NOTES ON f-HYPERLOGS

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The grand picture is that any QFT amplitude is given by algebraic integrals so that there should exist an isomorphism into the motivic 'f-alphabet' [3, 4, 6, 7],

$$\psi : \text{amplitude} \longrightarrow f\text{-alphabet},$$

where the right hand side is a shuffle algebra with deconcatenation as motivic coaction. Because the right hand side is free it can be considered as the solution of the integral on the left hand side. A futuristic dream would be to be able to construct ψ on any QFT amplitude.

Note that the f-alphabet is commonly used for pure numbers only. In the case of multiple-zeta-values (MZVs) the f-alphabet consists of words in letters of odd weights ≥ 3, one letter for each weight. One often chooses to represent the numbers $\zeta(3)$, $\zeta(5)$, $\ldots$ by f_3, f_5, $\ldots$ while (by shuffle) $\zeta(3)\zeta(5)$ is represented by $f_3 f_5 + f_5 f_3$. The word $f_3 f_5$ alone also exists and for HyperlogProcedures it correspond to the MZV $-\zeta(3,5)/5$ (in general there exists a $\mathbb{Q}\pi^8$ ambiguity). A freedom in the construction of ψ comes from the choice of an algebra basis of the considered numbers [4].

While, historically, the focus was on numbers (see e.g. [10] for first definitions), the f-alphabet should also exist for functions which are integrals of rational forms (over rational domains) where some variables are left unintegrated. These functions are called period functions, variations of periods, or algebraic integrals. In QFT, amplitudes are exactly of this type.

Very little is known on the f-alphabet for functions (although algebra bases are often evident). Let us be very specific and consider multiple polylogarithms $\mathrm{Li}_w(z)$ for some words w in 0 and 1. An algebra basis of multiple polylogarithms are Lis in Lyndon words. We call the ψ-image of multiple polylogarithms (or hyperlogarithms) f-hyperlogs.

At weight 1 we have logarithms which we use as weight one letters z_0, z_1:

$$\mathrm{Li}_0(z) = \log(z) \to z_0, \quad \mathrm{Li}_1(z) = \log(1-z) \to z_1.$$

With this definition the dilogarithm $\mathrm{Li}_{10}(z)$ (writing words from left to right) becomes in the f-alphabet

$$\mathrm{Li}_{10}(z) \to z_1 z_0.$$

To derive the above result one may use F. Brown's decomposition algorithm [4]. (To be precise, the translation depends on the sheet of the multi-valued polylogarithm.) In general, the translation is not so simple. We e.g. have [12]

$$\mathrm{Li}_{1010}(z) \to z_1z_0z_1z_0 - 2z_1f_3,$$

where we have the f-version of the zeta value $\zeta(3)$ on the right hand side. It is important to notice that the word $z_1z_0z_1z_0$ alone does not stand for a function. If one adds $2\zeta(3)\log(1-z)$ to the left hand side, then, by shuffle, the sum translates into $z_1z_0z_1z_0 + 2f_3z_1$.

The origin of this complication is that the left hand side has a non-trivial monodromy around 1 which involves $\zeta(3)$

$$\mathcal{M}_1\mathrm{Li}_{1010}(z) = \mathrm{Li}_{1010}(z) + 2\pi\mathrm{i}\mathrm{Li}_{010}(z) - 4\pi\mathrm{i}\zeta(2)\mathrm{Li}_0(z) - 4\pi\mathrm{i}\zeta(3).$$

While the monodromy is somewhat hidden in the multiple polylogarithm, it is more explicit in the f-alphabet: Modulo $\zeta(2)$ the monodromy at 1 picks words which begin in z_1 and replaces this z_1 by $2\pi\mathrm{i}$. This also holds for the dilog:

$$\mathcal{M}_1\mathrm{Li}_{10}(z) = \mathrm{Li}_{10}(z) + 2\pi\mathrm{i}\log(z).$$

The general situation is as follows: For multiple polylogarithms analytic differentiation is explicit (cut off the rightmost letter), monodromy is somewhat obscured. In the f-alphabet both are explicit: Monodromy is on the Betti side (left) while analytic differentiation is on the deRham side (right). The prize one has to pay is that one gets more terms in the f-alphabet. In this sense, one converts complexity into a proliferation of terms.

