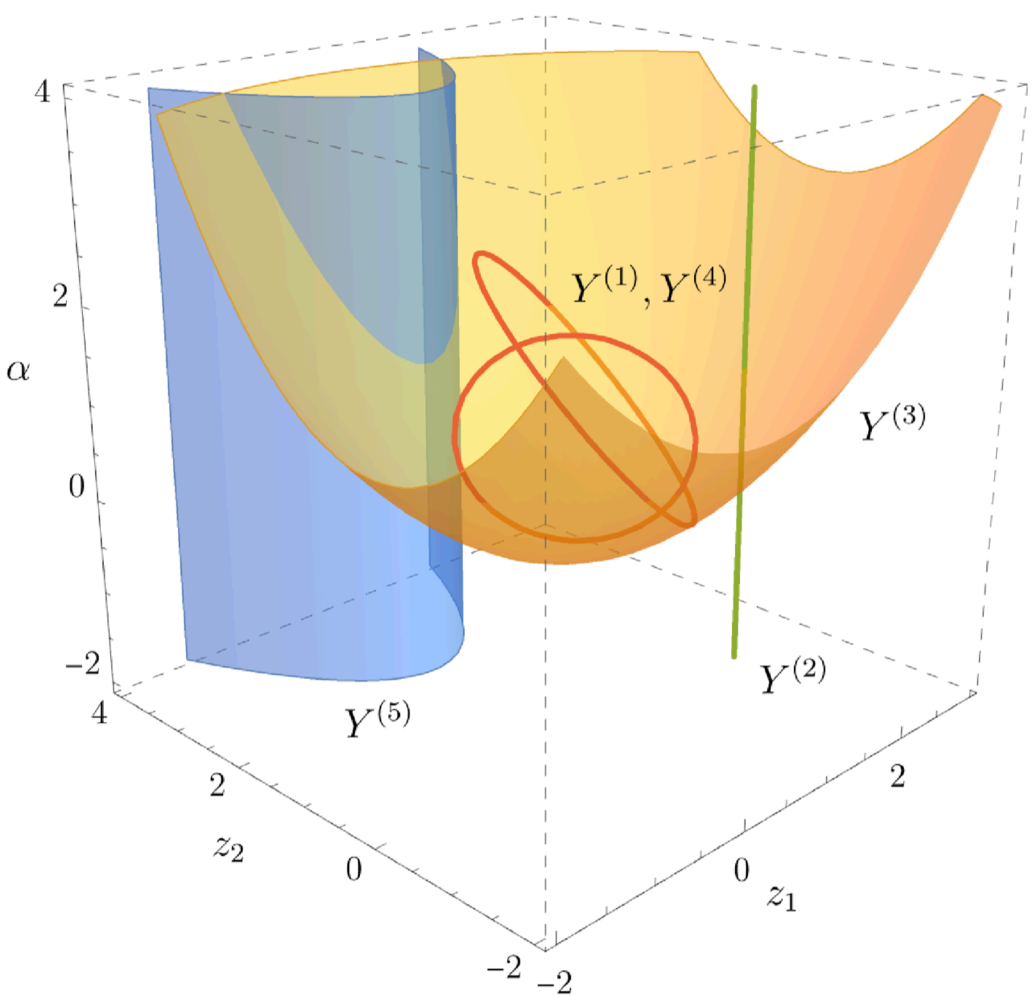
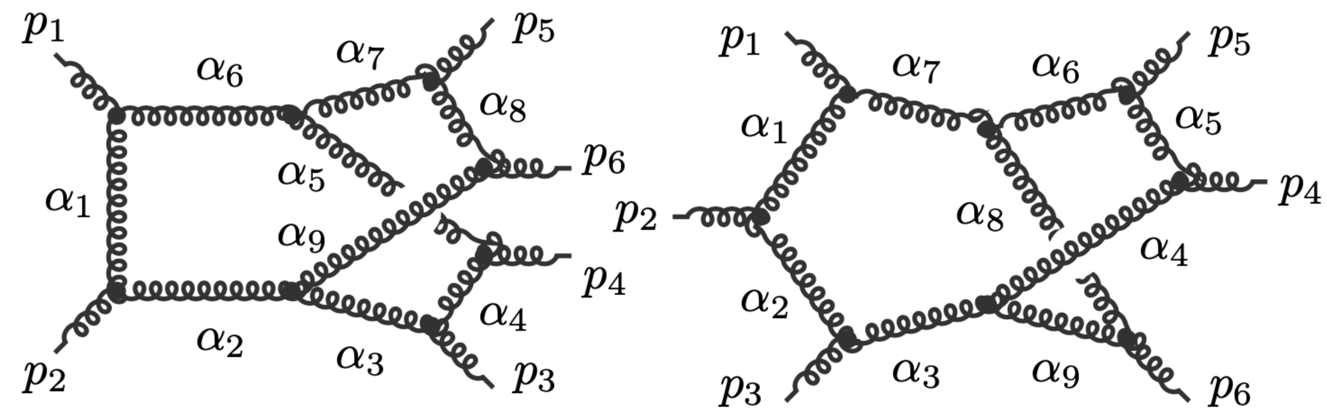
Computational algebraic geometry for Landau analysis

Simon Telen





Motives and period integrals in quantum field theory and string theory
University of Oxford, January 17, 2024


Joint with C. Fevola and S. Mizera

Euler integrals

Euler's Beta integral $\int_0^1 (1-x)^\mu \, x^\nu \, \frac{\mathrm{d}x}{x}$ converges for $\mathrm{Re}(\nu) \geq 0, \; \mathrm{Re}(\mu) \geq -1$

Its meromorphic extension to $\mathbb{C}^2$ is the Beta function

$$\mathrm{B}(\nu, 1+\mu) \; = \; \int_\Gamma (1-x)^\mu \, x^\nu \frac{\mathrm{d}x}{x} \; = \; \frac{\Gamma(\nu)\,\Gamma(1+\mu)}{\Gamma(\nu+1+\mu)}$$

Here Γ is a twisted cycle on $\mathbb{C}^* \backslash \{1\}$

A similar integral appears in Euler's integral formula for ${}_2F_1$:

$$\mathrm{B}(\nu, 1+\mu_1) \; {}_2F_1(-\mu_2, \nu, \mu_1+1+\nu \, ; \, z) \; = \; \int_\Gamma (1-x)^{\mu_1} (1-zx)^{\mu_2} \, x^\nu \, \frac{\mathrm{d}x}{x}$$

Euler integrals

$$\mathcal{I}_\Gamma(z) \;=\; \int_\Gamma \;(z_1\,\alpha^{m_1} + z_2\,\alpha^{m_2} + \cdots + z_s\,\alpha^{m_s})^{\mu}\;\alpha_1^{\nu_1}\,\cdots\,\alpha_n^{\nu_n}\;\frac{\mathrm{d}\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{\mathrm{d}\alpha_n}{\alpha_n}$$

$$f_A(\alpha;z) \;=\; z_1\,\alpha^{m_1} + z_2\,\alpha^{m_2} + \cdots + z_s\,\alpha^{m_s}$$

$$A \;=\; \begin{pmatrix} m_1 & m_2 & \cdots & m_s \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times s}$$

$$\mu \in \mathbb{C},\; \nu = (\nu_1,\ldots,\nu_n) \in \mathbb{C}^n$$

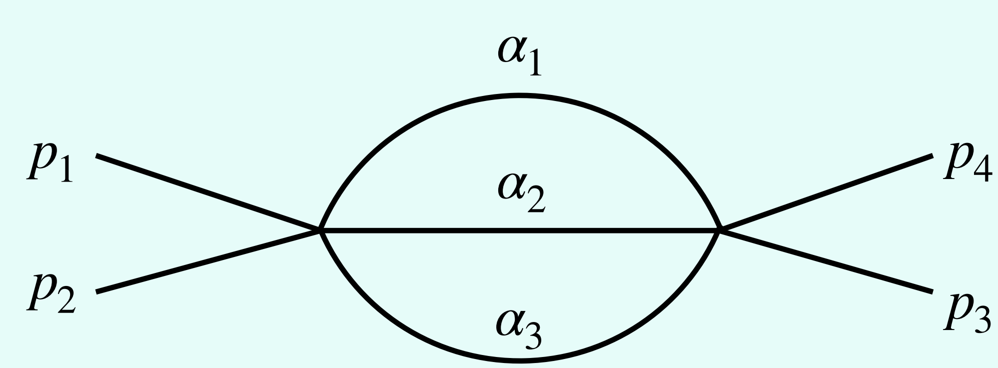
Generalized Euler Integrals and
A-Hypergeometric Functions

I. M. Gelfand,* M. M. Kapranov,† and A. V. Zelevinsky†

Γ is a twisted cycle on $X_z = (\mathbb{C}^*)^n \backslash V_{A,z}$, where $V_{A,z} \;=\; V_{(\mathbb{C}^*)^n}(f_{A(\alpha;z)})$

$$z_1\,\alpha_1\alpha_2 + z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 + z_{10}\,\alpha_1\alpha_2\alpha_3$$

$$A \;=\; \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



$\mathbf{G}_{\text{el'fand}}$-$\mathbf{K}_{\text{apranov}}$-$\mathbf{Z}_{\text{elevinsky}}$ systems

$$\mathscr{I}_\Gamma(z) = \int_\Gamma (z_1 \alpha^{m_1} + z_2 \alpha^{m_2} + \cdots + z_s \alpha^{m_s})^\mu \, \alpha_1^{\nu_1} \cdots \alpha_n^{\nu_n} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

The matrix $A \in \mathbb{Z}^{(n+1)\times s}$ defines a projective toric variety $\mathscr{X}_A \subset \mathbb{P}^{s-1}$

$$I(\mathscr{X}_A) = \langle \partial_z^u - \partial_z^v \, : \, u, v \in \mathbb{N}^s, \, A \cdot (u - v) = 0 \rangle$$

here $\partial_z^u = \partial_{z_1}^{u_1} \partial_{z_2}^{u_2} \cdots \partial_{z_s}^{u_s}$

The GKZ system or A-hypergeometric system of differential equations for $A, (-\mu, \nu)$ is

$$P(\partial_{z_1}, \partial_{z_2} \ldots, \partial_{z_s}) \bullet F(z) = 0 \quad \forall P \in I(\mathscr{X}_A), \qquad \left[A \cdot \begin{pmatrix} z_1 \partial_{z_1} \\ z_2 \partial_{z_2} \\ \vdots \\ z_s \partial_{z_s} \end{pmatrix} + \begin{pmatrix} -\mu \\ \nu_1 \\ \vdots \\ \nu_n \end{pmatrix} \right] \bullet F(z) = 0$$

Theorem (GKZ). Assuming that the parameters μ, ν are non-resonant, the local solutions of the A-hypergeometric system are $\mathscr{I}_\Gamma(z)$, for all twisted cycles Γ.

