The Tait Institute

IHES Workshop: Amplitudes and periods

Multi-parton Webs: non-abelian exponentiation theorem for the multi-parton case

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Webs

Diagrammatic soft-gluon exponentiation

Non-abelian, multiparton case (2010) :

EG-Laenen-Stavenga-White & Mitov-Sterman-Sung

Non-abelian, colour singlet case (1983) :

Gatheral, Frenkel-Taylor

Abelian case (1961):

Yennie-Frautschi-Suura

Diagrammatic exponentiation in multi-leg amplitudes

Motivation:

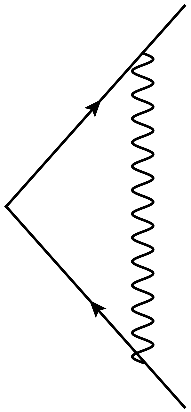
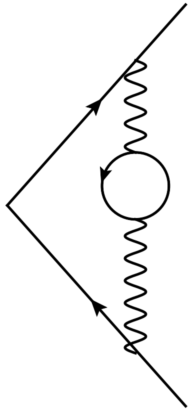
- The exponent is simpler – we wish to compute it directly!

- Non-Abelian exponentiation theorem for the colour-singlet case (cusp) :
  Diagrams contributing to the exponent have
  a) Irreducible (maximally non-Abelian) colour structure
  b) No subdivergences.

  How does this generalize to the multi-leg case?

Exponentiation in an Abelian theory

➢ The exponent only receives contributions from connected diagrams:

$$\mathcal{S} = \exp\Bigg\{$$

$$+$$

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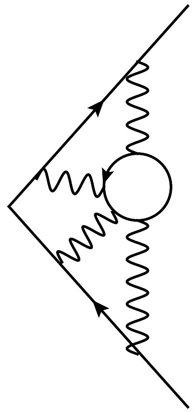
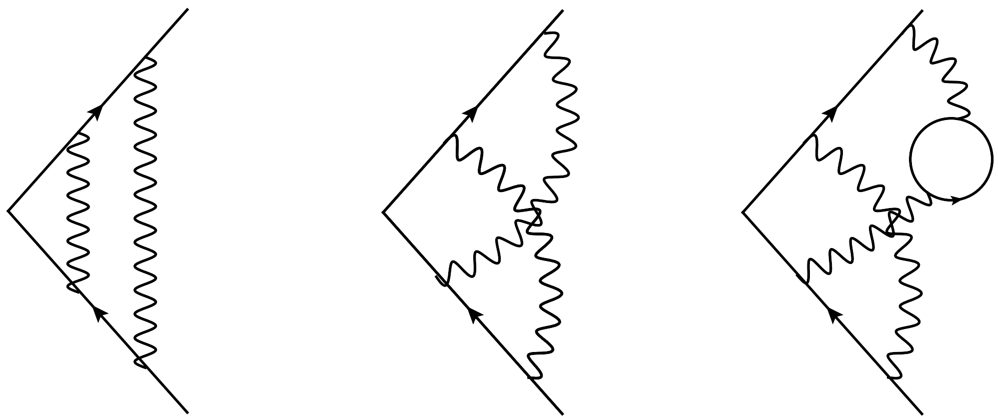

$$+\cdots\Bigg\}$$

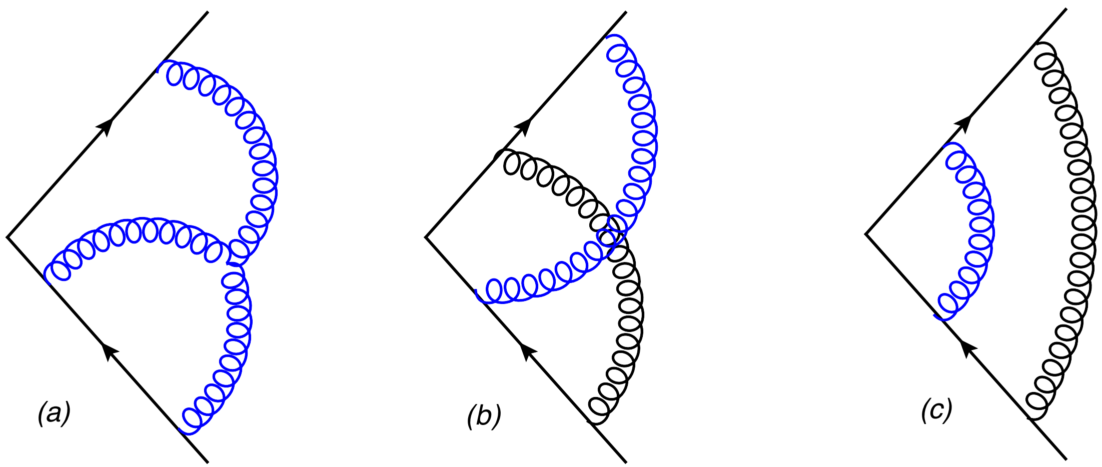
➢ Expanding the exponential exactly reproduces all disconnected diagrams:



Non-abelian exponentiation (colour-singlet case)

Two new features:

- 3 and 4 gluon vertices – more complicated connected diagrams
- Non-commuting generators for multiple emissions from a given Wilson line – colour-connected diagrams

