Attractor Points and Modular CY Threefolds

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"A One Parameter Family of Calabi-Yau Manifolds with Attractor Points of Rank Two",

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(hep-th/1912.96146)

Overview

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Attractor Mechanism

Review of Attractor Mechanism

Consider IIB supergravity with BPS black hole in 4 dimensions and a CY 3-fold X in the remaining 6.

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

Black hole has charge vector (dual to homology class wrapped by D3 brane)

$$\Gamma = p^a \alpha_a - q_a \beta^a \in H^3(X, \mathbb{Z})$$

and central charge given by

$$Z(\Gamma, \varphi) = e^{\frac{K}{2}} \int_X \Gamma \wedge \Omega(\varphi)$$

where Ω is the holomorphic 3-form and φ a coordinate on CS moduli space. We're assuming $h^{2,1}(X) = 1$ so $a, b, \in \{0, 1\}$.

Review of Attractor Mechanism

Preservation of SUSY requires that

$$\frac{dU(\rho)}{d\rho} = -e^{U(\rho)}|Z(\Gamma, \varphi)|$$

$$\frac{d\varphi(\rho)}{d\rho} = -2e^{U(\rho)}G^{\varphi\overline{\varphi}}\partial_{\overline{\varphi}}|Z(\Gamma, \varphi)|$$

where ρ is inverse of distance from horizon, φ is a coordinate on CS moduli space and $G_{\varphi\overline{\varphi}}$ is the metric on CS moduli space. Can solve this after computing periods Π of Ω.

Review of Attractor Mechanism

Each component of

$$\Pi(\varphi) = \begin{pmatrix} \int_{A^a} \Omega(\varphi) \\ \int_{B_b} \Omega(\varphi) \end{pmatrix}$$

is the solution of an ODE (Picard-Fuchs equation) $\mathcal{L}\Pi = 0$. This follows from the fact that

$$\{\Omega, \partial_\varphi \Omega, \partial_\varphi^2 \Omega, \partial_\varphi^3 \Omega, \partial_\varphi^4 \Omega\}$$

is linearly dependent in cohomology.
For example, operator AESZ 34 (math/0507430) is given by

$$\mathcal{L} = S_4 \theta^4 + S_3 \theta^3 + S_2 \theta^2 + S_1 \theta + S_0$$

where $\theta = \varphi \frac{d}{d\varphi}$ and S_i are polynomials in φ.

Review of Attractor Mechanism

Aside:

The Picard-Fuchs equation AESZ 34 appears in the study of Feynman integrals. For example, $\int_{B_1} \Omega$ is the maximally cut four loop banana graph integral with $m_i^2 = 1$ and $p^2 = \frac{1}{\varphi}$

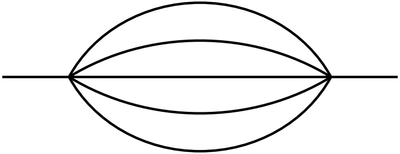


Figure: Four-loop banana graph.

At large p^2, this integral is given by

$$\int_{B_1} \Omega(\varphi) = \sum_{m=0}^{\infty} a_m \varphi^m \quad \text{where} \quad a_m = \sum_{n_1+n_2+n_3+n_4+n_5=m} \left(\frac{m!}{n_1!n_2!n_3!n_4!n_5!} \right)^2$$