Counting master integrals

The number of linearly independent functions $\mathcal{I}_\Gamma(z)$ in a neighbourhood of z^*

= the dimension of the space of local solutions of a GKZ system

= the number of master integrals

= the dimension of the n-th twisted (co)homology of $X_{z^*} = (\mathbb{C}^*)^n \setminus V_{A,z^*}$, $\mathrm{dlog} f^\mu x^\nu$

= the signed topological Euler characteristic of X_{z^*}

= …

[Submitted on 26 Dec 2017 (v1), last revised 16 Mar 2018 (this version, v2)]

Feynman integral relations from parametric annihilators

Thomas Bitoun, Christian Bogner, Rene Pascal Klausen, Erik Panzer

[Submitted on 9 Oct 2018 (v1), last revised 5 Mar 2019 (this version, v2)]

Feynman Integrals and Intersection Theory

Pierpaolo Mastrolia, Sebastian Mizera

[Submitted on 18 Aug 2022]

Vector Spaces of Generalized Euler Integrals

Daniele Agostini, Claudia Fevola, Anna-Laura Sattelberger, Simon Telen

SciPost | SciPost Phys. Lect. Notes 75 (2023)

Four lectures on Euler integrals

Saiei-Jaeyeong Matsubara-Heo[1*], Sebastian Mizera[2†] and Simon Telen[3‡]

A-discriminants

Where do the solutions $\mathscr{I}_\Gamma(z)$ to our GKZ system develop singularities?

We start with $z \in \mathbb{C}^s$ such that $V_{A,z} = V_{(\mathbb{C}^*)^n}(f_A(\alpha; z))$ is a singular hypersurface

$$Y_A \;=\; \{(\alpha, z) \in (\mathbb{C}^*)^n \times \mathbb{C}^s \;:\; f_A(\alpha; z) \;=\; \partial_\alpha f_A(\alpha; z) \;=\; 0\}$$

"Landau equations", "pinch singularities"

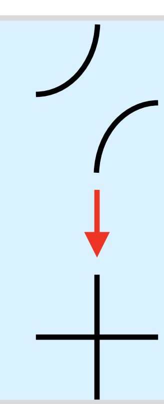
$$\nabla_A \;=\; \overline{\pi_{\mathbb{C}^s}(Y_A)}$$ *"A-discriminant variety"* (projectively dual to $\mathscr{X}_A$)

$$\nabla_A \;=\; \{\Delta_A = 0\}$$ *"A-discriminant (polynomial)"*

Example. $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\quad f_A(\alpha; z) \;=\; z_3\,\alpha^2 + z_2\,\alpha + z_1, \quad \Delta_A \;=\; z_2^2 - 4z_1z_3$

Example. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $\quad f_A(\alpha; z) \;=\; z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$

$$\Delta_A \;=\; z_1z_3 - z_2z_4$$



A-discriminants

$$f_A(\alpha; z) = \frac{1}{2} \cdot (1 \;\; \alpha_1 \;\; \cdots \;\; \alpha_n) \underbrace{\begin{pmatrix} 2z_{00} & z_{01} & \cdots & z_{0n} \\ z_{01} & 2z_{11} & \cdots & z_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{0n} & z_{1n} & \cdots & 2z_{nn} \end{pmatrix}}_{M(z)} \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \qquad \Delta_A = \det M(z)$$

$$n = 1: \; M(z) = \begin{pmatrix} 2z_{00} & z_{01} \\ z_{01} & 2z_{11} \end{pmatrix}, \qquad n = 2: \; M(z) = \begin{pmatrix} 2z_{00} & z_{01} & z_{02} \\ z_{01} & 2z_{11} & z_{12} \\ z_{02} & z_{12} & 2z_{22} \end{pmatrix},$$

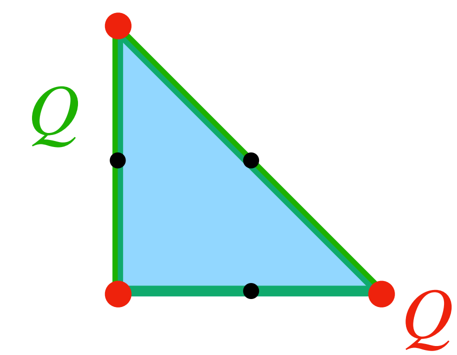
$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad z_3^2 z_4^2 - z_2 z_3 z_4 z_5 + z_1 z_4 z_5^2 + z_2^2 z_3 z_6 - 2 z_1 z_3 z_4 z_6 - z_1 z_2 z_5 z_6 + z_1^2 z_6^2$$

Principal A-determinants

… are built from A-discriminants

$$f_A(\alpha; z) = z_{00} + z_{01}\,\alpha_1 + z_{02}\,\alpha_2 + z_{11}\,\alpha_1^2 + z_{12}\,\alpha_1\alpha_2 + z_{22}\,\alpha_2^2$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad M(z) = \begin{pmatrix} 2z_{00} & z_{01} & z_{02} \\ z_{01} & 2z_{11} & z_{12} \\ z_{02} & z_{12} & 2z_{22} \end{pmatrix}, \quad \mathrm{conv}(A) =$$



$$A \cap Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A \cap Q = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The principal A-determinant is $E_A = \prod_{Q \subset \mathrm{conv}(A)} \Delta_{A\cap Q}^{e_Q}$

$$E_A = z_{00} \cdot z_{11} \cdot z_{22} \cdot (z_{01}^2 - 4z_{00}z_{11}) \cdot (z_{02}^2 - 4z_{00}z_{22}) \cdot (z_{12}^2 - 4z_{11}z_{22}) \cdot \det M(z)$$

They are computable via elimination!

Principal A-determinants

Where do the solutions $\mathcal{I}_\Gamma(z)$ to our GKZ system develop singularities?

Theorem (Cauchy-Kowalevskii-Kashiwara, GKZ). For a simply connected $U \subset \mathbb{C}^s \backslash \{E_A = 0\}$, the A-hypergeometric system has $\mathrm{vol}(\mathrm{conv}(A))$ holomorphic solutions.

Theorem (Amendola, Bliss, Burke, Gibbons, Helmer, Hoşten, Nash, Rodriguez, Smolkin)

$$|\chi(X_z)| < \mathrm{vol}(\mathrm{conv}(A)) \quad \Longleftrightarrow \quad E_A(z) = 0$$

This gives a nice algebraic description of the singularities of A-hypergeometric integrals.

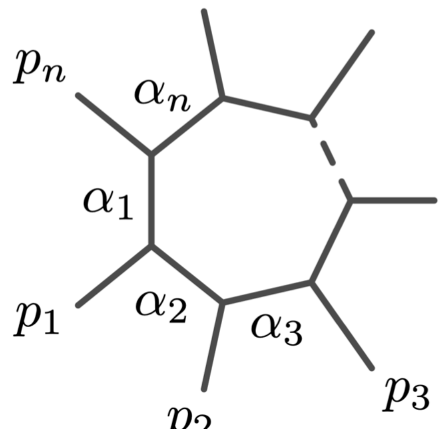
Landau analysis: singularities of Feynman integrals. These are specialized GKZ integrals

$$\mathcal{I}_G(z) = \int_\Gamma (\mathcal{U}_G + \mathcal{F}_G)^\mu \, \alpha_1^{\nu_1} \cdots \alpha_n^{\nu_n} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

$\mathcal{G}_G(\alpha; z) =$ Sum of first and second Symanzik polynomials of G

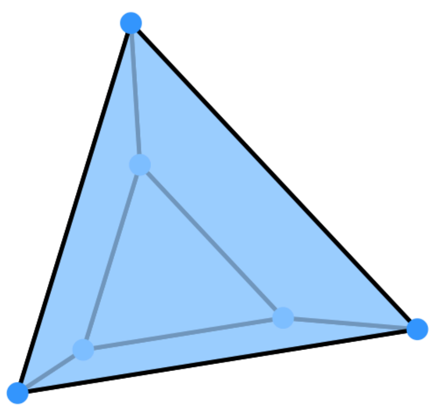
Coefficients z are restricted to lie in a linear subspace $\mathcal{K} \subset \mathbb{C}^s$, the kinematic space

One-loop diagrams

$\mathsf{A}_n =$ 

$$\mathcal{U}_{\mathsf{A}_n} = \alpha_1 + \ldots + \alpha_n$$

$$\mathcal{F}_{\mathsf{A}_n} = \sum_{i<n} (s_{i,i+1,\ldots,j-1} - \mathsf{m}_i - \mathsf{m}_j)\,\alpha_i\alpha_j - \sum_{i=1}^{n} \mathsf{m}_i\,\alpha_i^2$$



