Exact expressions for integrated correlators in $\mathcal{N}=4$ supersymmetric Yang-Mills

(with arbitrary classical gauge group)

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Online seminar on motives and period integrals in
Quantum Field Theory and String Theory
Feb 23 2022

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Based on recent papers arXiv:2102.08305, Phys. Rev. Lett. 126 (2021) no.16, 161601

arXiv:2102.09537, JHEP 05 (2021) 089

arXiv:2202.05784

Earlier work with large N Chester, MBG, Pufu, Wang, Wen arXiv:1912.13365 and arXiv:2008.02713

See also: Binder, Chester, Pufu, Wang, arXiv:1902.06263, Chester, Pufu, arXiv:2003.08412, Chester, arXiv:1908.05207

Alday, Chester, Hanson, arXiv:2110.13106 Collier, Perlmutter arXiv:2201.05093

In this talk **SUPERSYMMETRIC LOCALIZATION** will be used in order to determine the precise form of certain integrated correlation functions in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory.

ALTHOUGH THE INTEGRATION AVERAGES OVER THE SPATIAL DEPENDENCE IT PROVIDES A GREAT DEAL OF INFORMATION:

- Makes S-duality (Montonen-Olive duality) manifest. MODULAR INVARIANCE.
- Predicts SYM perturbation theory results to all orders – BOTH PLANAR AND NON-PLANAR – FOR ALL VALUES OF N.
- Includes detailed form of infinite set of INSTANTON and ANTI-INSTANTON contributions.
- The large-N expansion reproduces known facts about the low energy expansion of holographically dual IIB superstring.

The Main Results

For SU(N) gauge group

$$\boxed{\mathcal{G}_N(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B_N(t)dt}$$

where $\mathcal{G}_N(\tau,\bar{\tau}) \sim \int \prod_{i=1}^{4} dx_i \mu(\{x_r\}) \langle O_2(x_1)O_2(x_2)O_2(x_3)O_2(x_4)\rangle$ is an **integral of the correlator** of four superconformal primary operators of $\mathcal{N}=4, SU(N)$ supersymmetric Yang-Mills. (coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$)

$B_N(t) = \frac{\mathbb{Q}_N(t)}{(1+t)^{2N+1}}$ where $\mathbb{Q}_N(t)$ is a rational polynomial of order $(2N-1)$.

Satisfies Laplace difference equation – relates SU(N) to SU(N+1) and SU(N-1):

$$\boxed{(\Delta_\tau - 2)\,\mathcal{G}_N(\tau,\bar{\tau}) = N^2\left[\mathcal{G}_{N+1}(\tau,\bar{\tau}) - 2\mathcal{G}_N(\tau,\bar{\tau}) + \mathcal{G}_{N-1}(\tau,\bar{\tau})\right] - N\left[\mathcal{G}_{N+1}(\tau,\bar{\tau}) - \mathcal{G}_{N-1}(\tau,\bar{\tau})\right]}$$

where $\Delta_\tau = \tau_2^2\left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right)$ is the hyperbolic laplacian.

This is an exact formula for all values of (τ, N)

In this talk I will motivate these expressions and describe some of their remarkable properties.

and then describe the generalization to arbitrary classical gauge groups, **SO(2N), SO(2N+1), USp(2N).**

Supersymmetric Localization

- $\mathcal{N}=2^*$ supersymmetric YM $\to \mathcal{N}=4$ in the limit in which the mass of hypermultiplet in adjoint rep. vanishes

$$\mathcal{N}=2^* \underset{m\to 0}{\to} \mathcal{N}=4$$

- The $\mathcal{N}=2^*$ partition function on S^4 is determined by supersymmetric localization. [Pestun arXiv:0712.2824]

- Localized partition function of $\mathcal{N}=2^*$ with $SU(N)$ gauge group is a $(N-1)$-dimensional integral over the Lie algebra $\mathfrak{su}(N)$. $SU(N)$ hermitian matrix model (integrate over VEV's of coulomb branch vector multiplet).

$$Z_N(m,\tau,\bar{\tau}) = \int d^N a\, \delta(\sum_i a_i) \prod_{i<j}(a_i - a_j)^2\, e^{-\frac{8\pi^2}{g_{YM}^2}\sum_j a_j^2}\, \mathcal{Z}_{pert}(m,a_i)\, |\mathcal{Z}_{inst}(m,a_i,\tau)|^2$$

mass parameter

Vandermonde determinant

Perturbative factor

Nekrasov instanton partition function

$\mathcal{Z}_{pert}(m,a_i)$ is the one-loop determinant factor and is expressed in terms of a standard function (the **Barnes G-function**)

$\mathcal{Z}_{inst}(m,a_i)$ describes Coulomb branch instantons at the south pole and anti-instantons at the north pole of S^4 . (Express as a sum of Young diagrams)

[Nekrasov]

- The partition function of $\mathcal{N}=4$ SYM $\quad Z_N(0,\tau,\bar{\tau}). = 1$

- But the $m=0$ limit of derivatives of $Z_N(m,\tau,\bar{\tau})$ with respect to m may be nontrivial as we will see.