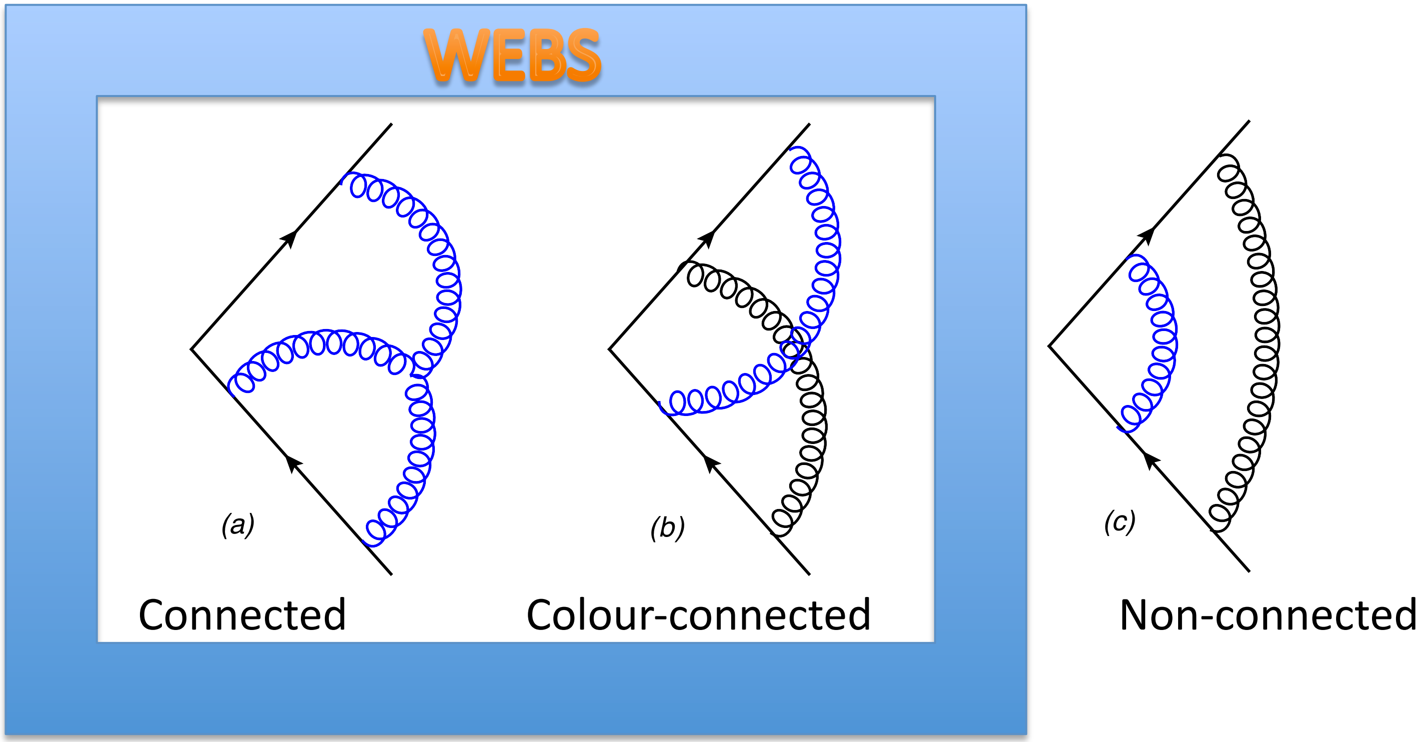


Connected | Colour-connected | Non-connected

Non-abelian exponentiation (colour-singlet case)

Two new features:

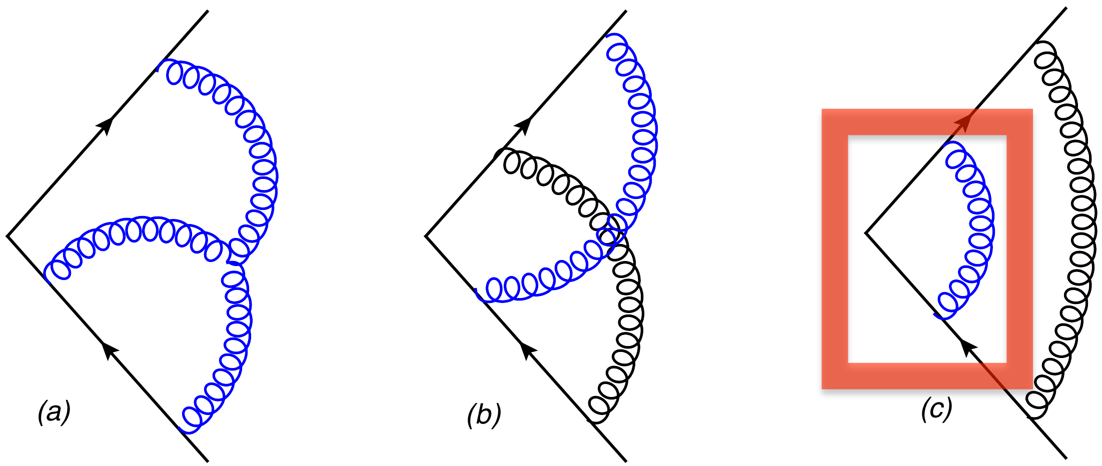
- ➢ 3 and 4 gluon vertices – more complicated connected diagrams
- ➢ Non-commuting generators for multiple emissions from a given Wilson line – colour-connected diagrams



Non-abelian exponentiation

colour-singlet case (cusp configuration)

- *Reducible colour structure:* Diagram D is reducible if it can be decomposed into two subdiagrams such that $C(D) = C(H_1)C(H_2)$