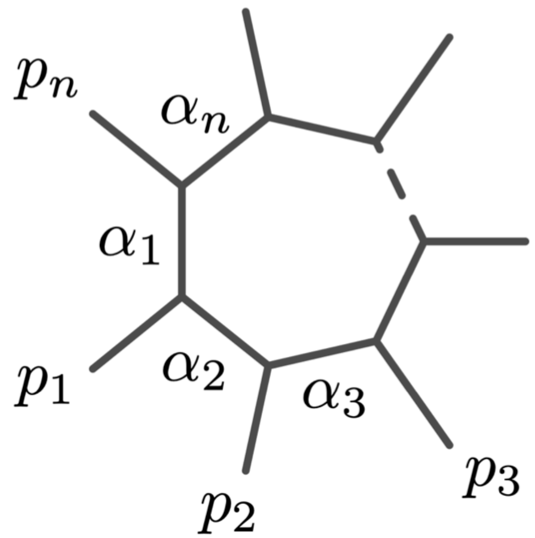
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -2\mathsf{m}_1 & \mathsf{M}_1 - \mathsf{m}_1 - \mathsf{m}_2 & \mathsf{M}_3 - \mathsf{m}_1 - \mathsf{m}_3 \\ 1 & \mathsf{M}_1 - \mathsf{m}_1 - \mathsf{m}_2 & -2\mathsf{m}_2 & \mathsf{M}_2 - \mathsf{m}_2 - \mathsf{m}_3 \\ 1 & \mathsf{M}_3 - \mathsf{m}_1 - \mathsf{m}_3 & \mathsf{M}_2 - \mathsf{m}_2 - \mathsf{m}_3 & -2\mathsf{m}_3 \end{pmatrix}$$

$$\begin{aligned} E_{\mathtt{A}_3}(\mathcal{K}) = \;& \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3 \prod_{i=1}^{3} \lambda(\mathsf{m}_i, \mathsf{m}_{i+1}, \mathsf{M}_i)\, \mathsf{M}_1\mathsf{M}_2\mathsf{M}_3 (\mathsf{m}_1^2\mathsf{M}_2 + \mathsf{m}_1\mathsf{M}_2^2 \\ & + \mathsf{m}_2\mathsf{m}_1\mathsf{M}_1 - \mathsf{m}_3\mathsf{m}_1\mathsf{M}_1 - \mathsf{m}_2\mathsf{m}_1\mathsf{M}_2 - \mathsf{m}_3\mathsf{m}_1\mathsf{M}_2 - \mathsf{m}_1\mathsf{M}_1\mathsf{M}_2 - \mathsf{m}_2\mathsf{m}_1\mathsf{M}_3 + \mathsf{m}_3\mathsf{m}_1\mathsf{M}_3 \\ & - \mathsf{m}_1\mathsf{M}_2\mathsf{M}_3 + \mathsf{m}_3\mathsf{M}_1^2 + \mathsf{m}_2\mathsf{M}_3^2 + \mathsf{m}_3^2\mathsf{M}_1 - \mathsf{m}_2\mathsf{m}_3\mathsf{M}_1 + \mathsf{m}_2\mathsf{m}_3\mathsf{M}_2 - \mathsf{m}_3\mathsf{M}_1\mathsf{M}_2 + \mathsf{m}_2^2\mathsf{M}_3 \\ & - \mathsf{m}_2\mathsf{m}_3\mathsf{M}_3 - \mathsf{m}_2\mathsf{M}_1\mathsf{M}_3 - \mathsf{m}_3\mathsf{M}_1\mathsf{M}_3 - \mathsf{m}_2\mathsf{M}_2\mathsf{M}_3 + \mathsf{M}_1\mathsf{M}_2\mathsf{M}_3)\, \lambda(\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3), \end{aligned}$$

This works because $\mathscr{K} \not\subset \{E_A = 0\}$

$\lambda(a,b,c) := a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$

Euler discriminants

$\mathsf{A}_n =$ 

$$\mathcal{U}_{\mathsf{A}_n} = \alpha_1 + \ldots + \alpha_n$$

$$\mathcal{F}_{\mathsf{A}_n} = \sum_{i<n} (s_{i,i+1,\ldots,j-1} - \mathsf{m}_i - \mathsf{m}_j)\,\alpha_i\alpha_j - \sum_{i=1}^{n} \mathsf{m}_i\,\alpha_i^2$$

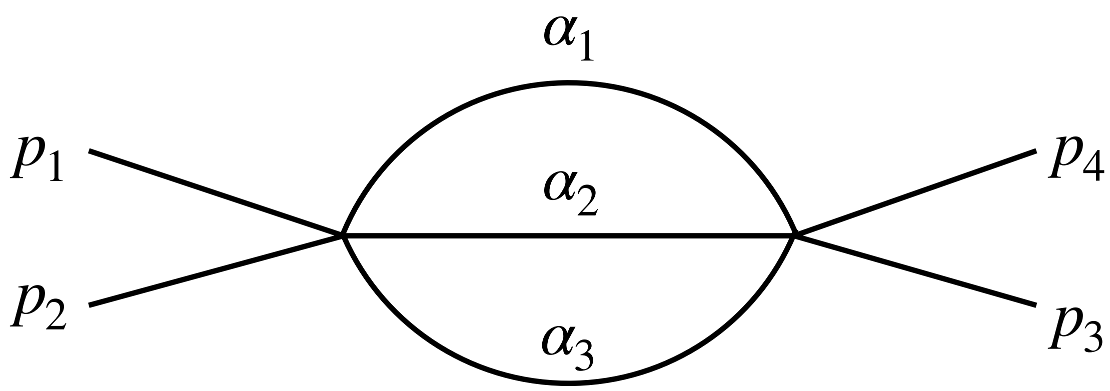
	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\chi^*(\mathscr{E})$	$2^n - 1$	$2^n - 1 - n$	$2^n - 1 - n$	$2^n - 1 + n - n^2$
$\mathrm{vol}(\mathrm{conv}(A(\mathscr{E})))$	$2^n - 1$	$2^n - 1 - n$	$2^n - 1$	$2^n - 1 + n - n^2$

Let $\mathscr{E} \subset \mathscr{K}$ be an irreducible subvariety. The generic signed Euler characteristic of X_z for $z \in \mathscr{E}$ is $\chi^*(\mathscr{E})$. The Euler discriminant of $\mathscr{E}$ is

$$\nabla_\chi(\mathscr{E}) \;=\; \{z \in \mathscr{E} \;:\; |\chi(X_z)| < \chi^*(\mathscr{E})\}$$

Differential equations are hard to compute. We expect their singular locus to be $\nabla_\chi(\mathscr{E})$

Sunrise problem



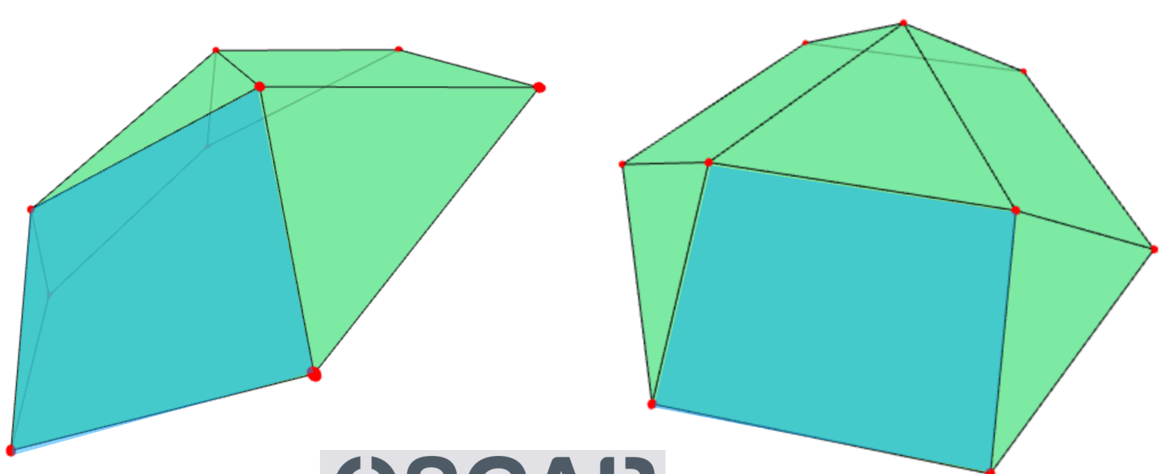
$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

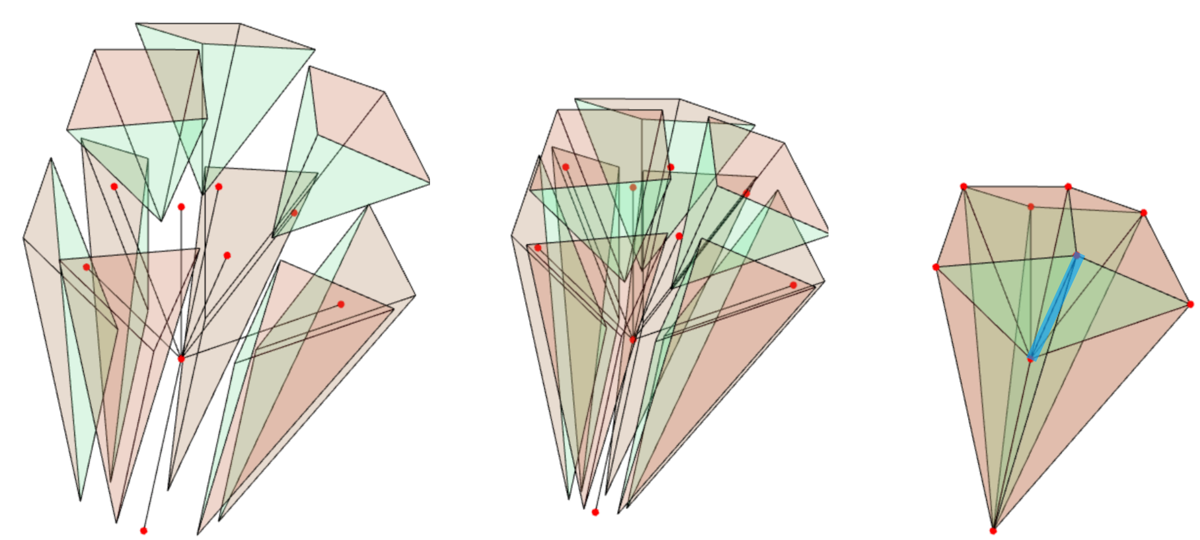
$$\mathcal{I} = \int_\Gamma [(1 - \textstyle\sum_{i=1}^3 \mathsf{m}_i \alpha_i)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + s\alpha_1\alpha_2\alpha_3]^\mu \alpha_1^{\nu_1}\alpha_2^{\nu_2}\alpha_3^{\nu_3} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \frac{\mathrm{d}\alpha_2}{\alpha_2} \wedge \frac{\mathrm{d}\alpha_3}{\alpha_3}$$

restrict to $\mathscr{E} = \mathscr{K}$

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

$$z_1\,\alpha_1\alpha_2 + \underline{z_2\,\alpha_1\alpha_3} + \underline{z_3\,\alpha_2\alpha_3} + z_4\,\alpha_1^2\alpha_2 + z_5\,\alpha_1^2\alpha_3 + z_6\,\alpha_2^2\alpha_3 + z_7\,\alpha_1\alpha_2^2 + \underline{z_8\,\alpha_1\alpha_3^2} + \underline{z_9\,\alpha_2\alpha_3^2} + z_{10}\,\alpha_1\alpha_2\alpha_3$$





Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 = 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 = 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 = 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 = 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 = 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) = (1, \underline{1}, \underline{1}, -\mathrm{m}_1, -\mathrm{m}_1, -\mathrm{m}_2, -\mathrm{m}_2, \underline{-\mathrm{m}_3}, \underline{-\mathrm{m}_3}, s - \mathrm{m}_1 - \mathrm{m}_2 - \mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

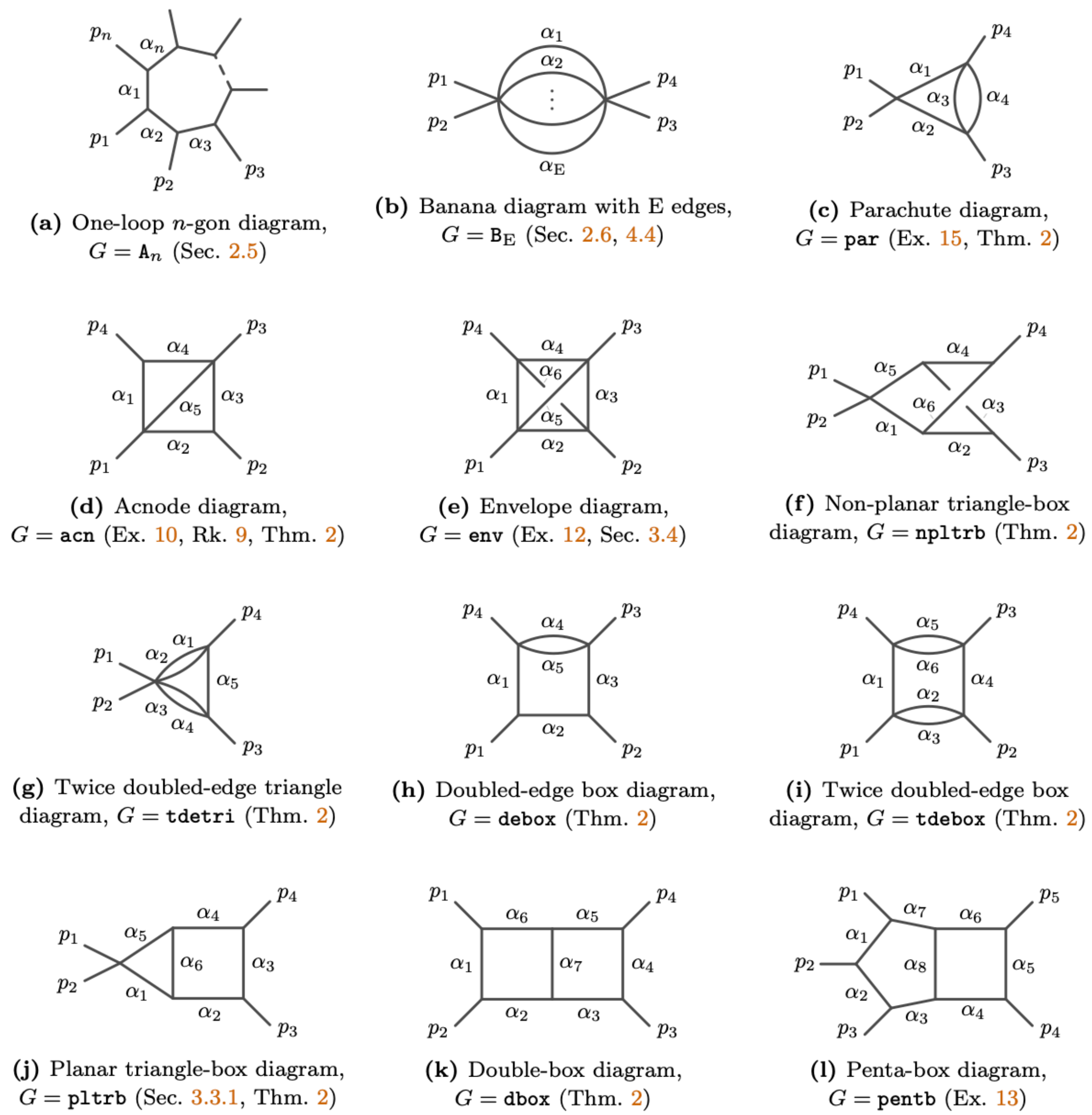
G	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	(15, 15)	(11, 11)	(11, 15)	(3, 3)
$\mathtt{B}_4$	(15, 35)	(1,1)	(15,35)	(1,1)
par	(19, 35)	(4, 8)	(13, 35)	(1, 3)
acn	(55, 136)	(20, 54)	(36, 136)	(3, 9)
env	(273, 1496)	(56, 262)	(181, 1496)	(10, 80)
npltrb	(116, 512)	(28, 252)	(77, 512)	(5, 61)
tdetri	(51, 201)	(4, 18)	(33, 201)	(1, 5)
debox	(43, 96)	(11, 33)	(31, 96)	(3, 10)
tdebox	(123, 705)	(11, 113)	(87, 705)	(3, 41)
pltrb	(81, 417)	(16, 201)	(61, 417)	(4, 80)
dbox	(227, 1422)	(75, 903)	(159, 1422)	(12, 238)
pentb	(543, 4279)	(228, 3148)	(430, 4279)	(62, 1186)

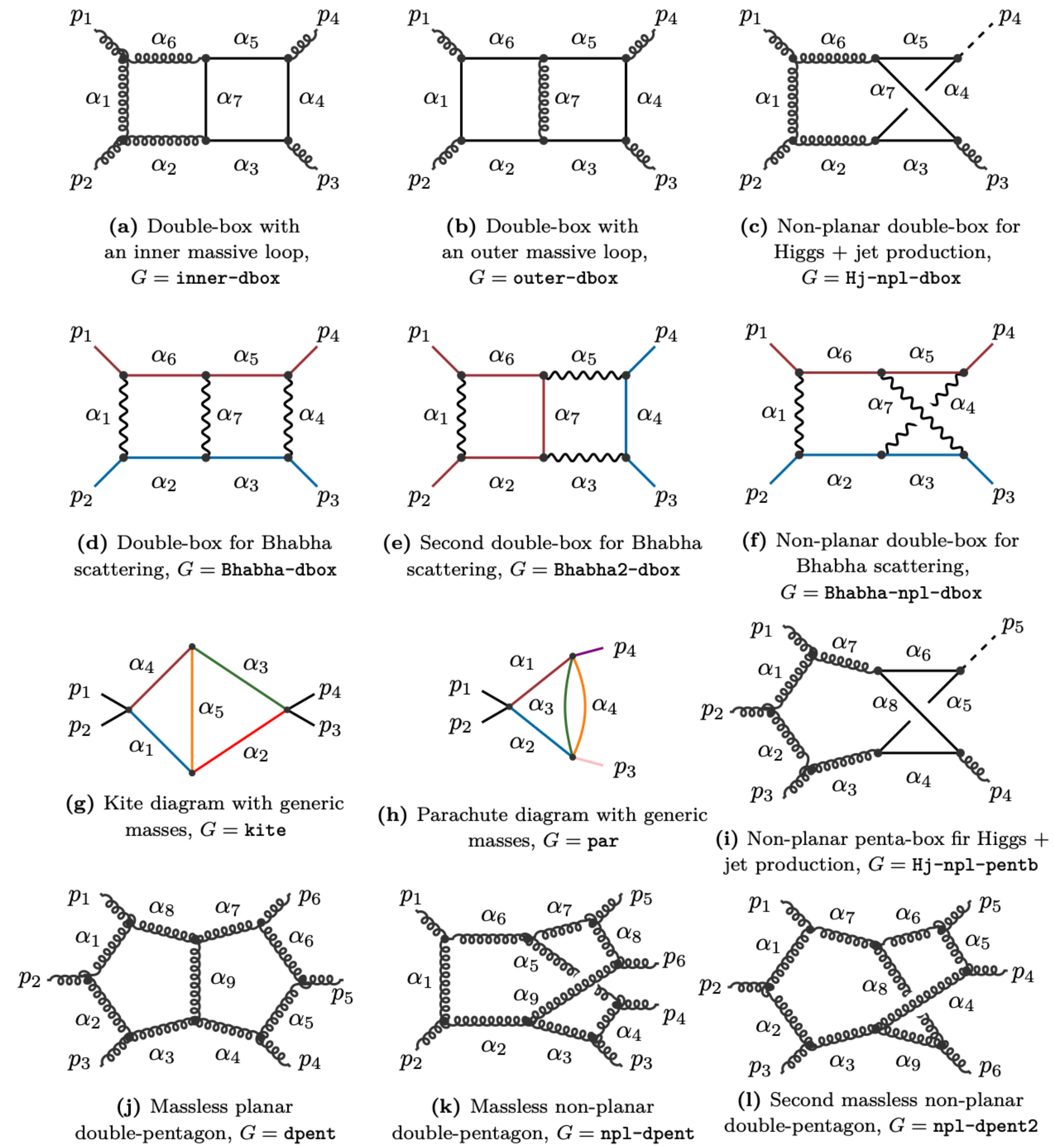
G	$\mathcal{E}$
inner-dbox	(43, 834)
outer-dbox	(64, 1302)
Hj-npl-dbox	(99, 1016)
Bhabha-dbox	(64, 774)
Bhabha2-dbox	(79, 910)
Bhabha-npl-dbox	(111, 936)
kite	(30, 136)
par	(19, 35)
Hj-npl-pentb	(330, 3144)
dpent	(281, 5511)
npl-dpent	(631, 5784)
npl-dpent2	(458, 5467)

The Euler discriminant can usually **not** be obtained by restricting the principal A-determinant

The **principal Landau determinant** is a computable subset of the Euler discriminant, whose definition is inspired by GKZ

A zoo of examples





Landau Discriminants

Sebastian Mizera,[1] Simon Telen[2]

Principal Landau Determinants

Claudia Fevola,[1] Sebastian Mizera,[2] Simon Telen[3]

A first guess

Why not $\prod_{(\Delta_{A\cap Q})_{|\mathscr{E}}\neq 0} (\Delta_{A\cap Q})_{|\mathscr{E}}$? $(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