The $\mathcal{N}=2^*$ Partition Function

$$Z(m,\tau,\bar{\tau}) = \int d^{N-1}a_i \prod_{i<j} \frac{a_{ij}^2 H^2(a_{ij})}{H(a_{ij}-m)H(a_{ij}+m)} e^{-\frac{8\pi^2}{g_{\text{YM}}^2}\sum_i a_i^2} |Z_{\text{inst}}(m,\tau,a_{ij})|^2 ,$$

Perturbative

$$H(z) = e^{-(1+\gamma)z^2} G(1+iz)\, G(1-iz)$$

Barnes G-function

$$\log G(1+z) = \frac{z}{2}\log 2\pi - \left(\frac{z+(1+\gamma)z^2}{2}\right) + \sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k+1} z^{k+1}$$

Instanton terms:

The Nekrasov instanton partition function: contribution from instantons at North Pole and anti-instantons at South Pole of S^4

$$Z_{inst}(m,\tau,a_{ij}) = \sum_{k=0}^{\infty} e^{2\pi i k\tau} Z_{inst}^{(k)}(m,a_{ij})$$

Fourier sum (sum over instanton number)

Using Nekrasov's result we have for k INSTANTONS:

$$Z_{inst}^{(k)}(m,a_{ij}) = \frac{1}{k!}\left(\frac{2m^2}{m^2+1}\right)^k \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi} \prod_{i=1}^{N} \frac{(\phi_I - a_I)^2 - m^2}{(\phi_I - a_i)^2 + 1} \prod_{I<J}^{k} \frac{\phi_{IJ}^2(\phi_{IJ}^2+4)(\phi_{IJ}^2-m^2)^2}{(\phi_{IJ}^2+1)((\phi_{IJ}-m)^2+1)((\phi_{IJ}+m)^2+1)}$$

where the integration contour circles the poles in a particular (and complicated) manner (sum of Young diagrams).

Integrated Four-Point Correlation Functions

- Superconformal primary of $\mathcal{N}=4$ stress tensor multiplet.

$$O_2(x,Y) = tr(\phi_{I_1}\,\phi_{I_2})\,Y^{I_1}\,Y^{I_2}$$

$I_1, I_2 = 1,\ldots,6$ $\quad Y\cdot Y = 0$ $\quad$ Encodes $SU(4)$ quantum numbers

- Four-point correlator of superconformal primaries,

$$\langle O_2(x_1)\,O_2(x_2)\,O_2(x_3)\,O_2(x_4)\rangle = \langle O_2(x_1)\,O_2(x_2)\,O_2(x_3)\,O_2(x_4)\rangle_{free} + \mathcal{I}_4(x_i,Y_i)\mathcal{T}_N(U,V,\tau,\bar\tau)$$

free correlator $\quad$ determined by symmetries $\quad$ [Eden, Petkou, Schubert, Sokatchev]

- Correlator is not supersymmetric but **integrated** correlator is **[Binder, Chester, Pufu, Wang]**

$$\mathcal{G}_N^i(\tau,\bar\tau) = \int dU\,dV\,\mu^i(U,V)\mathcal{T}_N(U,V,\tau,\bar\tau)$$

$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ $\quad$ cross ratios

where the measure $\mu^i(U,V)$ is designed to preserve supersymmetry,

Two examples of measures

$$\mathcal{G}_N^1(\tau,\bar\tau) = -\frac{8}{\pi}\int_0^\infty dr\int_0^\pi d\theta\frac{r\sin^2\theta}{U^2}\mathcal{T}_N(U,V,\tau,\bar\tau)$$

$U = 1 + r^2 - 2r\cos\theta\,, \quad V = r^2$

$$\mathcal{G}_N^2(\tau,\bar\tau) = -\frac{96}{\pi}\int_0^\infty dr\int_0^\pi d\theta\frac{r\sin^2\theta}{U^2}\,\bar D_{1111}(U,V)\,\left(\mathcal{T}_N(U,V,\tau,\bar\tau) + \mathcal{T}_{free}(U,V)\right)$$

box diagram

Relation to Localised $\mathcal{N}=2^*$ Partition Function

- Correlators are obtained by four derivatives acting on $Z_N(m,\tau,\bar\tau)$ the partition function of the $\mathcal{N}=2^*$ theory on S^4

$$\mathcal{G}_N^1(\tau,\bar\tau) = \tau_2^2 \partial_\tau \partial_{\bar\tau}\, \partial_m^2 \,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ Considered in this talk

$$\mathcal{G}_N^2(\tau,\bar\tau) = \partial_m^4 \,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ NOT Considered in this talk

- Equality with integrated correlators on R^4 shown in [Binder, Chester, Pufu, Wang, arXiv:1902.06263]

  Uses supersymmetric Ward identities and accounts for operator mixing on S^4.