Rank - 2 Attractors

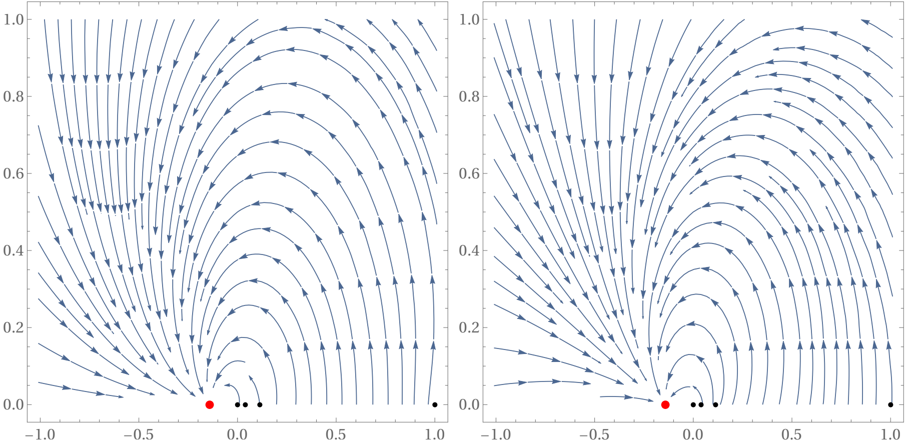


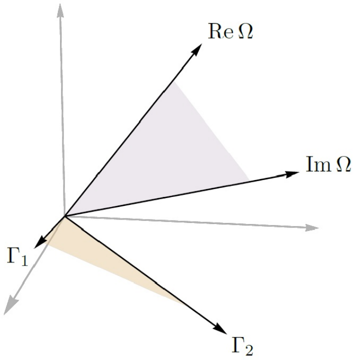
Figure: Flow in φ plane for $Q = (0, 0, 2, 1)^T$ (left) and $Q = (-4, 15, 5, 0)^T$ (right) leading to attractor point at $\varphi = -1/7$. Singularities are indicated in black.

Rank - 2 Attractors

An attractor point is characterised by

$$\Gamma \in H^{3,0} \oplus H^{0,3}$$

at the attractor point. It's helpful to visualise this as follows.



Rank - 2 Attractors

We find that

$$Q = \left(-\frac{4}{5}p^0 + \frac{8}{5}p^1, 3p^0 - 6p^1, p^0, p^1 \right)^T$$

leads to the attractor point $\varphi = -\frac{1}{7}$ for AESZ 34 $\forall p^0, p^1 \in \mathbb{Z}$.
Another way of saying this is

$$H^3\left(X_{-\frac{1}{7}}, \mathbb{Z}\right) \supset \Lambda \oplus \Lambda^\perp$$

where

$$\Lambda \otimes \mathbb{C} = H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda^\perp \otimes \mathbb{C} = H^{2,1} \oplus H^{1,2}.$$

Review of Zeta Functions

Zeta Functions

Review of Zeta Functions

Consider the following polynomial defined over $\mathbb{Q}$

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi\, x_0 x_1 x_2 x_3 x_4.$$

Can count number of points N_{p^r} over $\mathbb{F}_{p^r}$ and define Artin-Weil Congruent Zeta Function

$$Z_p(T) = \exp\left(\sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r\right).$$

This was the subject of the Weil Conjectures.

Review of Zeta Functions

$$Z_p(T) = \exp\left(\sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r\right).$$

Theorem (Weil Conjectures)

1. *Rationality*:

$$Z_p(T) = \frac{P_1^{(p)}(T)...P_{2n-1}^{(p)}(T)}{P_0^{(p)}(T)...P_{2n}^{(p)}(T)}$$

for integral polynomials $P_i^{(p)}(T)$ with (if p is a prime of good recuction) $\deg(P_i^{(p)}(T)) = b_i$ where b_i is the i^{th} Betti number.

$$P_i^{(p)} = \det\big(I - T\mathrm{Frob}_p\big); \quad \mathrm{Frob}_p : H^i(X) \to H^i(X)$$

Examples

Can define the related Hasse-Weil L-function for each polynomial e.g. for middle cohomology

$$L(X,s) = \prod_{good\ \ p} \frac{1}{P_n^{(p)}(p^{-s})} \ .$$

Example 1 - a point
Let X be a point.

$$Z_p(T) = \exp\left(\sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r\right) = \frac{1}{1-T}$$

$$\implies L(X,s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Example 2 - an elliptic curve

Similarly, we can study the Zeta/L-function of an elliptic curve (defined over $\mathbb{Q}$). Can show that

$$Z_p(E,T) = \frac{(1-a_pT+pT^2)}{(1-T)(1-pT)} \quad \text{and} \quad L(E,s) = \prod_{good\ \ p} \frac{1}{1-a_pp^{-s}+p^{1-2s}}.$$

The a_p's are the Fourier coefficients of a modular form f i.e.

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n;\ \ q = e^{2\pi i\tau}$$

that satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau); \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

for some $k \in \mathbb{N}$ known as the *weight* of the modular form.