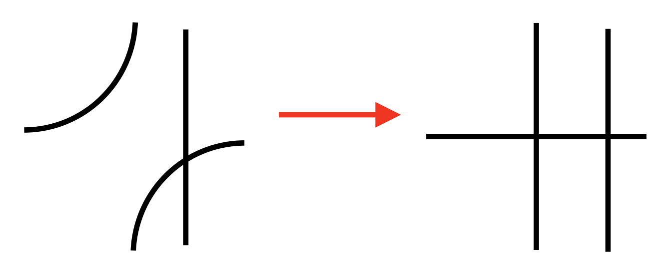
$$\begin{aligned} E_A &= z_1z_3z_4z_6(z_2^2-4z_1z_4)(z_5^2-4z_3z_6) \\ &\quad (z_3^2z_4^2 - z_2z_3z_4z_5 + z_1z_4z_5^2 + z_2^2z_3z_6 - 2z_1z_3z_4z_6 - z_1z_2z_5z_6 + z_1^2z_6^2) \end{aligned}$$

$$(z_1,z_2,z_3,z_4,z_5,z_6) = (a, a+b, c, b, c+d, d) \quad \Rightarrow \quad abcd(a-b)(c-d)$$

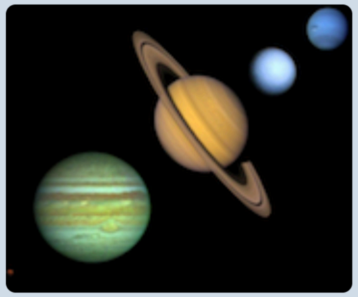
$$\begin{aligned} &\int_{\mathbb{R}_+^2} \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{[(1+\alpha_1)(a+b\alpha_1+c\alpha_2+d\alpha_1\alpha_2)]^2} \\ &\qquad = \frac{1}{(a-b)(c-d)} - \frac{1}{bc-ad}\left[\frac{b^2\log(a/b)}{(a-b)^2} + \frac{d^2\log(d/c)}{(c-d)^2}\right] \end{aligned}$$

R.P. Klausen, *Kinematic singularities of Feynman integrals and principal A-determinants*, *JHEP* **02** (2022) 004 [2109.07584].

C. Dlapa, M. Helmer, G. Papathanasiou and F. Tellander, *Symbol Alphabets from the Landau Singular Locus*, 2304.02629.



Incidence variety



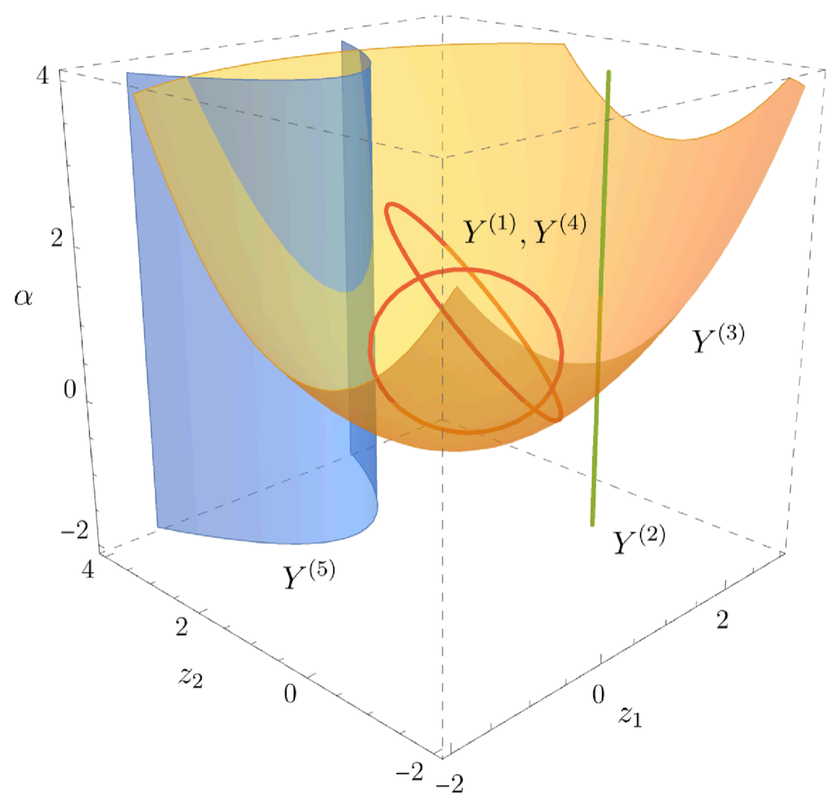
Welcome to the Macaulay2Web interface

```
i1 : R = QQ[x1,x2,a,b,c,d,y];

i2 : f = (1+x1)*(a + b*x1 + c*x2 + d*x1*x2);

i3 : I = ideal(f,diff(x1,f),diff(x2,f),x1*x2*y-1);

o3 : Ideal of R

i4 : PD = primaryDecomposition I;

i5 : netList PD

     +------------------------------------------------------------------------
o5 = |ideal (x1 + 1, a*y - b*y - c + d, x2*y + 1, x2*c - x2*d + a - b)
     +------------------------------------------------------------------------
     |ideal (a*y - d, x2*d + b, x1*d + c, b*c - a*d, x2*c + a, x1*b + a, x1*x2*y - 1)
     +------------------------------------------------------------------------
     |                           2           2                          2
     |ideal (a - b, x2*y + x1 + 2, c  - 2c*d + d , x1*c - x1*d + c - d, x1  + 2x1 + 1, 2x
     +------------------------------------------------------------------------

i6 : apply(PD,i->eliminate(i,{x1,x2,y}))
```

o6 = $\{$ideal(), ideal$(b\,c - a\,d)$, ideal $(a - b,\ c^2 - 2\,c\,d + d^2)\}$

Principal Landau determinant

$$Q \subset \mathrm{conv}(A) \quad \text{face}, \qquad \mathcal{G}_G = \sum_{a\in A} c_a\underbrace{(m,M,s,t)}_{z}\,\alpha^a, \qquad \mathcal{G}_{G,Q} = \sum_{a\in A\cap Q} c_a\underbrace{(m,M,s,t)}_{z}\,\alpha^a$$



$$Y_{G,Q}(\mathcal{E}) \;=\; \{(\alpha, z) \in (\mathbb{C}^*)^E \times \mathcal{E} \;:\; \mathcal{G}_{G,Q} \;=\; 0, \partial_\alpha \mathcal{G}_{G,Q} \;=\; 0\}$$

$$Y_{G,Q}(\mathcal{E}) \;=\; \bigcup_{i\in \mathbb{I}(G,Q)} Y^{(i)}_{G,Q}(\mathcal{E})$$ irreducible decomposition

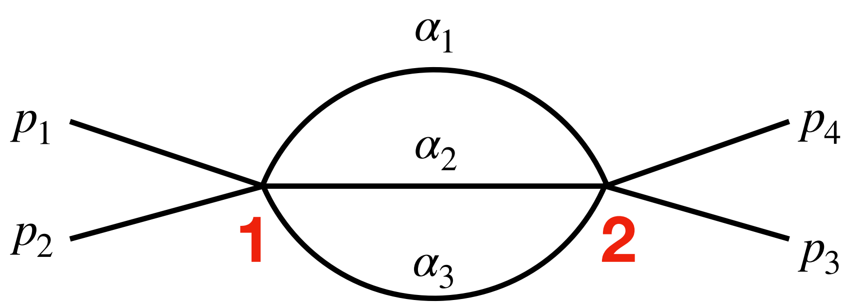
$$\nabla^{(i)}_{G,Q}(\mathcal{E}) \;=\; \overline{\pi_{\mathcal{E}}\left(Y^{(i)}_{G,Q}(\mathcal{E})\right)}$$ projection to $\mathcal{E}$

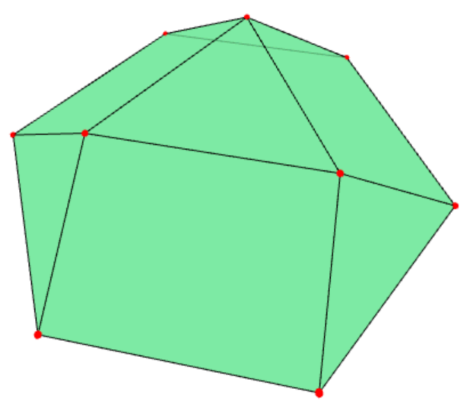
$$\mathbb{I}(G,Q)_1 \;=\; \{i \in \mathbb{I}(G,Q) \;:\; \mathrm{codim}\,\nabla^{(i)}_{G,Q}(\mathcal{E}) \;=\; 1\}$$

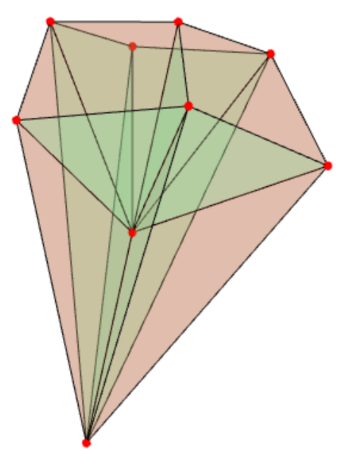
Definition. The **principal Landau determinant** of the diagram G with respect to the parameter space $\mathcal{E}$ is the defining polynomial E_G of

$$\mathrm{PLD}_G(\mathcal{E}) \;=\; \bigcup_{Q\subset \mathrm{conv}(A)} \; \bigcup_{i\in \mathbb{I}(G,Q)_1} \nabla^{(i)}_{G,Q}(\mathcal{E})$$