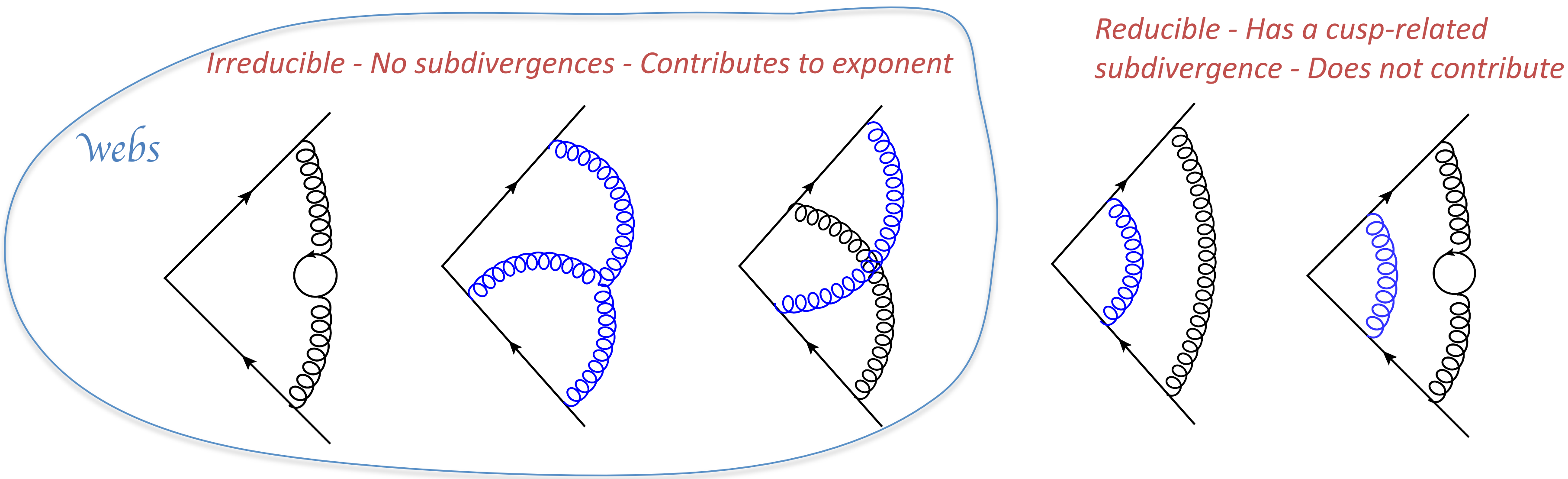


Irreducible Irreducible Reducible

- Reducible diagrams have subdivergences.
- Webs have irreducible colour structure; they have no subdivergences.

Non-abelian exponentiation (colour-singlet case)

- In the colour-singlet case diagrams fall into two classes

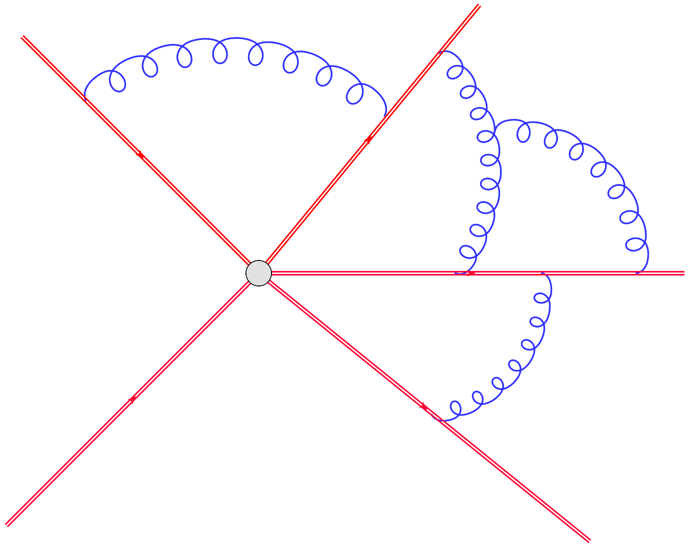


- This goes hand-in-hand with the renormalization of the cusp: webs have a single overall divergence – generating an $\mathcal{O}(1/\epsilon)$ singularity at each order (plus higher poles due to running coupling):

$$Z = \exp\left\{\frac{1}{2}\int_{\mu^2}^{\infty}\frac{d\lambda^2}{\lambda^2}\,\Gamma_{\mathcal{S}}(\alpha_s(\lambda^2))\right\} = \exp\left\{\frac{1}{2\epsilon}\Gamma_{\mathcal{S}}^{(1)}\,\alpha_s + \frac{1}{4\epsilon}\Gamma_{\mathcal{S}}^{(2)}\,\alpha_s^2 + \frac{1}{6\epsilon}\Gamma_{\mathcal{S}}^{(3)}\,\alpha_s^3 + \mathcal{O}(\alpha_s^4)\right\}$$

Diagrams contributing to the exponent have no subdivergences!