- Analysis of $\mathcal{G}_N \equiv \mathcal{G}_N^1$ is complicated.
  - Consider the exact perturbation expansion for many values of N.
  - Consider the exact 1-instanton contribution for many values of N.
  - Generalise to the k-instanton contribution.

    Only Young diagrams in the Nekrasov partition function with a single rectangular $k = p \times q$ block contribute (up to "partial transpositions").[Chester, MBG, Pufu, Wang, Wen]

- Leads to a remarkably simple conjectured expression for $\mathcal{G}_N(\tau,\bar\tau)$

2 Dim. Lattice Representation

$$\boxed{\mathcal{G}_N(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B_N(t)dt}$$

where $B_N(t) = \dfrac{\mathbb{Q}_N(t)}{(1+t)^{2N+1}}$ and $\mathbb{Q}_N(t)$ is a rational polynomial of order $(2N-1)$.

- $SL(2,\mathbb{Z})$ invariance is manifest: $\tau \to \dfrac{a\tau+b}{c\tau+d}$ $\quad a,b,c,d\in\mathbb{Z}$, $ad-bc=1$

  [Montonen-Olive]

Relates theories at different values of coupling constant – holographic image of S-duality in type IIB superstring.

- It is important that $B_N(t) = \dfrac{1}{t} B_N(1/t)$, as well as $\displaystyle\int_0^\infty B_N(t)dt = \frac{N(N-1)}{4}$ and $\displaystyle\int_0^\infty B_N(t)\frac{1}{\sqrt{t}}dt = 0$.

e.g. $SU(2)$: $B_2(t) = \dfrac{9t^3-30t^2+9t}{(1+t)^5}$ $\qquad SU(3)$: $B_3(t) = \dfrac{18t^5-99t^4+126t^3-99t^2+18t}{(1+t)^7}$

- General N:

$$\mathbb{Q}_N(t) = -\frac{1}{4}N(N-1)(1-t)^{N-1}(1+t)^{N+1}\left[(3+(8N+3t-6)t)\,P_N^{(1,-2)}(z) + \frac{3t^2-8Nt-3}{t+1}P_N^{(1,-1)}(z)\right]$$

where $z = \dfrac{1+t^2}{1-t^2}$ and $P_N^{(\alpha,\beta)}(z)$ is a Jacobi polynomial.

Perturbation Expansion

't Hooft expansion $a = \frac{g_{YM}^2 N}{4\pi^2} = \frac{N}{\pi}\tau_2^{-1}$

First non-planar contribution

$$\mathcal{G}_{N,0}(\tau_2) = (N^2-1)\left[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32}\right.$$
$$+\frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024}$$
$$\left.+\frac{32207175\,\zeta(15)\left(1+\frac{55}{13}N^{-2}+\frac{332}{143}N^{-4}\right)a^7}{2048} + \mathcal{O}(a^8)\right],$$

- Coefficients are RATIONAL MULTIPLES OF ODD ZETA VALUES.

- Recall the UNINTEGRATED CORRELATOR has very complicated dependence on cross ratios involving polylogs,

e.g. $L = 1,\ 2$

$$f^{(L)}(z,\bar{z}) = \sum_{r=0}^{L} \frac{(-1)^r(2L-r)!}{r!(L-r)!L!}\log^r(z\,\bar{z})\left(\mathrm{Li}_{2L-r}(z) - \mathrm{Li}_{2L-r}(\bar{z})\right)$$

$z\bar{z} = U \qquad (1-z)(1-\bar{z}) = V$

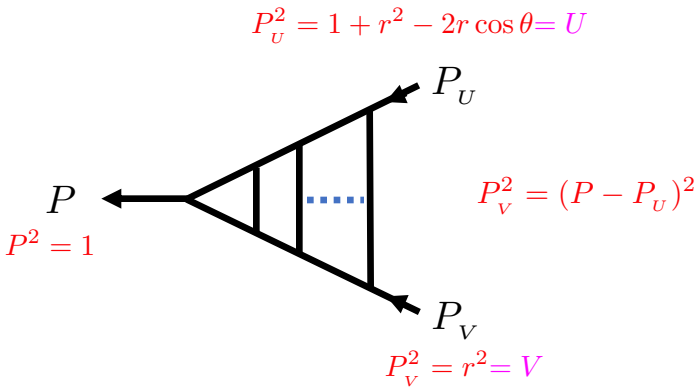
- The **INTEGRATED CORRELATOR is much simpler.** The coefficients of terms up to $O(a^3)$ can be compared with calculations from Feynman diagrams. [Belokurov and Usyukina, 1983] [Usyukina, 1991]
  Higher order coefficients are predicted by the above expansion.