Sunrise solution: PLD.jl



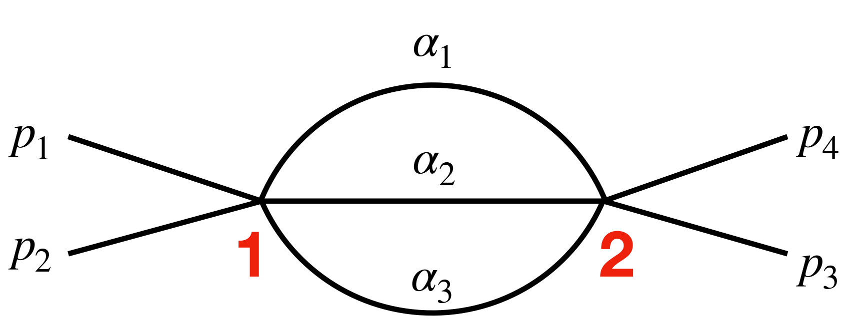


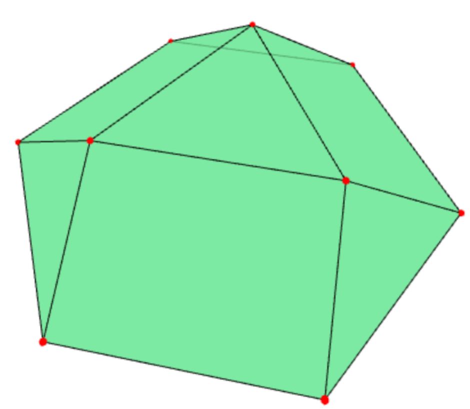


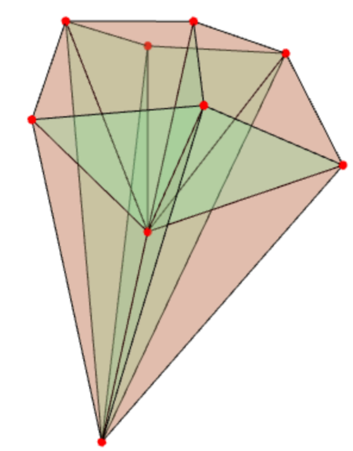
```
julia> edges = [[1,2],[1,2],[1,2]]; nodes = [1,1,2,2]; @var m[1:3] M[1:4];

julia> getPLD(edges, nodes; internal_masses = m, external_masses = M, method = :num)
------- codim = 3, 9 faces
codim: 3, face: 1/9, weights: [-1, 0, 1], discriminant: m1
New discriminants after codim 3, face 1/9. The list is: m1
codim: 3, face: 2/9, weights: [-1, 1, 0], discriminant: m1
codim: 3, face: 3/9, weights: [0, -1, 1], discriminant: m2
New discriminants after codim 3, face 3/9. The list is: m1, m2
codim: 3, face: 4/9, weights: [2, 2, 4], discriminant: 1
New discriminants after codim 3, face 4/9. The list is: 1, m1, m2
codim: 3, face: 5/9, weights: [0, 1, -1], discriminant: m3
New discriminants after codim 3, face 5/9. The list is: 1, m1, m2, m3
codim: 3, face: 6/9, weights: [2, 4, 2], discriminant: 1
codim: 3, face: 7/9, weights: [1, -1, 0], discriminant: m2
codim: 3, face: 8/9, weights: [1, 0, -1], discriminant: m3
codim: 3, face: 9/9, weights: [4, 2, 2], discriminant: 1
Unique discriminants after codim 3: 1, m1, m2, m3
------- codim = 2, 15 faces
codim: 2, face: 1/15, weights: [-1, 0, 0], discriminant: m1
codim: 2, face: 2/15, weights: [-1, -1, 0], discriminant: 1
codim: 2, face: 3/15, weights: [0, 1, 2], discriminant: 1
codim: 2, face: 4/15, weights: [-1, 0, -1], discriminant: 1
codim: 2, face: 5/15, weights: [0, 2, 1], discriminant: 1
codim: 2, face: 6/15, weights: [1, 0, 2], discriminant: 1
codim: 2, face: 7/15, weights: [0, -1, 0], discriminant: m2
codim: 2, face: 8/15, weights: [1, 2, 2], discriminant: 1
codim: 2, face: 9/15, weights: [2, 1, 2], discriminant: 1
codim: 2, face: 10/15, weights: [1, 2, 0], discriminant: 1
codim: 2, face: 11/15, weights: [0, 0, -1], discriminant: m3
codim: 2, face: 12/15, weights: [2, 2, 1], discriminant: 1
codim: 2, face: 13/15, weights: [0, -1, -1], discriminant: 1
codim: 2, face: 14/15, weights: [2, 0, 1], discriminant: 1
codim: 2, face: 15/15, weights: [2, 1, 0], discriminant: 1
Unique discriminants after codim 2: 1, m1, m2, m3
```

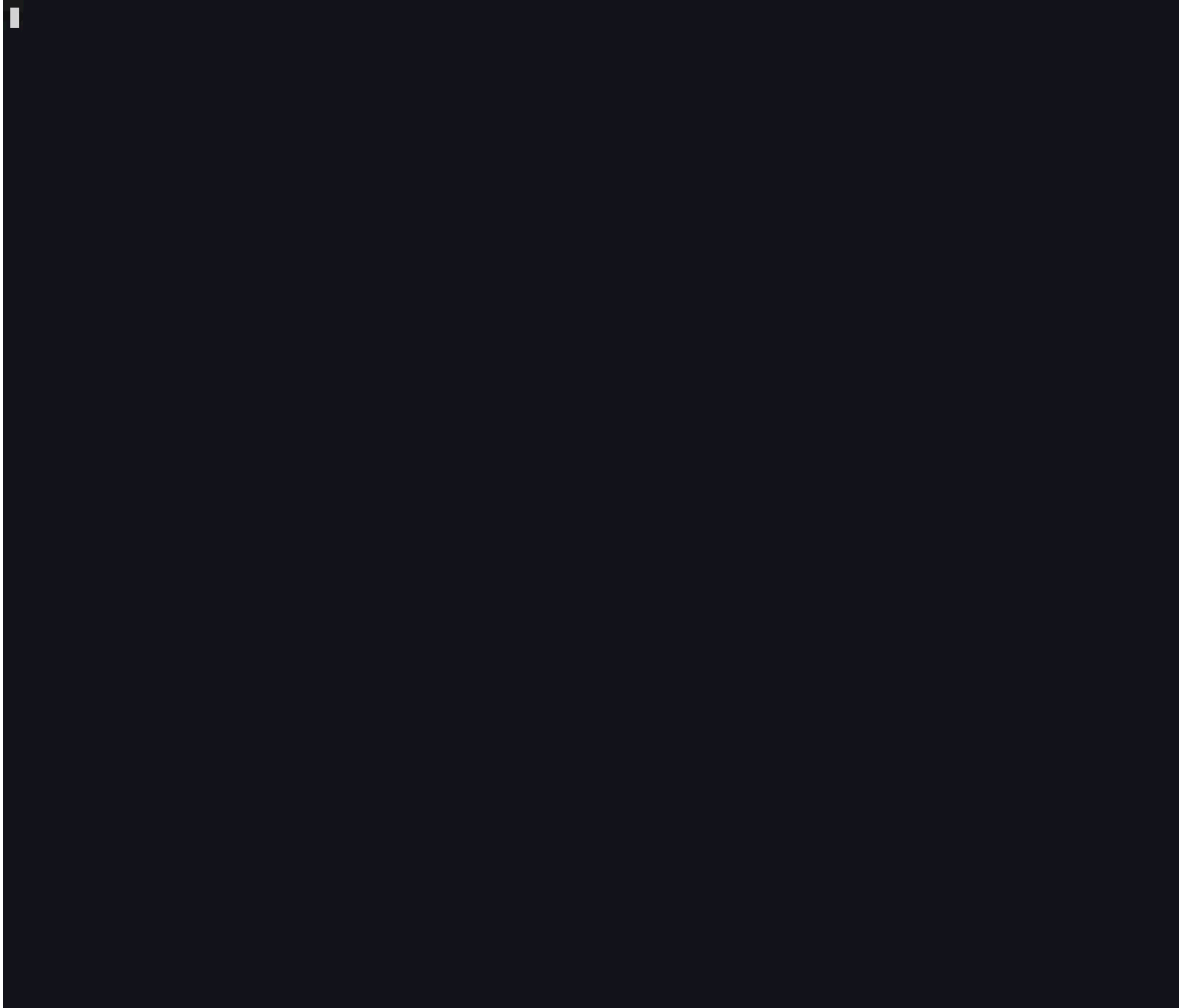
Sunrise solution: PLD.jl

```
-------- codim = 1, 8 faces
codim: 1, face: 1/8, weights: [-1, -1, -1], discriminant: m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ +
4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*
m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s
^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s
New discriminants after codim 1, face 1/8. The list is: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*
m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s
 + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m
₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
codim: 1, face: 2/8, weights: [0, 1, 1], discriminant: 1
codim: 1, face: 3/8, weights: [0, 0, 1], discriminant: 1
codim: 1, face: 4/8, weights: [0, 1, 0], discriminant: 1
codim: 1, face: 5/8, weights: [1, 0, 1], discriminant: 1
codim: 1, face: 6/8, weights: [1, 1, 1], discriminant: 1
codim: 1, face: 7/8, weights: [1, 1, 0], discriminant: 1
codim: 1, face: 8/8, weights: [1, 0, 0], discriminant: 1
Unique discriminants after codim 1: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
-------- codim = 0, 1 faces
codim: 0, face: 1/1, weights: [0, 0, 0], discriminant: s
Unique discriminants after codim 0: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
```