Examples

Example 2 - an elliptic curve

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Modularity Theorem

Let E be an elliptic curve defined over $\mathbb{Q}$ then, for all but finitely many primes p, a_p is the p^{th} Fourier coefficient of a weight 2 cusp form for a congruence subgroup $\Gamma_0(N)$.

Examples

Example 3 - a one-parameter CY

Expect that the zeta function takes the form

$$Z_p(X,T) = \frac{1 + a_p T + b_p p T^2 + a_p p^3 T^2 + p^6 T^4}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}. \quad (1)$$

Recall that, at a rank - 2 attractor $\varphi_\star$, we can find

$$H^3(X_{\varphi_\star}, \mathbb{Z}) \supset \Lambda \oplus \Lambda^\perp$$

where

$$\Lambda \otimes \mathbb{C} = H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda^\perp \otimes \mathbb{C} = H^{2,1} \oplus H^{1,2}.$$

This would be visibile in the zeta function which would take the form

$$Z_p(X_{\varphi_\star}, T) = \frac{(1 - \alpha_p p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$

and the α_p's and β_p's would come from weight - 2 and weight - 4 modular forms respectively (arXiv:0902.1466).

Persistent Factorisations

Zeta-functions have been computed for one-parameter CY 3-folds by Candelas, de la Ossa, Thorne, van Straten, Villegas,...

Strategy:

Look for polynomials

$$G(\varphi) = c_n\varphi^n + c_{n-1}\varphi^{n-1} + \ldots + c_0; \qquad c_i \in \mathbb{Z}$$

such that the zeta function factors as above for $G(\varphi) = 0 \bmod p$.

Persistent Factorisations

Figure: Number of $\varphi \in \mathbb{F}_p$ such that the Frobenius polynomial factorises for each prime $7 \leq p \leq 3583$. AESZ 34 is plotted above and the mirror quintic below.

Persistent Factorisations

We find that the zeta function of AESZ 34 factorises over $\mathbb{Z}$

$$G(\varphi) = (7\varphi + 1)(\varphi^2 - 66\varphi + 1) = 0 \bmod p$$

so there are rank two attractors at

$$\varphi \in \left\{ -\frac{1}{7}, 33 \pm 8\sqrt{17} \right\}$$

Modularity of AESZ 34 at $\varphi = -\frac{1}{7}$

As expected, at $\varphi = -\frac{1}{7}$, α_p appears in weight 2 form for $\Gamma_0(14) \subset SL(2,\mathbb{Z})$ with Fourier expansion

$$q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 - 2q^{12} - 4q^{13} - q^{14} + q^{16} + 6q^{17} - q^{18} + \ldots$$

Whereas, β_p appears in Fourier expansion of weight 4 form for $\Gamma_0(14) \subset SL(2,\mathbb{Z})$ with Fourier expansion

$$q - 2q^2 + 8q^3 + 4q^4 - 14q^5 - 16q^6 - 7q^7 - 8q^8 + 37q^9 + 28q^{10} - 28q^{11} + 32q^{12} + \ldots$$

The weight 2 and weight 4 modular forms have LMFDB labels **14.2.a.a** and **14.4.a.a** respectively..

Modularity of AESZ 34 $\varphi = 33 \pm 8\sqrt{17}$

Find that α_p and β_p appear as Fourier coefficients of modular form for $\Gamma_1(34) \subset SL(2, \mathbb{Z})$.
As expected, α_p appears in weight 2 form with Fourier expansion

$$q - q^2 + 2\sqrt{-2}q^3 + q^4 - 2\sqrt{-2}q^5 - 2\sqrt{-2}q^6 - q^8 - 5q^9 + 2\sqrt{-2}q^{10} + \ldots$$

Whereas, β_p appears in Fourier expansion of weight 4 form with Fourier expansion

$$q - 2q^2 + 2iq^3 + 4q^4 + 8iq^5 - 4iq^6 + 34iq^7 - 8q^8 + 23q^9 - 16iq^{10} - 30iq^{11} + \ldots$$

The weight 2 and weight 4 modular forms have LMFDB labels **34.2.b.a** and **34.4.b.a** respectively..