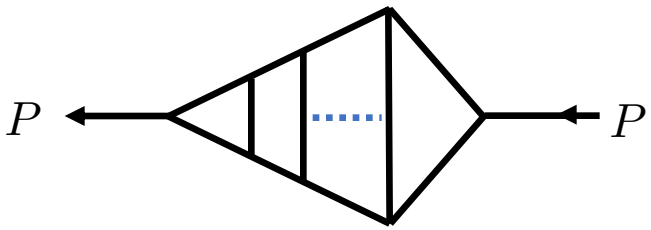
- **NON-PLANAR CORRECTIONS** BEGIN AT FOUR LOOPS – as is known from Feynman perturbation theory.
  Interesting pattern of non-planarity determined to arbitrary order.
  [Eden, Heslop, Korchemsky, Sokatchev] [Boels, Kniehl, Tarasov, Yang] [Fleury and Pereira]

Integrated Perturbative (Ladder) Diagrams

Cross-ratios $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = 1 + r^2 - 2r\cos\theta$ $\quad V = \frac{x_{12}^2 x_{14}^2}{x_{23}^2 x_{24}^2} = r^2$

Integrated correlation function $\int_0^\infty dr \int_0^\pi d\theta\, r^3 \sin^2\theta\, \frac{\mathcal{T}'_N(U,V)}{U\,V}$ $\quad \mathcal{T}'_N(U,V) = \mathcal{T}_N(U,V)\,\frac{V}{U}$





ℓ - loop ladder diagram
unintegrated correlator

$(\ell + 1)$ - loop ladder diagram
integrated correlator

$$= \text{rational} \times \left(\frac{g_{YM}^2 N}{4\pi^2}\right)^\ell \zeta(2\ell + 1)$$

Calculation of ladder diagrams at arbitrary order. [Belokurov and Usyukina, 1983, Usyukina, 1991]

Differential Recurrence Relation

- Using the differential recurrence relation of Jacobi functions we find that

$$t\,\frac{d^2}{dt^2}(t\,B_N(t)) = N(N-1)B_{N+1}(t) - 2(N^2-1)B_N(t) + N(N+1)B_{N-1}(t)$$

recall $\mathcal{G}_N(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B_N(t)dt$

- From which one can show that the integrated correlator satisfies a

Laplace difference equation:

$$\boxed{(\Delta_\tau - 2)\,\mathcal{G}_N(\tau,\bar{\tau}) = N^2\left[\mathcal{G}_{N+1}(\tau,\bar{\tau}) - 2\mathcal{G}_N(\tau,\bar{\tau}) + \mathcal{G}_{N-1}(\tau,\bar{\tau})\right] - N\left[\mathcal{G}_{N+1}(\tau,\bar{\tau}) - \mathcal{G}_{N-1}(\tau,\bar{\tau})\right]}$$

where $\Delta_\tau = \tau_2^2\left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right)$ is the hyperbolic laplacian.

- Since $\mathcal{G}_1 = 0$ this equation determines $\mathcal{G}_N(\tau,\bar{\tau})$ for all $N > 2$ In terms of $\mathcal{G}_2(\tau,\bar{\tau})$.

- Solutions can be expressed in terms of NON-HOLOMORPHIC EISENSTEIN SERIES

$SL(2,\mathbb{Z})$ Modular Forms and Non-Holomorphic Eisenstein Series

A modular form transforms as

$$f^{(w,\hat{w})}(\tau,\bar{\tau}) \underset{SL(2,\mathbb{Z})}{\rightarrow} (c\tau+d)^{w}\,(c\bar{\tau}+d)^{\hat{w}}\, f^{(w,\hat{w})}(\tau,\bar{\tau})$$

Holomorphic anti-holomorphic

weights $(w,\hat{w})$

- When $\hat{w}=-w$ the pre-factor is a phase. $U(1)$ transformation charge $q_U=2w$.
- Operators in the stress tensor multiplet have specific $U(1)$ charges: e.g. $q_{O_2}=0$,

Real analytic (non-holomorphic) Eisenstein series

Modular function $\hat{w}=w=0$

$$E(s,\tau,\bar{\tau}) = \frac{1}{\pi^s}\sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{|m+n\tau|^{2s}} = \sum_{(m,n)\neq(0,0)}\frac{1}{\Gamma(s)}\int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}}t^{s-1}dt \;=\sum_{k\in\mathbb{Z}}\mathcal{F}_k(s;\tau_2)\,e^{2\pi i k\tau_1} \qquad s\in\mathbb{C}$$

Fourier modes

- Zero mode (perturbative)

$$\mathcal{F}_0(s;\tau_2) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\pi^s\Gamma(s)}\tau_2^{1-s}$$

$(4\pi/g_{YM}^2)^s$ $\qquad$ $(g_{YM}^2/4\pi)^{s-1}$

two perturbative terms

- Non-zero modes (instantons)

divisor sum $\qquad$ Bessel

$$\mathcal{F}_k(s;\tau_2) = \frac{4}{\Gamma(s)}|k|^{s-\frac{1}{2}}\,\sigma_{1-2s}(|k|)\sqrt{\tau_2}\,K_{s-\frac{1}{2}}(2\pi|k|\tau_2)\,, \quad k\neq 0 \qquad \sigma_r(k)=\sum_{d|k}d^r$$

$$\underset{\tau_2\to\infty}{\sim}(\dots)\,e^{-2\pi|k|\tau_2}$$

characteristic of instanton or anti-instanton

- **Laplace eigenvalue equation** $(\Delta_\tau - s(s-1))\,E(s;\tau,\bar{\tau})=0$