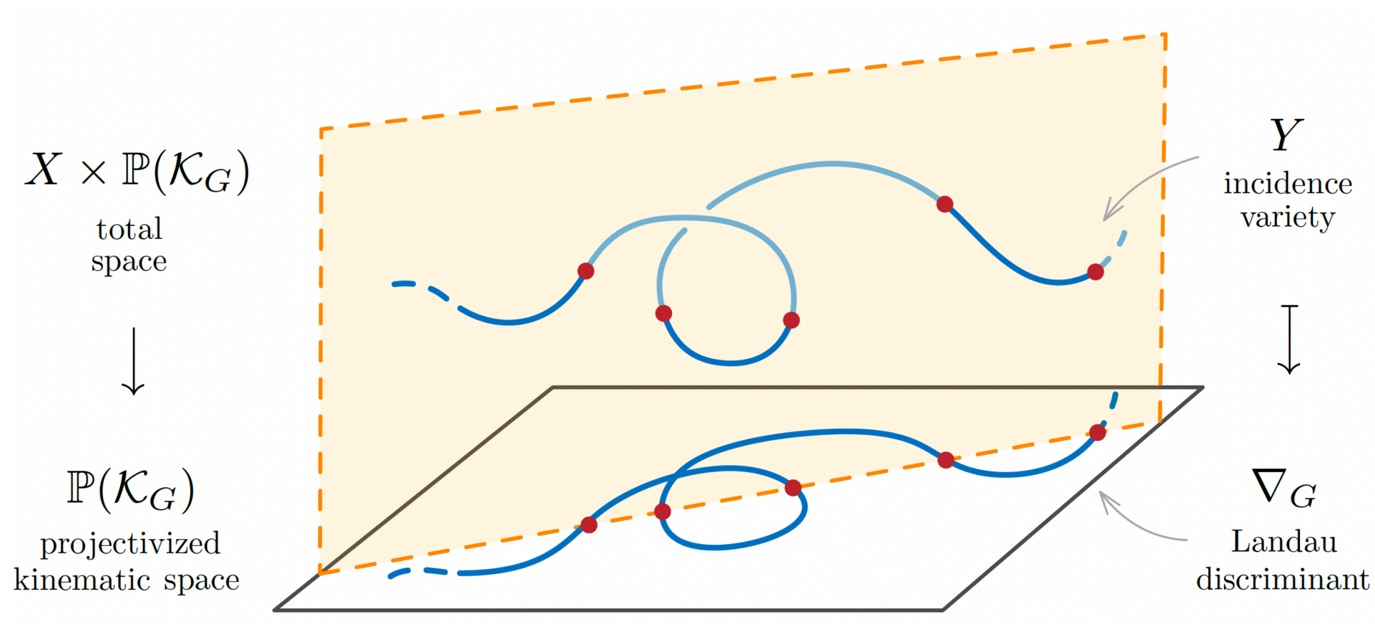






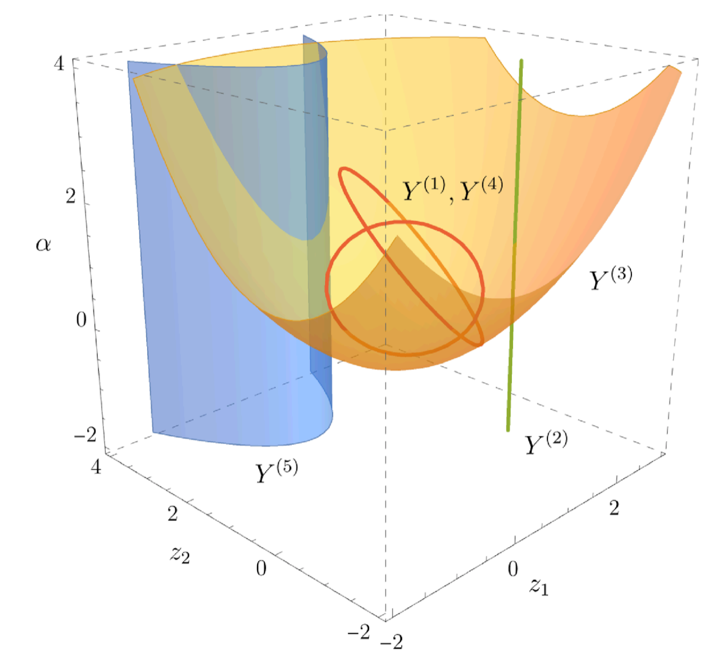


method = :num



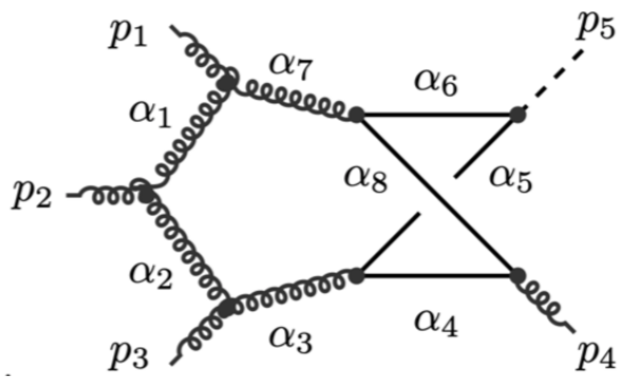
S. Mizera and S. Telen, *Landau discriminants*, *JHEP* **08** (2022) 200 [2109.08036].

Homotopy Continuation.jl **+ numerical interpolation**



1. Compute samples on the incidence variety as regular solutions to systems of polynomial equations. We pick up points on desired components + dominant components
2. Filter out such dominant points, and continue with the remaining samples
3. Divide the samples into groups corresponding to their incidence components
4. Deduce the degree of the projected components from the number of samples
5. Collect enough samples to find a unique interpolant

Hj-npl-pentb



arXiv > hep-ph > arXiv:2306.15431

High Energy Physics – Phenomenology

[Submitted on 27 Jun 2023]

All Two-Loop Feynman Integrals for Five-Point One-Mass Scattering

Samuel Abreu, Dmitry Chicherin, Harald Ita, Ben Page, Vasily Sotnikov, Wladimir Tschernow, Simone Zoia

M2^10*s34^2 - 2*M2^9*m2*s12*s34 + 4*M2^9*m2*s34*s45 - 4*M2^9*s12*s34^2 - 4*M2^9*s34^3 - 4*M2^9*s34^2*s45 +
M2^8*m2^2*s12^2 + 8*M2^8*m2*s12^2*s34 + 40*M2^8*m2*s12*s34^2 - 12*M2^8*m2*s12*s34*s45 - 12*M2^8*m2*s34^2*s45 -
12*M2^8*m2*s34*s45^2 + 6*M2^8*s12^2*s34^2 + 12*M2^8*s12*s34^3 + 16*M2^8*s12*s34^2*s45 + 6*M2^8*s34^4 + 12*M2^8*s34^3*s45 +
6*M2^8*s34^2*s45^2 - 4*M2^7*m2^2*s12^3 - 68*M2^7*m2^2*s12^2*s34 - 4*M2^7*m2^2*s12^2*s45 + 136*M2^7*m2^2*s12*s34*s45 -
12*M2^7*m2*s12^3*s34 - 88*M2^7*m2*s12^2*s34^2 + 8*M2^7*m2*s12^2*s34*s45 - 140*M2^7*m2*s12*s34^3 -
108*M2^7*m2*s12*s34^2*s45 + 48*M2^7*m2*s12*s34*s45^2 + 24*M2^7*m2*s34^3*s45 + 24*M2^7*m2*s34^2*s45^2 +
12*M2^7*m2*s34*s45^3 - 4*M2^7*s12^3*s34^2 - 12*M2^7*s12^2*s34^3 - 24*M2^7*s12^2*s34^2*s45 - 12*M2^7*s12*s34^4 -
36*M2^7*s12*s34^3*s45 - 24*M2^7*s12*s34^2*s45^2 - 4*M2^7*s34^5 - 12*M2^7*s34^4*s45 - 12*M2^7*s34^3*s45^2 - 4*M2^7*s34^2*s45^3 +
32*M2^6*m2^3*s12^3 + 6*M2^6*m2^2*s12^4 + 140*M2^6*m2^2*s12^3*s34 + 16*M2^6*m2^2*s12^3*s45 + 518*M2^6*m2^2*s12^2*s34^2 -
144*M2^6*m2^2*s12^2*s34*s45 + 6*M2^6*m2^2*s12^2*s45^2 - 432*M2^6*m2^2*s12*s34^2*s45 - 408*M2^6*m2^2*s12*s34*s45^2 +
48*M2^6*m2^2*s34^2*s45^2 + 8*M2^6*m2*s12^4*s34 + 56*M2^6*m2*s12^3*s34^2 + 8*M2^6*m2*s12^3*s34*s45 -
40*M2^6*m2*s12^2*s34^3 + 268*M2^6*m2*s12^2*s34^2*s45 - 72*M2^6*m2*s12^2*s34*s45^2 + 168*M2^6*m2*s12*s34^4 +
556*M2^6*m2*s12*s34^3*s45 + 132*M2^6*m2*s12*s34^2*s45^2 - 52*M2^6*m2*s12*s34*s45^3 - 40*M2^6*m2*s34^4*s45 -
48*M2^6*m2*s34^3*s45^2 - 12*M2^6*m2*s34^2*s45^3 - 4*M2^6*m2*s34*s45^4 + M2^6*s12^4*s34^2 + 4*M2^6*s12^3*s34^3 +
16*M2^6*s12^3*s34^2*s45 + 6*M2^6*s12^2*s34^4 + 36*M2^6*s12^2*s34^3*s45 + 36*M2^6*s12^2*s34^2*s45^2 +
4*M2^6*s12*s34^5 + 24*M2^6*s12*s34^4*s45 + 36*M2^6*s12*s34^3*s45^2 + 16*M2^6*s12*s34^2*s45^3 + M2^6*s34^6 +
4*M2^6*s34^5*s45 + 6*M2^6*s34^4*s45^2 + 4*M2^6*s34^3*s45^3 + M2^6*s34^2*s45^4 - 64*M2^5*m2^3*s12^4 - 640*M2^5*m2^3*s12^3*s34 -
128*M2^5*m2^3*s12^3*s45 + 1280*M2^5*m2^3*s12^2*s34*s45 - 4*M2^5*m2^2*s12^5 - 76*M2^5*m2^2*s12^4*s34 -
24*M2^5*m2^2*s12^4*s45 + 116*M2^5*m2^2*s12^3*s34^2 - 128*M2^5*m2^2*s12^3*s34*s45 - 24*M2^5*m2^2*s12^3*s45^2 -
324*M2^5*m2^2*s12^2*s34^3 - 2200*M2^5*m2^2*s12^2*s34^2*s45 + 840*M2^5*m2^2*s12^2*s34*s45^2 - 4*M2^5*m2^2*s12^2*s45^3 -
352*M2^5*m2^2*s12*s34^3*s45 + 1512*M2^5*m2^2*s12*s34^2*s45^2 + 408*M2^5*m2^2*s12*s34*s45^3 - 96*M2^5*m2^2*s34^3*s45^2 -
96*M2^5*m2^2*s34^2*s45^3 - 2*M2^5*m2*s12^5*s34 - 8*M2^5*m2*s12^4*s34^2 - 12*M2^5*m2*s12^4*s34*s45 -
76*M2^5*m2*s12^3*s34^3 - 164*M2^5*m2*s12^3*s34^2*s45 + 48*M2^5*m2*s12^3*s34*s45^2 + 120*M2^5*m2*s12^2*s34^4 +
224*M2^5*m2*s12^2*s34^3*s45 - 348*M2^5*m2*s12^2*s34^2*s45^2 + 88*M2^5*m2*s12^2*s34*s45^3 - 66*M2^5*m2*s12*s34^5 -
676*M2^5*m2*s12*s34^4*s45 - 894*M2^5*m2*s12*s34^3*s45^2 - 100*M2^5*m2*s12*s34^2*s45^3 + 18*M2^5*m2*s12*s34*s45^4 +
36*M2^5*
12*M2^5*
24*M2^5*