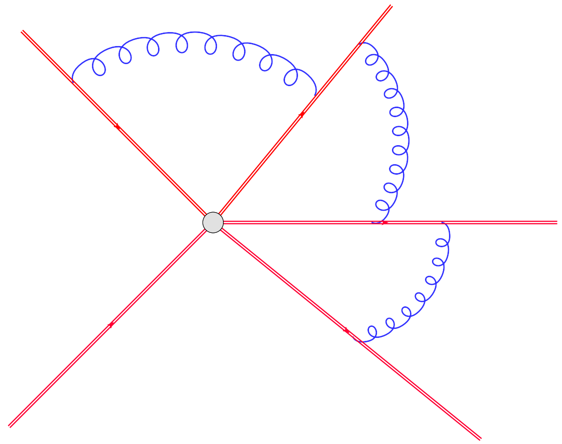
Non-abelian exponentiation: the multi-leg case



In the multi-leg case this separation breaks down:
reducible diagrams do contribute to the exponent …

- Is the exponent still ``maximaly non-abelian''?
- Reducible diagrams have subdivergences – is this consistent with renormalizability?
- What is the proper generalization of WEBS to the multi-leg case?
  How can one compute directly the exponent?

Consequences of renormalization in multi-leg amplitudes



In the multiparton case:

$$Z = \exp\left\{ \frac{1}{2\epsilon}\Gamma_{\mathcal{S}}^{(1)}\,\alpha_s + \left(\frac{1}{4\epsilon}\Gamma_{\mathcal{S}}^{(2)} - \frac{b_0}{4\epsilon^2}\Gamma_{\mathcal{S}}^{(1)}\right)\alpha_s^2 \right.$$
$$\left. + \left(\frac{1}{6\epsilon}\Gamma_{\mathcal{S}}^{(3)} + \frac{1}{48\epsilon^2}\left[\Gamma_{\mathcal{S}}^{(1)},\Gamma_{\mathcal{S}}^{(2)}\right] - \frac{1}{6\epsilon^2}\left(b_0\Gamma_{\mathcal{S}}^{(2)} + b_1\Gamma_{\mathcal{S}}^{(1)}\right) + \frac{b_0^2}{6\epsilon^3}\Gamma_{\mathcal{S}}^{(1)}\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4)\right\}$$

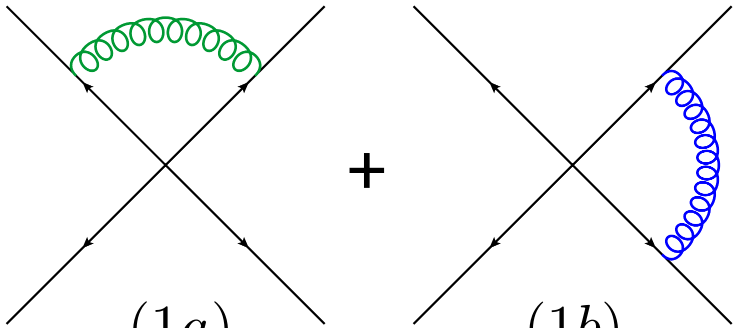
multiple poles occur due to two distinct reasons:

1) running coupling
2) commutators (only in the multi-leg case, and beyond the planar limit)

Specific subdivergences of the multi-eikonal vertex survive in the exponent, BUT all multiple poles are predicted by lower orders. Only $\mathcal{O}(1/\epsilon)$ are new.
In particular, there no $1/\epsilon^n$ at $\mathcal{O}(\alpha_s^n)$.

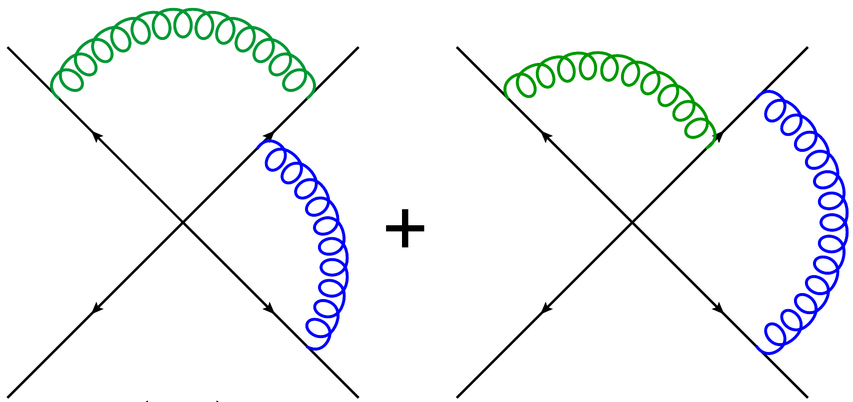
Non-abelian exponentiation: the multi-leg case

Exponentiating:



$(1a)$ $(1b)$

At 2-loops, do we get



$(2a)$ $(2b)$

?

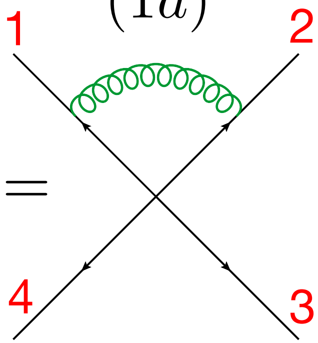
Non-abelian exponentiation: the multi-leg case

Exponentiating:

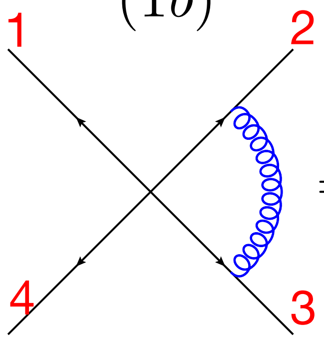
$(1a)$

$(1b)$

$$D_{(1a)} + D_{(1b)} =$$



\+



$$= \mathcal{F}(1a)\,\mathrm{T}_1^{(a)}\mathrm{T}_2^{(a)} + \mathcal{F}(1b)\,\mathrm{T}_2^{(b)}\mathrm{T}_3^{(b)}$$