Expression for Integrated Correlator

Formal Infinite sum of Eisenstein series (with integer-index)

$$\boxed{\mathcal{G}_N(\tau,\bar\tau) = \frac{N(N-1)}{8} + \frac{1}{2}\sum_{s=2}^{\infty} c_N(s)\, E(s,\tau,\bar\tau)}$$

where the coefficients are given by $B_N(t) = \sum_{s=2}^{\infty} c_N(s)\,\frac{t^{s-1}}{\Gamma(s)}$ with $B_N(t) = \frac{1}{t}\,B_N(1/t)$

See Collier and Perlmutter for a general discussion

e.g. for $SU(2)$ $\quad c_2(s) = \frac{(-1)^s}{2}(s-1)(1-2s)^2\,\Gamma(s+1)$

- Perturbative terms.: Infinite sum of $\tau_2^s = (4\pi/g_{YM}^2)^s$ terms = infinite sum of $\tau_2^{1-s} = (g_{YM}^2/4\pi)^{s-1}$ terms!

  after Borel resummation

- Instanton Contributions

  e.g. $k=1$ in $SU(2)$

$$\mathcal{G}_{2,k=1}(\tau,\bar\tau) = e^{2\pi i\tau}\left[12y^2 - 3\sqrt{\pi}e^{4y}y^{3/2}(1+8y)\,\mathrm{erfc}\,(2\sqrt{y})\right]$$

$$\underset{g_{YM}^2\to 0}{\sim} e^{2\pi i\tau}\left[-\frac{3}{8} + \frac{9}{32y} - \frac{135}{512y^2} + \frac{315}{1024y^3} + \cdots\right] \qquad y = \pi\tau_2 = \frac{4\pi^2}{g_{YM}^2}$$

General $k = \hat m n$ and N

$$\mathcal{G}_{N,k}(\tau,\bar\tau) = \frac{1}{2}\sum_{\substack{\hat m\neq 0,\, n\neq 0\\ \hat m n = k}} e^{2\pi(-|k|\tau_2 + ik\tau_1)}\int_0^\infty \exp\left[-\left(\frac{|\hat m|}{\sqrt{t}} - |n|\sqrt{t}\right)^2\pi\tau_2\right]\sqrt{\frac{\tau_2}{t}}\,B_N(t)\,dt\,.$$

Large-N Expansion

't Hooft Expansion

Suppresses instantons so duality is not manifest

$$\mathcal{G}_N(\tau,\bar{\tau}) \sim \sum_{g=0}^{\infty} N^{2-2g}\,\mathcal{G}^{(g)}(\lambda)$$

$\lambda = g_{YM}^2\, N = 4\pi\tau_2^{-1}\, N$

't Hooft coupling

1. Small-λ expansion

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n$$

Radius of convergence $|\lambda| \leq \pi^2$

Proportional to N^2

Planar diagrams

Borel sum $= \lambda \int_0^{\infty} dw\, w^3 \frac{{}_1F_2\left(\frac{5}{2};2,4;-\frac{w^2\lambda}{\pi^2}\right)}{4\pi^2\,\sinh^2(w)}\,.$

2. Large-λ expansion

$$\mathcal{G}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\lambda^{n+1/2}}$$

Not Borel summable

- Asymptotic series that is **not Borel summable**. Requjres non-perturbative completion (resurgence)

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \frac{489\mathrm{Li}_3(e^{-2\sqrt{\lambda}})}{16\lambda^{3/2}} + \cdots\right]$$

- The behaviour $e^{-2\sqrt{\lambda}}$ is characteristic of a **world-sheet instanton** in string theory since $e^{-2\sqrt{\lambda}} = e^{-2L^2/\alpha'}$.

- Similar analysis for terms with higher powers of $1/N^2$

Large-N Expansion

Fixed - g^2_{YM} Expansion

instantons not suppressed – S-duality is manifest.

- The $1/N$ expansion is holographically related to the low energy expansion of the dual IIB superstring amplitude in $AdS_5 \times S^5$.