M2^2*s34^2*(M2^2 - M2*s12 - M2*s34 - M2*s45 + s12*s45)^4

4*M2^5*s12*s34^2*s45^4 + 256*M2^4*m2^4*s12^4 + 32*M2^4*m2^3*s12^5 - 64*M2^4*m2^3*s12^4*s34 + 256*M2^4*m2^3*s12^4*s45 +
160*M2^4*m2^3*s12^3*s34^2 + 1216*M2^4*m2^3*s12^3*s34*s45 + 192*M2^4*m2^3*s12^3*s45^2 + 1600*M2^4*m2^3*s12^2*s34^2*s45 -
3840*M2^4*m2^3*s12^2*s34*s45^2 - 2656*M2^4*m2^3*s12*s34^2*s45^2 + M2^4*m2^2*s12^6 + 4*M2^4*m2^2*s12^5*s34 + 16*M2^4*m2^2*s12^5*s45 +
134*M2^4*m2^2*s12^4*s34^2 + 144*M2^4*m2^2*s12^4*s34*s45 + 36*M2^4*m2^2*s12^4*s45^2 - 252*M2^4*m2^2*s12^3*s34^3 -

Conjectures

$\mathrm{PLD}_{\mathrm{G}}(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$ but we know $\mathrm{PLD}_{\mathrm{G}}(\mathscr{E}) \not\supset \nabla_\chi(\mathscr{E})$

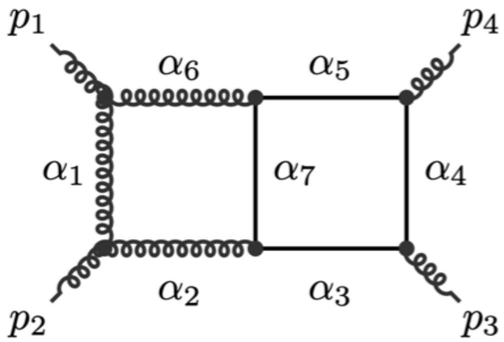
```
julia> χdiscriminantQ(U+F,pp,vv,unique(vcat(Δ...)))
Generic |Euler characteristic|, χ* = 7
candidates = Any[m₁, m₂, m₃, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^
2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 -
 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂
^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 -
 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s]
Subspace m₁ has χ = 4 < χ*
Subspace m₂ has χ = 4 < χ*
Subspace m₃ has χ = 4 < χ*
Subspace m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2
 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m
₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 +
 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s
^2 - 4*m₃*s^3 + s^4 has χ = 6 < χ*
Subspace s has χ = 4 < χ*
```

$$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E}) = \text{Landau variety} \subset \text{result of } \texttt{HyperInt}$$

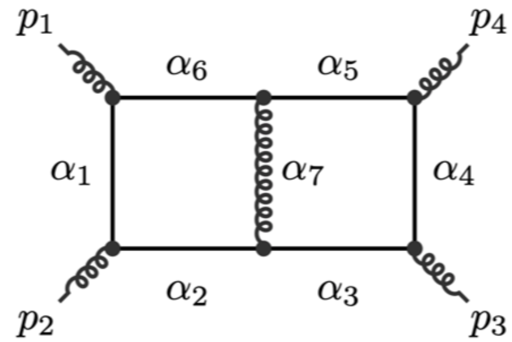
F.C.S. Brown, *On the periods of some Feynman integrals*, 0910.0114.

E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Comput. Phys. Commun. **188** (2015) 148 [1403.3385].

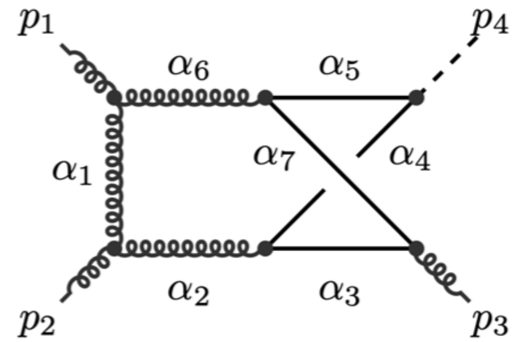
Can we compute $\nabla_\chi(\mathscr{E})$ from the primary decomposition of an ideal in the Cox ring of $\mathscr{X}_A$?



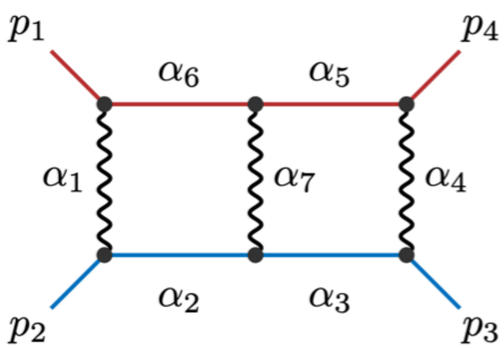
(a) Double-box with an inner massive loop, $G =$ `inner-dbox`



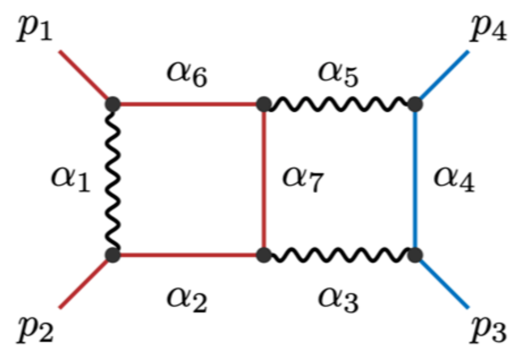
(b) Double-box with an outer massive loop, $G =$ `outer-dbox`



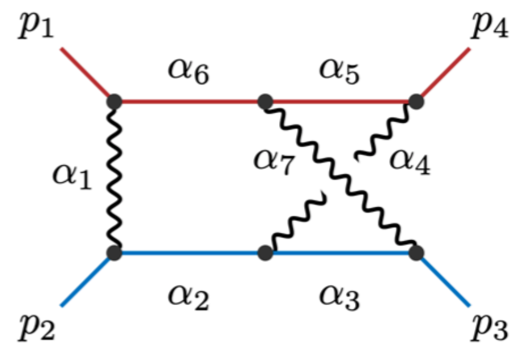
(c) Non-planar double-box for Higgs + jet production, $G =$ `Hj-npl-dbox`



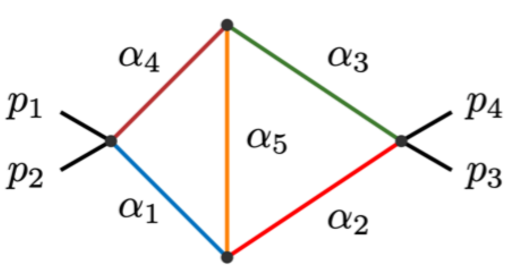
(d) Double-box for Bhabha scattering, $G =$ `Bhabha-dbox`



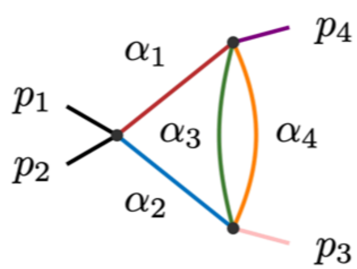
(e) Second double-box for Bhabha scattering, $G =$ `Bhabha2-dbox`



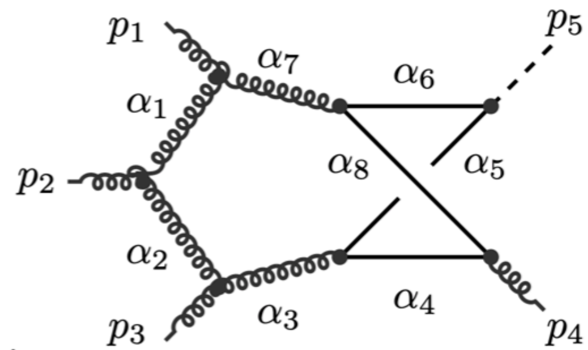
(f) Non-planar double-box for Bhabha scattering, $G =$ `Bhabha-npl-dbox`



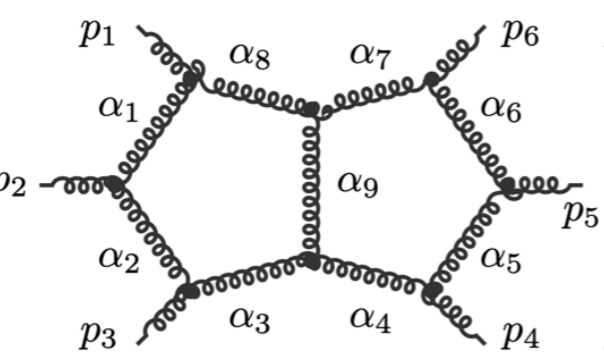
(g) Kite diagram with generic masses, $G =$ `kite`



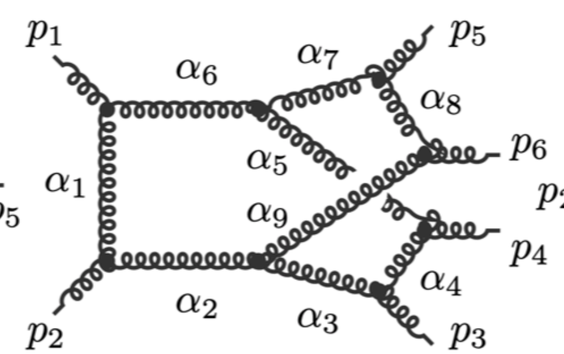
(h) Parachute diagram with generic masses, $G =$ `par`



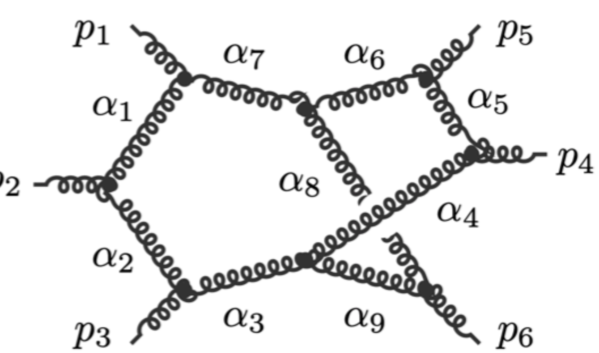
(i) Non-planar penta-box fir Higgs + jet production, $G =$ `Hj-npl-pentb`



(j) Massless planar double-pentagon, $G =$ `dpent`



(k) Massless non-planar double-pentagon, $G =$ `npl-dpent`



(l) Second massless non-planar double-pentagon, $G =$ `npl-dpent2`

Database available at

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TENORS
Tensor modEliNg, geOmetRy and optimiSation
Marie Skłodowska-Curie Doctoral Network
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