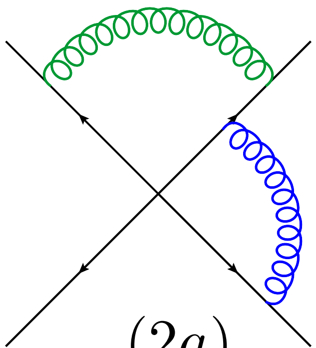
At 2-loops we get:

$$\frac{1}{2}\left[D_{(1a)} + D_{(1b)}\right]^2 = \frac{1}{2}\left[D_{(1a)}D_{(1b)} + D_{(1b)}D_{(1a)} + \cdots\right]$$

$$= \frac{1}{2}\mathcal{F}(1a)\mathcal{F}(1b)\;\mathrm{T}_1^{(a)}\left[\mathrm{T}_2^{(a)}\mathrm{T}_2^{(b)} + \mathrm{T}_2^{(b)}\mathrm{T}_2^{(a)}\right]\mathrm{T}_3^{(b)}$$

$$= \frac{1}{2}\Big[\mathcal{F}(2a) + \mathcal{F}(2b)\Big]\quad\Big[C(2a) + C(2b)\Big]$$

While



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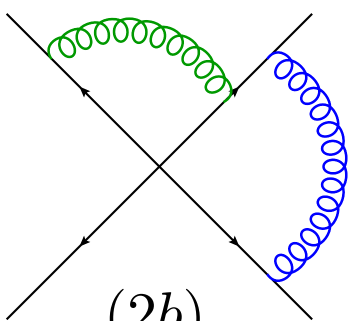


$(2a)$

$(2b)$

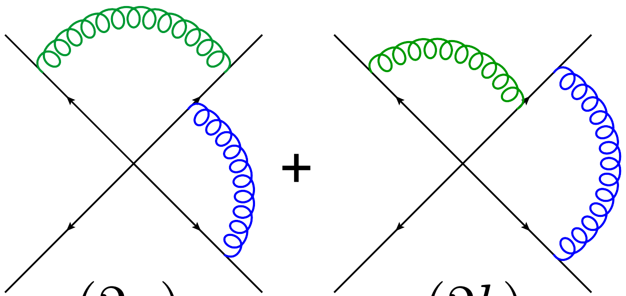
$$= \mathcal{F}(2a)\,C(2a) + \mathcal{F}(2b)\,C(2b)$$

Non-abelian exponentiation: the multi-leg case

Exponentiating 1-loop diagrams yields:

$$\frac{1}{2}\left[D_{(1a)} + D_{(1b)}\right]^2 = \frac{1}{2}\Big[\mathcal{F}(2a) + \mathcal{F}(2b)\Big] \quad \Big[C(2a) + C(2b)\Big]$$

While the 2-loop amplitude is:



$(2a)$ $(2b)$

$$= \mathcal{F}(2a)\, C(2a) + \mathcal{F}(2b)\, C(2b)$$

The 2-loop contribution to the exponent is therefore:

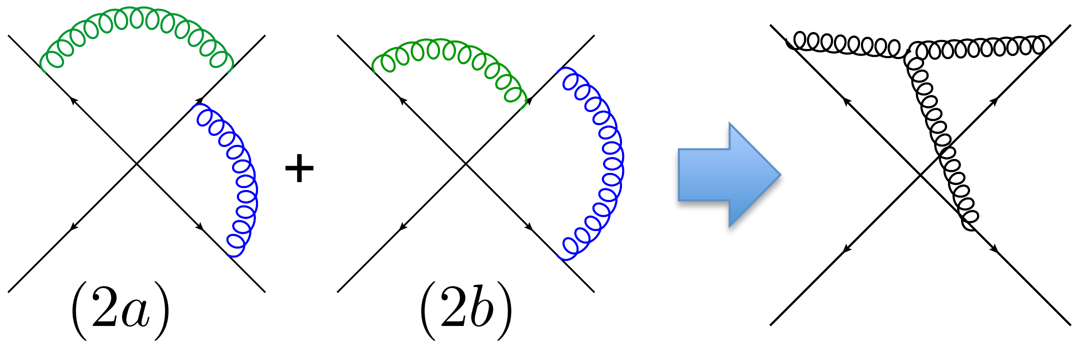
$$\frac{1}{2}\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big] \quad \Big[C(2a) - C(2b)\Big]$$

In the multi-leg case, *reducible diagrams do contribute to the exponent*!

Non-abelian exponentiation: the multi-leg case

The 2-loop contribution to the exponent

is the *anti-symmetric part* of



$$= \frac{1}{2}\underbrace{\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big]}_{\mathcal{O}(1/\epsilon)} \times \underbrace{\Big[C(2a) - C(2b)\Big]}_{\mathrm{i}f^{abc}\mathrm{T}_1^{(a)}\mathrm{T}_3^{(b)}\mathrm{T}_2^{(c)}}$$

These properties (single pole, maximally non-Abelian colour structure) are familiar from the colour singlet case.