- Substitute the large-N expansion of $B_N(t)$ (determined by the differential recurrence relation) :

Supergravity

$$\mathcal{G}_N(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac12}}{2^4}\overset{R^4}{E(\tfrac32;\tau,\bar\tau)} + \frac{45}{2^8 N^{\frac12}}\overset{d^4R^4}{E(\tfrac52;\tau,\bar\tau)}$$
$$+ \frac{3}{N^{\frac32}}\left[\frac{1575}{2^{15}}E(\tfrac72;\tau,\bar\tau) - \frac{13}{2^{13}}E(\tfrac32;\tau,\bar\tau)\right] + \frac{225}{N^{\frac52}}\left[\frac{441}{2^{18}}E(\tfrac92;\tau,\bar\tau) - \frac{5}{2^{16}}E(\tfrac52;\tau,\bar\tau)\right]$$
$$+ \frac{63}{N^{\frac72}}\left[\frac{3898125}{2^{27}}E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}}E(\tfrac72;\tau,\bar\tau) + \frac{73}{2^{22}}E(\tfrac32;\tau,\bar\tau)\right] + O(N^{-\frac92}),$$

Extends the earlier analysis in [Chester, MBG, Pufu, Wang, Wen]

- Series of **½-integer** index Eisenstein series.

- Close connection to well-established BPS terms in low energy expansion of IIB superstring in flat space.

- Note the absence of terms with integer powers of $1/N$, such as the term of order d^6R^4.
  Such terms arise in the $1/N$ expansion of $\mathcal{G}^2_N(\tau,\bar\tau) = \partial^4_m \log Z_N(m,\tau,\bar\tau)|_{m=0}$.

Integrated Correlators for SO(N), USp(N)

[Dorigoni, MBG, Wen, arXiv:2202.05784]]
(See Alday, Chester and Hansen)

Goddard-Nuyts-Olive duality of magnetic monopoles and electric charges (c.f. Langlands)

(Correlators are not sensitive to global factors)

- Simply-laced Self-duality $SU(N)\,,\quad SO(2N)$ $\quad S:\ \tau \to -\frac{1}{\tau}\qquad T:\ \tau \to \tau+1$

  generate $SL(2,\mathbb{Z})\ :\quad \tau \to \frac{a\tau+b}{c\tau+d}\qquad a,b,c,d\in\mathbb{Z}\qquad ad-bc=1$

- Non simply-laced $SO(2N+1)\,,\quad USp(2N)$ $\quad \hat{S}:\ \tau \to -\frac{1}{2\tau}\qquad T:\ \tau \to \tau+1$

  $\left(\frac{\text{long roots}}{\text{short roots}}\right)^2 = 2$

  $\hat{S}\,T\,\hat{S}$ and T generate $\Gamma_0(2):\quad c = 0 \bmod 2$

  maps $SO(2N+1) \to SO(2N+1)$ and $USp(2N) \to USp(2N)$

  $\hat{S}$ maps $USp(2N) \leftrightarrow SO(2N+1)$

Previously $\mathcal{G}_N(\tau,\bar{\tau})$

Integrated correlator $\quad \frac{1}{4}\mathbb{C}_{G_N}(\tau,\bar{\tau}) = \Delta_\tau\,\partial_m^2 \log Z_{G_N}(m,\tau,\bar{\tau})|_{m\to 0}$

Results: Beautiful extension of the SU(N) case

- GNO duality explicit.
- Large-N limit gives results consistent with expected string theory results.
- Set of Laplace difference equations highly constrain results fo all N.

Correlators with General Classical Gauge Group G_N

With :

$$\boxed{\mathbb{C}_{G_N}(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2}\int_0^\infty dt\left(B^1_{G_N}(t)e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} + B^2_{G_N}(t)e^{-t\pi\frac{|m+2n\tau|^2}{2\tau_2}}\right)}$$

- $SU(N), \quad SO(2N)$ Invariance under $SL(2,\mathbb{Z})$ generated by $S:\tau\to-\frac{1}{\tau}, \quad T:\tau\to\tau+1$

$$B^2_{SU(N)}(s) = B^2_{SO(2N)}(s) = 0$$

- $SO(2N+1), \quad USp(2N),$ Invariance under $\Gamma_0(2)$ generated by $\hat S\, T\, \hat S$ and T, where

$$\hat S:\tau\to-\frac{1}{2\tau}, \quad T:\tau\to\tau+1 \qquad \text{not} \subset SL(2,\mathbb{Z})$$

$$B^1_{SO(2N+1)}(t) = B^2_{USp(2N)}(t)\,, \qquad B^1_{USp(2N)}(t) = B^2_{SO(2N+1)}(t)$$

- $\hat S$ Interchanges $B^1_{G_N}$ with $B^2_{G_N}$ GNO duality $\longrightarrow$ $\mathbb{C}_{SO(2N+1)} \to \mathbb{C}_{USP(2N)}$

- **Formal expansion** $\boxed{\mathbb{C}_{G_N}(\tau,\bar\tau) = -b_{G_N}(0) + \sum_{s=2}^{\infty}\left[b^1_{G_N}(s)\,E(s;\tau,\bar\tau) + b^2_{G_N}(s)\,E(s;2\tau,2\bar\tau)\right]}$

noting that $E(s;\tau,\bar\tau) \underset{\hat S}{\to} E\left(s;-\frac{1}{2\tau},-\frac{1}{2\bar\tau}\right) = E(s;2\tau,2\bar\tau)$

$$B^i_{G_N}(t) = \sum_{s=2}^{\infty}\frac{b^i_{G_N}(s)}{\Gamma(s)}\,t^{s-1} \qquad i=1,2$$