Fixed Points of Involutions

Fixed Points of Involutions

Sometimes, the splitting of Hodge structure is due to a symmetry of the moduli space e.g. AESZ 101 with Riemann symbol

$$\mathcal{P}\left\{\begin{array}{cccccc} -1 & 0 & \frac{1}{2}(123-55\sqrt{5}) & 1 & \frac{1}{2}(123+55\sqrt{5}) & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\ \varphi\right\}$$

admits the involution

$$\frac{1}{\varphi}\Pi\left(\frac{1}{\varphi}\right) = A\Pi(\varphi) \qquad \text{where} \qquad A = \begin{pmatrix} -4 & 0 & 0 & 15 \\ 7 & 4 & -15 & 0 \\ 0 & 1 & -4 & 7 \\ -1 & 0 & 0 & 4 \end{pmatrix} \in \ Sp(4,\mathbb{Z})\ .$$

Fixed Points of Involutions

$$\frac{1}{\varphi}\Pi\left(\frac{1}{\varphi}\right) = A\Pi(\varphi) \qquad \text{where} \qquad A = \begin{pmatrix} -4 & 0 & 0 & 15 \\ 7 & 4 & -15 & 0 \\ 0 & 1 & -4 & 7 \\ -1 & 0 & 0 & 4 \end{pmatrix} \in \ Sp(4,\mathbb{Z}) \ .$$

A has eigenvalues $(1, 1, -1, -1)$ and, at the fixed point $\varphi = 1$, its eigenvectors span $\Lambda_\pm \subset H^3(X,\mathbb{Z})$. We confirm that the fixed point is a rank two attractor i.e.

$$\Lambda_+ \otimes \mathbb{C} = H^{3,0} \oplus H^{0,3} \qquad \text{and} \qquad \Lambda_- \otimes \mathbb{C} = H^{2,1} \oplus H^{1,2}$$

by numerically checking that the following hold to high precision

$$\int_X \Gamma_+ \wedge D_\varphi \Omega = Q_+^T \Sigma D_\varphi \Pi = 0 \qquad \text{and} \qquad \int_X \Gamma_- \wedge \Omega = Q_-^T \Sigma \Pi = 0$$

Periods and L-Function Values

Can compute L-function of modular form via Mellin transform

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n \quad \longleftrightarrow \quad L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We find that

$$\Pi\left(-\frac{1}{7}\right) = -\frac{1}{2}\frac{L_4(1)}{2\pi i}\begin{pmatrix} 8 \\ -30 \\ 0 \\ 5 \end{pmatrix} - 14\frac{L_4(2)}{(2\pi i)^2}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} .$$

where L_4 is the L-function associated to the weight - 4 modular form.

Periods and L-Function Values

Consider AESZ 34 at $\varphi = -\frac{1}{7}$. The periods of the harmonic $(2,1)$-form are given by

$$D_\varphi \Pi\left(-\tfrac{1}{7}\right) = -\frac{147}{8}\frac{L_2(1)}{(2\pi i)^2}\begin{pmatrix}-7\\14\\-10\\-5\end{pmatrix} + \frac{147}{16}\frac{L_2(1)}{(2\pi i)^2}\frac{1}{i\mathrm{Im}\ \tau}\begin{pmatrix}-3\\6\\0\\-1\end{pmatrix}.$$

where τ is the parameter of the elliptic curve associated to the weight - 2 form and L_2 is it's Mellin transform.

Bekenstein - Hawking Entropy

Let

$$Q_{k,l} = k(4, -15, -5, 0)^T + l(0, 0, 2, 1)^T \qquad k, l \in \mathbb{Z}.$$

A black hole with charge $Q_{k,l}$ will have horizon area

$$\frac{A(Q_{k,l})}{4} = \frac{\pi(5k - 2l)^2}{8}\left(\frac{\pi L_4(1)}{L_4(2)}\right) + \frac{49\pi k^2}{2}\left(\frac{\pi L_4(1)}{L_4(2)}\right)^{-1}$$

This implies that the growth of black hole degeneracies $n_{k,l}$ is controlled by L-values i.e.

$$\log(n_{k,l}) \sim \frac{\pi(5k - 2l)^2}{8}\left(\frac{\pi L_4(1)}{L_4(2)}\right) + \frac{49\pi k^2}{2}\left(\frac{\pi L_4(1)}{L_4(2)}\right)^{-1}$$

Thank You!