To make the above statements precise, note that by unipotence the monodromy can be written as the exponential of a 'infinitesimal monodromy'

$$\mathcal{M}_a = \exp(2\pi\mathrm{i}m_a),$$

where (for any $a \in \mathbb{C}$) the infinitesimal monodromy around a is a derivative. Here, we need a deRham version of m_a,

$$m_a^{\mathrm{dR}} = m_a \mod \zeta(2)$$

which makes sense in the motivic context. The deRham infinitesimal monodromy is the Betty analog of the analytic (deRham) derivative. In the f-alphabet it translates into cutting off the weight 1 left letter z_a

$$m_a^{\mathrm{dR}} \to \delta_{z_a}^{\mathrm{B}}$$

while the analytic derivative translates into cutting off a weight 1 right letter z_a

$$\partial_z \to \sum_a \frac{1}{z-a}\delta_{z_a}^{\mathrm{dR}}.$$

Here, we have introduced the shuffle differentials (for any letter x)

$$\delta_x^{\mathrm{B}} w = \begin{cases} v & \text{, if } w = xv \\ 0 & \text{, otherwise} \end{cases}$$

$$\begin{tikzcd}
 & \mathcal{HL} & \\
\partial_z\mathcal{HL} \arrow[ur, "\int_0 \mathrm{d}z"] & & m\mathcal{HL} \arrow[ul, "m^{-1}"'] \\
\partial_z\mathcal{HL} \arrow[u, "\pi_\partial"] \arrow[dr, "m"'] & & m\mathcal{HL} \arrow[u, "\pi_m"'] \\
 & m\partial_z\mathcal{HL} \arrow[ur, "\int_0 \mathrm{d}z"'] &
\end{tikzcd}$$

FIGURE 1. The inductive construction of hyperlogarithms in the f-alphabet by a commutative hexagon.

$$\begin{tikzcd}
 & \mathcal{G} & \\
\partial_z\mathcal{G} \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"] & & \partial_{\overline{z}}\mathcal{G} \arrow[ul, "\int_{\mathrm{sv}} \mathrm{d}\overline{z}"'] \\
\partial_z\mathcal{G} \arrow[u, "\pi_\partial"] \arrow[dr, "\partial_{\overline{z}}"'] & & \partial_{\overline{z}}\mathcal{G} \arrow[u, "\pi_{\overline{\partial}}"'] \\
 & \partial_{\overline{z}}\partial_z\mathcal{G} \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"'] &
\end{tikzcd}$$

FIGURE 2. The inductive construction of GSVHs by a commutative hexagon. Here $\mathcal{G}$ is the space of GSVHs, $\int_{\mathrm{sv}}$ is single-valued integration, and π_∂ ($\pi_{\overline{\partial}}$) is the projection onto the (anti-)residue-free subspace (subtracting (anti-)residues).

and

$$\delta_x^{\mathrm{dR}} w = \begin{cases} v & \text{, if } w = vx \\ 0 & \text{, otherwise.} \end{cases}$$

This admixture of monodromy information leads to the problem that integration is non-trivial in the f-alphabet (in contrast to integrating hyperlogarithms). The analogy between the infinitesimal (deRham) monodromy and the differential can be used for an intrinsic construction of integration of f-hyperlogs. There exists a commutative hexagon for (normal and f-) hyperlogs, see Figure 1 (compare Figure 2).

The operation π_m is the monodromy analog of subtracting residues in π_0 while m^{-1} is the analog of integration.

Theorem 1 (Schnetz 2021, [11]). *The hexagons in Figures 1 and 2 commute.*

For the intrinsic construction of f-hyperlogs one needs the transition to m^{dR} by nullifying $2\pi\mathrm{i}$ on the bottom and the right of Figure 1. It is easy to see that the bottom right path in Figure 1 suffices to construct f-hyperlogs if the integrand has no words with constant letters $(f_3, f_5, \ldots)$ on the left hand side. This can always be achieved by un-shuffling constants.