Yang-Mills Perturbation Expansion

Expansion parameters

$$a_{SU(N)} = \frac{N g_{YM}^2}{4\pi^2}\,, \qquad a_{SO(n)} = \frac{(n-2) g_{YM}^2}{4\pi^2}\,, \qquad a_{USp(n)} = \frac{(n+2) g_{YM}^2}{8\pi^2}$$

$n = 2N \; or \; 2N+1$

$$\mathbb{C}^{pert}_{G_N}(\tau_2) = -4c_{G_N}\left[\frac{3\,\zeta(3) a_{G_N}}{2} - \frac{75\,\zeta(5) a_{G_N}^2}{8} + \frac{735\,\zeta(7) a_{G_N}^3}{16} - \frac{6615\,\zeta(9)\,(1+P_{G_N,1})\,a_{G_N}^4}{32} + \frac{114345\,\zeta(11)\,(1+P_{G_N,2})\,a_{G_N}^5}{128} - \frac{3864861\,\zeta(13)\,(1+P_{G_N,3})\,a_{G_N}^6}{1024} + \frac{32207175\,\zeta(15)\,(1+P_{G_N,4})\,a_{G_N}^7}{2048} + \mathcal{O}(a_{G_N}^8)\right],$$

$$P_{SU(N),1} = \frac{2}{7N^2}\,, \qquad P_{SU(N),2} = \frac{1}{N^2}\,,$$

$$P_{SU(N),3} = \frac{25N^2+4}{11N^4}\,, \qquad P_{SU(N),4} = \frac{605N^2+332}{143N^4}$$

Etc.

- **The "planar" pieces are identical for all gauge groups.**
- Non-planar terms first enter at four loops.
- The transformation $(N\,, g_{YM}^2) \leftrightarrow (-N, -g_{YM}^2)$ **Symmetry of** $\mathbb{C}_{SU(N)}(\tau_2)$

  Interchanges $\mathbb{C}_{SO(2N)}(\tau_2)$ **and** $\mathbb{C}_{USp(2N)}(\tau_2)$

Laplace Difference Equations

Central charges

$$c_{SU(N)} = \frac{N^2-1}{4}\,,$$
$$c_{SO(n)} = \frac{n(n-1)}{8}\,,$$
$$c_{USp(n)} = \frac{n(n+1)}{8}\,.$$

$n = N$ *or* $2N$

$$\Delta_\tau \mathbb{C}_{SO(n)}(\tau,\bar\tau) - 2c_{SO(n)}\Big[\mathbb{C}_{SO(n+2)}(\tau,\bar\tau) - 2\,\mathbb{C}_{SO(n)}(\tau,\bar\tau) + \mathbb{C}_{SO(n-2)}(\tau,\bar\tau)\Big] - n\,\mathbb{C}_{SU(n-1)}(\tau,\bar\tau) + (n-1)\,\mathbb{C}_{SU(n)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathbb{C}_{USp(n)}(\tau,\bar\tau) - 2c_{USp(n)}\Big[\mathbb{C}_{USp(n+2)}(\tau,\bar\tau) - 2\,\mathbb{C}_{USp(n)}(\tau,\bar\tau) + \mathbb{C}_{USp(n-2)}(\tau,\bar\tau)\Big] + n\,\mathbb{C}_{SU(n+1)}(2\tau,2\bar\tau) - (n+1)\,\mathbb{C}_{SU(n)}(2\tau,2\bar\tau) = 0$$

- Identities $\mathbb{C}_{SO(3)}(\tau,\bar\tau) = \mathbb{C}_{SU(2)}(\tau,\bar\tau)\,,\qquad \mathbb{C}_{SO(4)}(\tau,\bar\tau) = 2\,\mathbb{C}_{SU(2)}(\tau,\bar\tau)\,,\qquad \mathbb{C}_{SO(6)}(\tau,\bar\tau) = \mathbb{C}_{SU(4)}(\tau,\bar\tau)$

- All integrated correlators can be related to $SU(N)$ correlators, and hence to the $SU(2)$ case.

  e.g. $$\mathbb{C}_{SO(7)}(\tau,\bar\tau) = \left[\frac{8}{5}\mathbb{C}_{SU(2)}(\tau,\bar\tau) - \frac{12}{5}\mathbb{C}_{SU(3)}(\tau,\bar\tau) + \frac{3}{5}\mathbb{C}_{SU(4)}(\tau,\bar\tau) + \frac{4}{5}\mathbb{C}_{SU(5)}(\tau,\bar\tau)\right] + \left[\frac{3}{5}\mathbb{C}_{SU(2)}(2\tau,2\bar\tau) - \frac{12}{5}\mathbb{C}_{SU(3)}(2\tau,2\bar\tau) + \frac{8}{5}\mathbb{C}_{SU(4)}(2\tau,2\bar\tau)\right]\,,$$