Non-abelian exponentiation: the multi-leg case

- In contrast to the colour-singlet case, reducible diagrams enter the exponent. Each diagram enters the exponent with a *modified colour factor* $\widetilde{C}(D)$

- Individual diagrams do not have "web properties", but only particular linear combinations that do - enter the exponent:

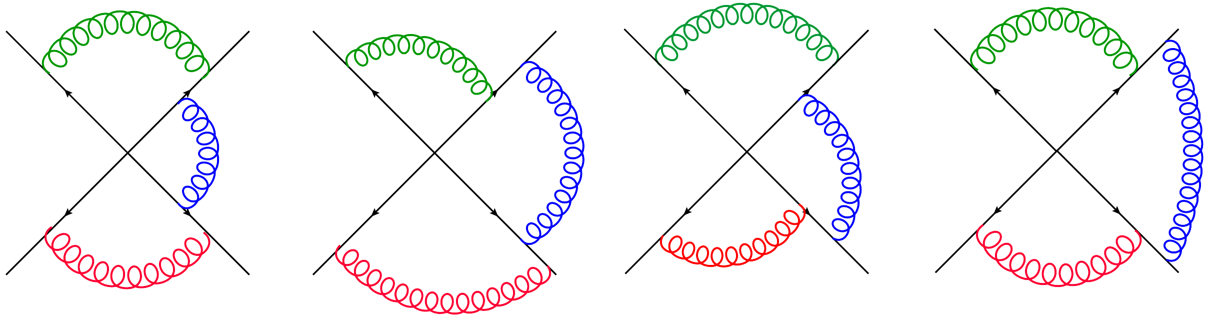
$$\mathcal{S} = \exp\left[\sum_i W_i\right] \qquad\qquad W_i = \sum_{\{D\}_i} \mathcal{F}(D)\, \widetilde{C}(D)$$

- *modified colour factors* $\widetilde{C}(D)$ are linear combination of (ordinary) colour factors of diagrams that are obtained by permuting attachments to the Wilson lines, so:

$$W_i = \sum_D \mathcal{F}(D) \underbrace{\sum_{D'} R_{DD'}\, C(D')}_{\widetilde{C}(D)} = \mathcal{F}^T \underbrace{R}_{\text{web mixing matrix}} C$$

- Using the replica trick we derived a general combinatorial formula for R

Three-loop example



The entire web contributes:

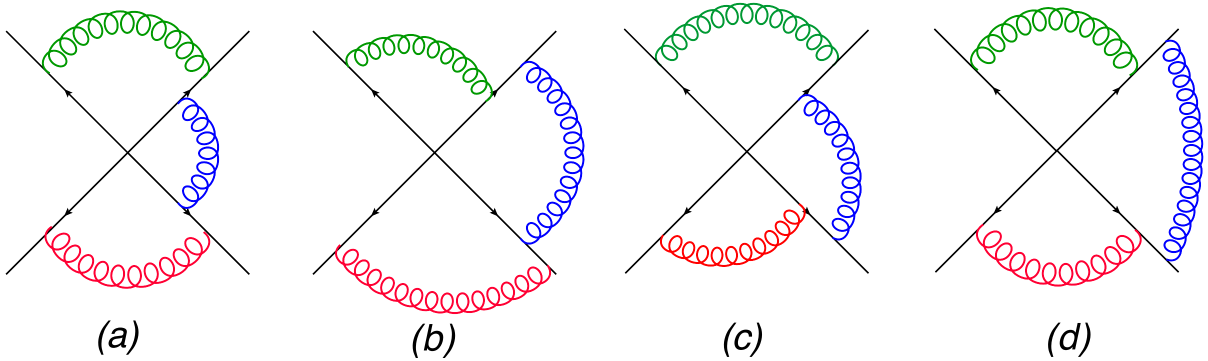
$$= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}$$

Kinematics

Web mixing matrix R

Colour

Three-loop example



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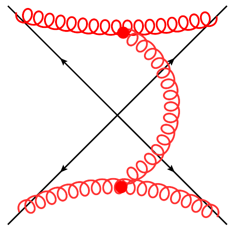
$$= \underbrace{\frac{1}{6}\Big(\mathcal{F}(3a) - 2\mathcal{F}(3b) - 2\mathcal{F}(3c) + \mathcal{F}(3d)\Big)}_{\text{subdivergences cancel}} \times \underbrace{\Big(C(3a) - C(3b) - C(3c) + C(3d)\Big)}$$



Properties of web mixing matrices

Any web mixing matrix R admits:

A. It is *idempotent:*

$$R^2 = R$$

R is diagonalisable, with all its eigenvalues 0 or 1.

B. Its rows sum to zero:

$$\sum_{D'} R_{DD'} = 0$$

C. Its columns, weighted by a symmetry factor s(D), sum to zero:

$$\sum_{D} s(D)\, R_{DD'} = 0$$

s(D) counts the number of way of sequentially shrinking subdiagrams to the vertex.

Properties of web mixing matrices

Any web mixing matrix R admits:

A. It is *idempotent:*

$$R^2 = R$$

R is a projection operator

R is diagonalisable, with all its eigenvalues 0 or 1.

B. Its rows sum to zero:

$$\sum_{D'} R_{DD'} = 0$$

Colour-symmetric terms are projected out

C. Its columns, weighted by a symmetry factor s(D), sum to zero:

$$\sum_{D} s(D)\, R_{DD'} = 0$$

Subdivergences cancel

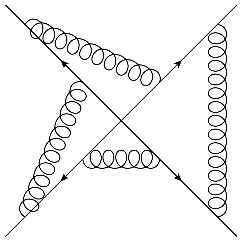
s(D) counts the number of way of sequentially shrinking subdiagrams to the vertex.

