrightarrow y) \right], \quad \phi_{w_1,w_2} \equiv 0, \text{ if } w_1 \setminus w_2 \neq e \]

with the start of the recursion defined as \( \phi_{i,j} = \delta_{ij} \). The \( n \)-point amplitude is

\[ m_n(w_1n|w_2n) = (-1)^{(n-3)} s_{w_1} \varphi_{w_1,w_2} \]
Weyl algebra

- Take \( w_1 n = 12 \ldots n \)
- Ring of Mandelstam invariants
  \[
  s_{ijk\ldots} := (p_i + p_j + p_k + \ldots)^2
  \]
- Examples
  \[
  m_4 = -\frac{1}{s_{12}} - \frac{1}{s_{23}}, \quad m_5 = \frac{1}{s_{12}s_{13}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{123}s_{23}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{234}s_{34}}
  \]
- Kinematic invariants \( S_n = \{ s_w \mid w \in B_n \} \), where \(|S_n| = \frac{1}{2} n(n - 3) = N\), so its associated ring is \( \mathbb{C}[S_n] \).
- We then define the corresponding set of operators by
  \[
  \partial_{S_n} := \{ \partial_{s_w} \mid w \in B_n \}
  \]
so the associated Weyl algebra is
  \[
  D_N = \mathbb{C}[S_n] \langle \partial_{S_n} \rangle
  \]
Differential equations for biajoint scalars

- Annihiliators of amplitudes \( m_n = f/g \)

\[
P_i = gf \partial_i + (f \partial_i g - g \partial_i f), \quad i = 1, \ldots, N,
\]

\[
H_n = \left[ \sum_{w \in B_n} \theta_{sw} + (n - 3) \right]
\]

\[
\langle P_1, \ldots, P_N, H_n \rangle \subset D_N.
\]

- Canonical holonomic representation \(^{\text{Zeilberger, '90}}\)

\[
I_n = \langle A_n \theta_n - \kappa_n \rangle \Rightarrow s_w \theta_{sw} m_n(S) = \kappa_w, \quad \forall w \in B_n
\]

\[
A_n = m_n \text{diag}(s_{12}, s_{23}, \ldots), \quad \theta_n = (\theta_{s_{12}}, \theta_{s_{23}}, \ldots)^T, \quad \kappa_n = (\iota_n(12), \iota_n(23), \ldots)^T,
\]

for \( 2 \leq |w| \leq n - 2 \) and zero otherwise

- Boundary condition \( m_n \big|_{S_n \to \infty} = 0. \)
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Summary

- GKZ systems are the most general tools to study hypergeometric functions
- Generalized Feynman integrals are A-hypergeometric,
- SST canonical series provide the tool to evaluate Feynman integrals
- Output of canonical series method equivalent to Mellin-Barnes
- Holonomicity is key to extend the approach to scattering amplitudes

Outlook

- Elephant in the room: efficiency and scaling (restriction of D-modules) (Henn-Pratt-Sattelberger-Zoia, '23)
- Relation between GKZ and PDEs from Griffiths-Dwork (LDLC-Vanhove, '24)
- Generalized unitarity gives us hint to extend this approach to general scattering amplitudes

Thanks!
Canonical series algorithm (Saito-Sturmfels-Takayama)

Input: Matrix $A$, weight vector $w$, and complex parameters $\kappa$. Output: Roots of the fake indicial ideal $\text{fin}_w(H_A(\kappa))$.

1. Compute the toric ideal associated with $A$

   $$I_A = \langle \partial^u - \partial^v : Au = Av, \quad u, v \in \mathbb{N}^n \rangle.$$ 

2. Let $w \in \mathbb{R}^n$ be a generic weight vector. Compute the initial ideal $\text{in}_w(I_A)$ with respect to $w$ and obtain its standard pairs $S(\text{in}_w(I_A))$.