- Invariance under $(N\,,g^2_{YM}) \leftrightarrow (-N,-g^2_{YM})$

- Instanton sectors confirm these equations

Large-N Expansions

Expansion parameters $\hat{N}_{SU(N)} = N$ $\quad \hat{N}_{SO(n)} = \frac{n}{2} - \frac{1}{4}$ $\quad \hat{N}_{USp(n)} = \frac{n}{2} + \frac{1}{4}$ $\quad n = 2N \text{ or } 2N+1$

- 't Hooft expansion $\mathbb{C}_{G_N}(\lambda) \sim \sum_{g=0}^{\infty} (\hat{N}_{G_N})^{2-2g} f^{(g)}_{G_N}(\lambda_{G_N})$

where $\lambda_{SU(N)} := g^2_{YM}\, N\,, \quad \lambda_{SO(n)} := g^2_{YM}\left(\frac{n}{2} - \frac{1}{4}\right), \quad \lambda_{USp(n)} := g^2_{YM}\left(\frac{n}{2} + \frac{1}{4}\right)$

= $g^2_{YM} \times$ Ramond-Ramond flux

in holographically dual string theory in $AdS_5 \times S^5/Z_2$ (orientifold)

- Fixed- g^2_{YM}

$$
\begin{aligned}
2\,\mathbb{C}_{SO(n)}(\tau,\bar{\tau}) = {} & \frac{(2\hat{N}_{SO(n)})^2}{4} - \frac{3(2\hat{N}_{SO(n)})^{\frac{1}{2}}}{2^4} E(\tfrac{3}{2};\tau,\bar{\tau}) + \frac{45(2\hat{N}_{SO(n)})^{-\frac{1}{2}}}{2^8} E(\tfrac{5}{2};\tau,\bar{\tau}) \\
& + (2\hat{N}_{SO(n)})^{-\frac{3}{2}}\left[\frac{4725}{2^{15}} E(\tfrac{7}{2};\tau,\bar{\tau}) - \frac{111}{2^{13}} E(\tfrac{3}{2};\tau,\bar{\tau})\right] + (2\hat{N}_{SO(n)})^{-\frac{5}{2}}\left[\frac{99225}{2^{18}} E(\tfrac{9}{2};\tau,\bar{\tau}) - \frac{3825}{2^{16}} E(\tfrac{5}{2};\tau,\bar{\tau})\right] \\
& + (2\hat{N}_{SO(n)})^{-\frac{7}{2}}\left[\frac{245581875}{2^{27}} E(\tfrac{11}{2};\tau,\bar{\tau}) - \frac{10749375}{2^{25}} E(\tfrac{7}{2};\tau,\bar{\tau}) + \frac{40239}{2^{22}} E(\tfrac{3}{2};\tau,\bar{\tau})\right]\Bigg] + O(\hat{N}^{-\frac{9}{2}}_{SO(n)})\,.
\end{aligned}
$$

- Furthermore $\mathbb{C}_{USp(n)}(\tau,\bar{\tau}) \sim \mathbb{C}_{SO(n)}(2\tau, 2\bar{\tau})$ with $\hat{N}_{SO(n)} \to \hat{N}_{USp(n)}$

- Coefficients of highest-index Eisenstein series at each order same as in $\mathbb{C}_{SU(N)}(\tau,\bar{\tau})$ expansion.

Comments

- We have determined the functional form of the integrated correlators

$$\frac{1}{4}\mathbb{C}_{G_N}(\tau,\bar\tau) = \Delta_\tau\, \partial_m^2 \log Z_{G_N}(m,\tau,\bar\tau)|_{m\to 0}$$

  for all values of N and $\tau = \theta/2\pi + i\, 4\pi/g_{YM}^2$.

- Also generalization to Maximal $U(1)$-violating n-point correlators.

- Integration over positions of operators washes out detailed information about the spatial dependence.

- Extension to the second correlator $\mathcal{G}_N^2(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)|_{m=0}$

  This was considered in the large N expansion by [Chester, MBG, Pufu, Wang, Wen].
  Involves integer powers of $1/N$ with coefficients that are "generalised Eisenstein series" satisfying

$$(\Delta_\tau - r(r+1))\mathcal{E}(r,s_1,s_2;\tau,\bar\tau) = -E(s_1,\tau,\bar\tau)\, E(s_2,\tau,\bar\tau)$$

  which arises in the discussion of the d^6R^4 term [MBG and Pierre Vanhove] in the low energy expansion of type IIB string theory

- These results add to our knowledge of superstring scattering amplitudes in $AdS_5 \times S^5$ expanded around the large-radius (flat-space) and low energy limits.
- But one more integrated correlator is needed in order to reproduce the three leading (BPS) terms in flat space type IIB superstring amplitudes.
- Try generalisation to squashed S^4 with squashing parameter b. [Chester, MBG, Pufu, Wang, Wen].
- Many possible extensions to other models such as ABJM and to exceptional gauge groups.