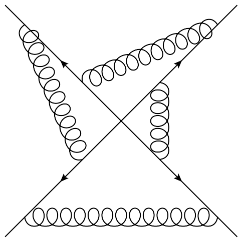
Connected colour factors: generalised non-Abelian exponentiation theorem

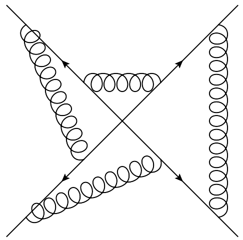
$$S = \exp\{w\} = \langle \phi_{\beta_1} \, \phi_{\beta_2} \, \ldots \, \phi_{\beta_n} \rangle$$

<u>Theorem</u>: all colour structures in the exponent w correspond to connected graphs

Mixing matrices: four-loop example


[[1,2],[3,1],[3,4],[2,4]]


[[1,2],[2,3],[4,3],[4,1]]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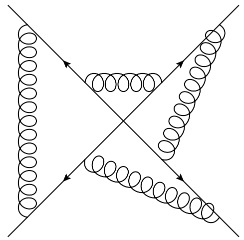
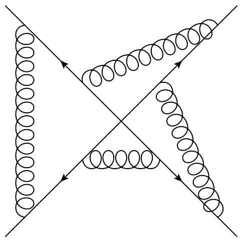

[[1,2],[2,3],[3,4],[1,4]]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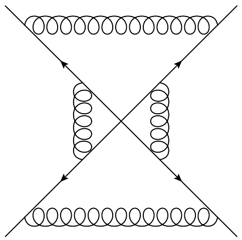
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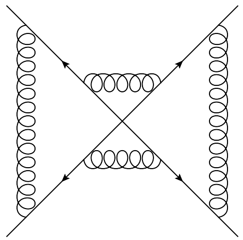
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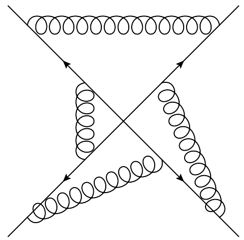
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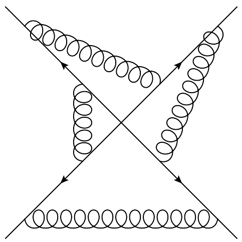
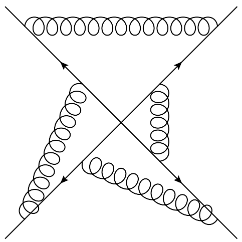

[[1,2],[3,1],[4,3],[4,2]]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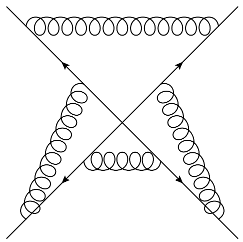
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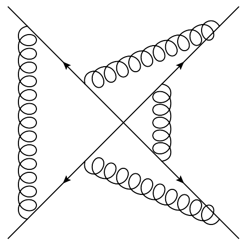
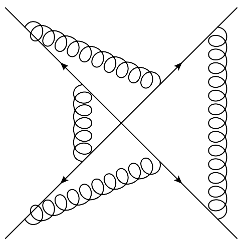

[[1,2],[2,3],[4,3],[1,4]]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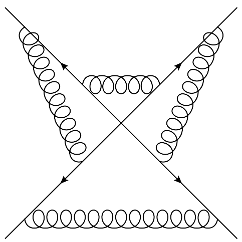
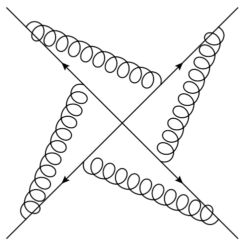
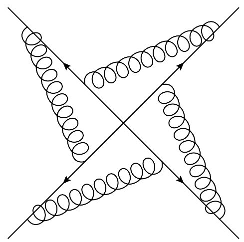

[[1,2],[3,2],[4,3],[4,1]]


[[1,2],[3,1],[4,3],[2,4]]


[[1,2],[2,3],[3,4],[4,1]]