Going from f-hyperlogs to single-valued f-hyperlogs is trivial. With the Ihara action the single-valued map (sv-map) in the f-alphabet is [5]

$$\mathrm{sv}: w \mapsto \sum_{w=uv} \overline{\tilde{u}} ш v, \tag{1}$$

where $\tilde{u}$ is u in reversed order, $\overline{\bullet}$ is complex conjugation, and ш is the shuffle product. With the new letters $\overline{z}_0$ and $\overline{z}_1$ for $\log \overline{z}$ and $\log(1-\overline{z})$ (respectively) we get in the above examples (also see [1, 2]):

$$\begin{aligned}\mathrm{svLi}_{10}(z) &\equiv \mathcal{L}_{10}(z) = \mathrm{Li}_{10}(z) + \mathrm{Li}_1(\overline{z})\mathrm{Li}_0(z) + \mathrm{Li}_{01}(\overline{z})\\ &\to z_1z_0 + z_0\overline{z}_1 + \overline{z}_1z_0 + \overline{z}_0\overline{z}_1 = (z_0+\overline{z}_0)\overline{z}_1 + (z_1+\overline{z}_1)z_0\end{aligned}$$

and

$$\begin{aligned}&\mathrm{svLi}_{1010}(z) \equiv \mathcal{L}_{1010}(z) = \mathrm{Li}_{1010}(z) + \mathrm{Li}_1(\overline{z})\mathrm{Li}_{010}(z)\\ &+ \mathrm{Li}_{01}(\overline{z})\mathrm{Li}_{10}(z) + \mathrm{Li}_{101}(\overline{z})\mathrm{Li}_0(z) + \mathrm{Li}_{0101}(\overline{z}) - 4\zeta(3)\mathrm{Li}_1(\overline{z})\\ &\to (z_1+\overline{z}_1)z_0z_1z_0 + \ldots + (z_0+\overline{z}_0)\overline{z}_1\overline{z}_0\overline{z}_1 - 2(z_1+\overline{z}_1)f_3 - 4f_3\overline{z}_1.\end{aligned}$$

Again, we have a proliferation of terms in the f-alphabet that compensates for structural simplicity. In the hyperlog case we had to do a non-trivial calculation (e.g. using the commutative hexagon in Figure 2) to obtain the $\zeta(3)$ contribution in $\mathcal{L}_{1010}(z)$. In the f-hyperlog case we obtain the f_3-terms directly from applying the sv-map to $-2z_1f_3$:

$$\mathrm{sv}(-2z_1f_3) = -2z_1f_3 - 2\overline{z}_1 ш f_3 - 2f_3\overline{z}_1.$$

The single-valuedness of the expressions in the f-alphabet are evident from the fact that the leftmost log letters can always be written as the single-valued combinations

$$z_0 + \overline{z}_0 = \log(z\overline{z}) \quad \text{and} \quad z_1 + \overline{z}_1 = \log((1-z)(1-\overline{z})).$$

The situation, however, is non-trivial and not yet fully understood for GSVHs which are not single-valued hyperlogarithms. One of the simplest GSVHs which is not a single-valued hyperlogarithm is the single-valued primitive of $(\log z\overline{z})/(z-\overline{z}^{-1})$,

$$\int_{\mathrm{sv}} \frac{\log(z\overline{z})}{z-\overline{z}^{-1}}\mathrm{d}z \equiv \mathcal{L}_{0\overline{z}^{-1}}(z) = \mathrm{Li}_{0\overline{z}^{-1}}(z) + \mathrm{Li}_0(\overline{z})\mathrm{Li}_{\overline{z}^{-1}}(z).$$

The GSVH-character of the above expression is evident from the letter $\overline{z}^{-1}$ in $\mathcal{L}$ or in the Lis with argument z. The f-version of $\mathrm{Li}_{0\overline{z}^{-1}}(z)$ is

$$z_0z_{\overline{z}^{-1}} - z_{\overline{z}^{-1}}\overline{z}_0.$$

while the f-version of $\mathcal{L}_{0\overline{z}^{-1}}(z)$ is merely

$$(z_0+\overline{z}_0)z_{\overline{z}^{-1}}.$$

The latter cannot be derived from the first by using (1).