3. Use the standard pairs to construct the indicial ideal

   $$\text{ind}_w(I_A) = \bigcap_{(\partial^a, F) \in S(\text{in}_w(I_A))} \langle (\theta_j - a_j), j \notin F \rangle \subset \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_n],$$ 

4. Write the ideal $\langle A\theta - \kappa^T \rangle \subset \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_n]$.

5. The fake indicial ideal with respect to $w$ is given by

   $$\text{fin}_w(H_A(\kappa)) := \text{ind}_w(I_A) + \langle A\theta - \kappa^T \rangle.$$

6. Compute the roots of $\text{fin}_w(H_A(\kappa))$. These are called fake exponents and we denote them by $\gamma$. 
Standard Pairs

Let $R = \mathbb{K}[\partial_1, \ldots, \partial_n]$ and let $I$ be a monomial ideal in $R$. Furthermore, let $\partial^\alpha$ be a monomial and $F \subseteq \{1, \ldots, n\}$, where $\alpha \in \mathbb{N}^n$. A standard pair of a monomial ideal $I$ is a pair $(\partial^\alpha, F)$ satisfying three conditions:

1. $\alpha_i = 0$ for all $i \in F$,
2. for all choices of integers $\beta_j \geq 0$, the monomial $\partial^\alpha \prod_{j \in F} \partial_j^{\beta_j} \notin I$,
3. for all $l \notin F$, there exist $\beta_j \geq 0$ such that $\partial^\alpha \partial_l^{\beta_l} \prod_{j \in F} \partial_j^{\beta_j} \in I$.

Let us denote by $S(I)$ the set of all standard pairs of $I$. The decomposition of $I$ into irreducible monomial ideals can be obtained from the identity.

$$ I = \bigcap_{(\partial^\alpha, F) \in S(I)} \langle \partial^\alpha_{i}^{\alpha_i + 1} : i \in F \rangle. $$
Example: Cantaloupe or dealing with deformation

\[ g(z_1, \ldots, z_{L+1}) = \sum_{i=1}^{L+1} \prod_{j \neq i} z_j + s \prod_{i=1}^{L+1} z_i, \]

where \( s = -p^2 \). The integral to be computed reads

\[ I(\alpha) = \int_{\mathbb{R}_{+}^{L+1}} \frac{z_1^{\alpha_1} \cdots z_{L+1}^{\alpha_{L+1}}}{g(z)^\beta}. \]
In order to perform such deformation systematically, let us introduce some notation. Let $1_i$ denote a sequence of $1$’s of length $i$ and similarly for $0_j$. We have the relation $i + j = L + 1$. Furthermore, let

$$v := (1_{L-1}, 0_2).$$

At each loop, we set a deformation monomial

$$r(z) = c_1 z^v,$$

hence we have

$$g_r(c, z) = c_1 z^v + \sum_{i=1}^{L+1} c_{L+3-i} \prod_{j \neq i}^L z_j + c_{L+3} \prod_{i=1}^{L+1} z_i,$$

where $c_{L+3} = s$. Let us give an example. For $L = 3$, $v = (1, 1, 0, 0)$ and $r(z) = c_1 z_1 z_2$, then we have the deformed toric polynomial

$$g_r(c, z) = c_1 z_1 z_2 + c_2 z_1 z_2 z_3 + c_3 z_1 z_2 z_4 + c_4 z_1 z_3 z_4 + c_5 z_2 z_3 z_4 + c_6 z_1 z_2 z_3 z_4.$$
\[
A = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
1_{L+1} & 0 & 1_1 \\
1_L & 0 & 1_2 \\
\vdots & & & & \\
1_3 & 0 & 1_{L-1} \\
0 & 1 & 0 & 1_L \\
0 & 0 & 1 & 1_L
\end{pmatrix}
\]

\[
I_{gr}(\kappa) = \int_{\Omega} d\eta_{L+1} \frac{z_1^{\alpha_1} \cdots z_{L+1}^{\alpha_{L+1}}}{g_r(c, z)^\beta},
\]