As second example, we take the single-valued primitive of the Bloch-Wigner dilogarithm D over $z-\overline{z}$ [14, 9]. More conveniently,

$$\begin{aligned}\int_{\mathrm{sv}}\frac{4\mathrm{i}D}{z-\overline{z}}\mathrm{d}z &= \mathcal{L}_{10\overline{z}}(z)-\mathcal{L}_{01\overline{z}}(z)\\ &= \mathrm{Li}_{10\overline{z}}(z)-\mathrm{Li}_{01\overline{z}}(z)+\mathrm{Li}_1(\overline{z})\mathrm{Li}_{0\overline{z}}(z)-\mathrm{Li}_0(\overline{z})\mathrm{Li}_{1\overline{z}}(z)\\ &\quad+\mathrm{Li}_{01}(\overline{z})\mathrm{Li}_{\overline{z}}(z)-\mathrm{Li}_{10}(\overline{z})\mathrm{Li}_{\overline{z}}(z)+\mathrm{Li}_{101}(\overline{z})-\mathrm{Li}_{100}(\overline{z}).\end{aligned}$$

The f-version of $\mathrm{Li}_{10\overline{z}}(z)-\mathrm{Li}_{01\overline{z}}(z)$ is

$$\begin{aligned}&z_1z_0z_{\overline{z}}-z_0z_1z_{\overline{z}}+z_1z_{\overline{z}}\overline{z}_0-z_0z_{\overline{z}}\overline{z}_1+z_{\overline{z}}\overline{z}_1\overline{z}_0-z_{\overline{z}}\overline{z}_0\overline{z}_1\\ &+z_0z_1\overline{z}_1-z_0z_1\overline{z}_0+z_1\overline{z}_0\overline{z}_0-z_1\overline{z}_1\overline{z}_0.\end{aligned}$$

The f-version of the single-valued $\mathcal{L}_{10\overline{z}}(z)-\mathcal{L}_{01\overline{z}}(z)$ is

$$(z_0+\overline{z}_0)(-z_1z_{\overline{z}}+\overline{z}_1z_{\overline{z}}+z_1\overline{z}_1-z_1\overline{z}_0)+(z_1+\overline{z}_1)(z_0z_{\overline{z}}-\overline{z}_0z_{\overline{z}}+\overline{z}_0\overline{z}_1-\overline{z}_0\overline{z}_0).$$

Again, we cannot use (1) for the single-valued map.

In general, we need to derive an intrinsic algorithm for single-valued integration of f-hyperlogs. Is there something more efficient than a naive combination of the two commutative hexagons? Does there exist a modified Ihara action that works for all GSVHs?

With an integration prescription at hand one can use single-valued f-hyperlogs to express graphical functions. So far, two possible formats for GSVHs exist:

- The representation in terms of $\mathrm{Li}_\bullet(\overline{z})\mathrm{Li}_\bullet(z)$. It is the simplest representation and all operations of GSVHs are reasonably straight forward here (using the commutative hexagon Figure 2).
- The representation in terms of the single-valued $\mathcal{L}_\bullet(z)$. This representation is significantly shorter (as it is manifestly single-valued). However, the frequently needed evaluation at certain values of z often requires a transform back to the previous representation in terms of Lis. Still, this version is the one that is very efficiently used in HyperlogProcedures.

The f-representation is the third option. Expressions will have more terms but the terms are structurally simpler. Can an implementation of f-hyperlogs be more efficient? Possibly not for an implementation in Maple or Mathematica but maybe in a C++ or FORM transcript which can handle large expressions much more efficiently.

In QFT hyperlogarithms do not suffice. In the end one has to handle more complex structures. The c_2 analysis done in 2012 with F. Brown [8] and recently refined in [13] gives an impression which geometries are to be expected in QFT. Is it possible to generalize the f-alphabet to these geometries?

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