where \(\kappa = (-\beta, -\alpha_1, \ldots, -\alpha_{L+1})\). Computing the kernel of the above matrix leads to

\[
\mathcal{L} = \mathbb{Z}(1, -1, -1, 0_{L-1}, 1),
\]

where by definition \(0_0 := \emptyset\). We choose \(w = (1, 0_{L+2})\), thus obtaining

\[
\text{fin}_w(H_A(\kappa)) = \langle \theta_1 \theta_{L+3} \rangle + \langle A\theta - \kappa^T \rangle.
\]
The roots can be written as
\[ \{\gamma_i\} = \left\{0, \alpha_{L+1} - \beta, \ldots, \alpha_1 - \beta, L\beta - \sum_{i=1}^{L+1} \alpha_i \right\}, \]
\[ \left(\sum_{i=1}^{L} \alpha_i - L\beta, (L - 1)\beta - \sum_{i=1}^{L} \alpha_i, (L - 1)\beta - \sum_{i=1}^{L+1} \alpha_i, -\beta + \alpha_{L-1}, \ldots,\right. \]
\[ \left. -\beta + \alpha_1, 0 \right\}, \]
which lead to the canonical series
\[ \phi_1 = c^{\gamma_1} \, _2F_1 \left(\beta - \alpha_{L+1}, \beta - \alpha_L, L\beta - \sum_{i=1}^{L+1} \alpha_i + 1; x \right), \]
\[ \phi_2 = c^{\gamma_2} \, _2F_1 \left((-L - 1)\beta + \sum_{i=1}^{L} \alpha_i, -(L - 1)\beta + \sum_{i=1}^{L+1} \alpha_i; \sum_{i=1}^{L} \alpha_i - L\beta + 1; x \right), \]
where \( x = \frac{c_1 c_{L+3}}{c_2 c_3} \). The relevant integration constant reads
\[ K_1 = \frac{\Gamma(-L\beta + \sum_{i=1}^{L+1} \alpha_i)}{\Gamma(\beta)} \prod_{i=1}^{L+1} \Gamma(\beta - \alpha_i), \]
Let \( g_r(c, z) \) be the deformed polynomial in \( N \) variables obtained from \( g(c, z) = U(c) + F(c) \), where \( F(c) \) and \( U(c) \) are obtained by considering the coefficients appearing in the Symanzik polynomials as variables. \( g_r(c, z) \) is obtained by introducing a deformation \( r(c, z) \) demanding that its matrix satisfies \( \text{co}(A) > 0 \). Let \( A = (a_1 \ a_2 \cdots \ a_n) \) be the configuration matrix associated with \( g_r(c, z) \) and consider the polynomial with indeterminate generic coefficients

\[
g_r(c, z) = \sum_{i=1}^{n} c_i z^{a_i}, \quad c_i \in \mathbb{C}_*
\]

Let \( A \) be its associated \((N + 1) \times n\) matrix

\[
A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}
\]
Theorem

The Euler-Mellin integral

\[ I_{gr}(\kappa) = \int_{\Omega} \frac{z^\alpha}{g_r(c, z)^{d/2}} \, d\eta_N \]

is a solution of the A-hypergeometric system \( H_A(\kappa) \) of degree \( \kappa = (-d/2, -\alpha) \).
Noncompact cycles \( \Omega \) can be obtained by taking the coamoeba of \( g_r(c, z) \) and choosing representatives \( \theta \) of connected components \( \Theta \in \mathbb{R}^N \setminus \overline{A'} \). Proof. Show that the above integral satisfies GKZ system. Validity for non-compact cycles demonstrated by Berkesh-Forsgård-Passare.

Remark on cycles

Noncompact cycles for A-hypergeometric functions from coamoebas of \( A'_g \) simply gives \( \Omega = \mathbb{R}^N_+ \) thanks to positivity of coefficients in \( g(z) \).