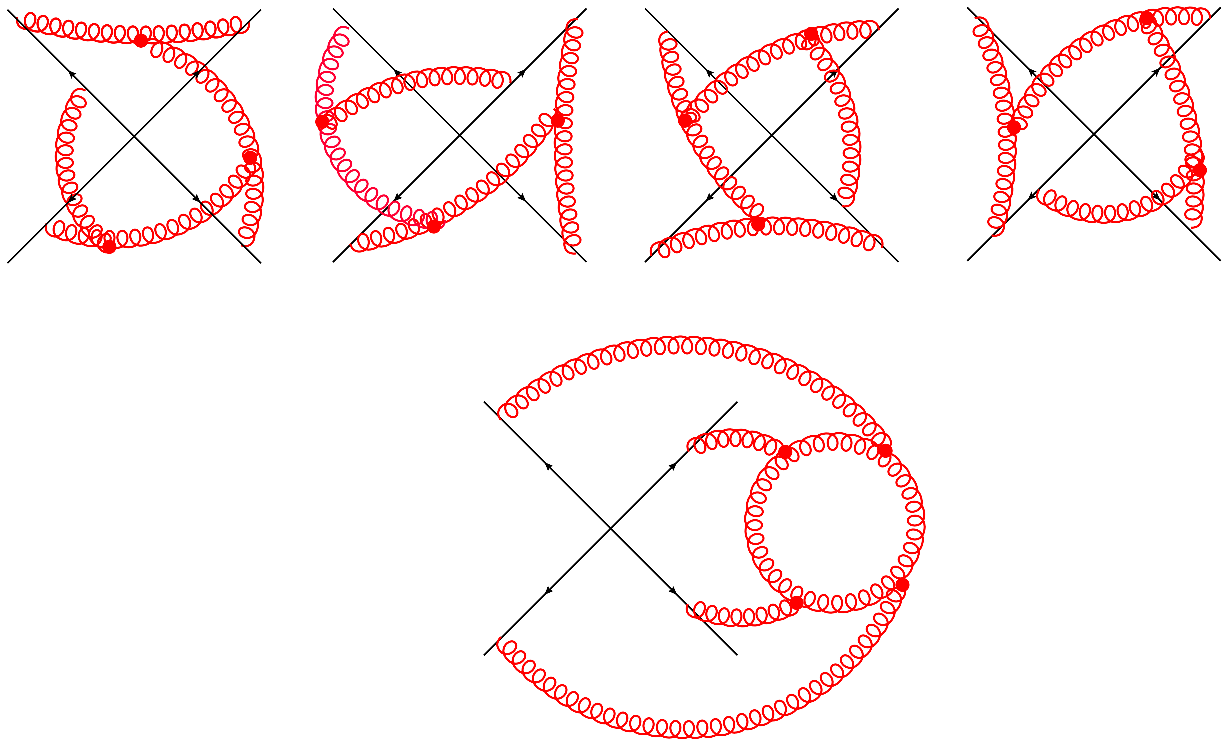
Mixing matrices: four-loop example

The resulting mixing matrix:

$$\widetilde{C} = RC = \frac{1}{24}\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}
\begin{pmatrix}
C[[1,2],[3,1],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[4,1]] \\
C[[1,2],[3,2],[3,4],[4,1]] \\
C[[1,2],[2,3],[3,4],[1,4]] \\
C[[1,2],[3,2],[4,3],[1,4]] \\
C[[1,2],[1,3],[4,3],[4,2]] \\
C[[1,2],[3,2],[3,4],[1,4]] \\
C[[1,2],[1,3],[3,4],[4,2]] \\
C[[1,2],[3,1],[4,3],[4,2]] \\
C[[1,2],[1,3],[4,3],[2,4]] \\
C[[1,2],[1,3],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[1,4]] \\
C[[1,2],[3,1],[3,4],[4,2]] \\
C[[1,2],[3,2],[4,3],[4,1]] \\
C[[1,2],[3,1],[4,3],[2,4]] \\
C[[1,2],[2,3],[3,4],[4,1]]
\end{pmatrix}$$

Mixing matrices: four-loop example

This mixing matrix has rank 5 (5 eigenvectors with eigenvalue 1, the rest 0) corresponding to the colour factors:



Diagrammatic exponetiation: conclusions

- Diagrammatic exponentiation has been extended to the multi-parton case.
  We can now compute the exponent directly!

  - ✓ Webs are formed by sets of (reducible) diagrams, related by permutations. Contributions to the exponent appear through mixing between kinematic and colour factors of the diagrams in the set.

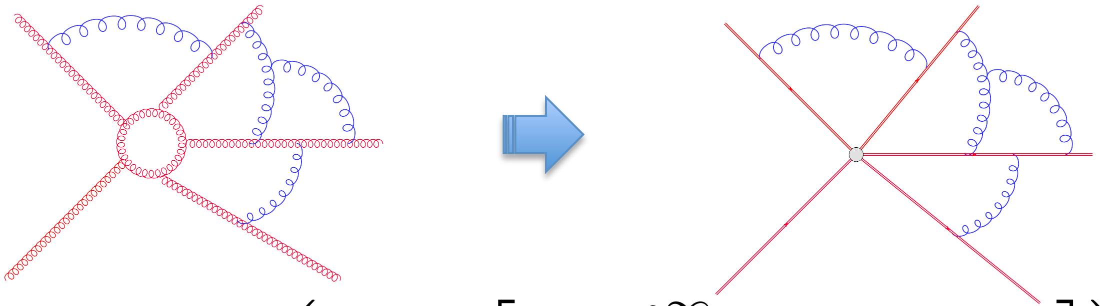
  - ✓ A general formula for the mixing matrices was derived using the replica trick.

  - ✓ Mixing matrices have interesting mathematical properties. This structure leads to non-Abelian colour structure and a singularity structure that is consistent with renormalization (verified explicitly at three loops).

- Theorem:
  All colour factors appearing in the exponent correspond to <u>connected graphs</u>

Non-abelian exponentiation using the replica trick

- Each hard parton, $l = 1..L,$ is represented by a Wilson line ray:



$$\Phi^{(l)}_{a_l b_l} \equiv \left(\mathcal{P}\exp\left[\mathrm{i}g_s \int_0^\infty dt\beta_l \cdot A(t\beta_l)\right]\right)_{a_l b_l}$$

- Generating functional for the Eikonal amplitude:
  Wilson lines are sources for the soft gluon field A_s^μ

$$\mathcal{M}_{b_1\ldots b_L}(p_1,\ldots,p_L) = H_{a_1\ldots a_L}\mathcal{Z}_{a_1\ldots a_L, b_1\ldots b_L}$$

$$\mathcal{Z} = \int [\mathcal{D}A_s^\mu]\, e^{\mathrm{i}S[A_s^\mu]}\left[\Phi^{(1)} \otimes \Phi^{(2)} \otimes \cdots \otimes \Phi^{(L)}\right]$$

- We want to write $\mathcal{Z} = \exp\{\cdots\}$: we wish to compute $\ln \mathcal{Z}$ directly

Non-abelian exponentiation using the replica trick

- Replicate the theory N times, such that different replicas do not interact:

$$S[A_\mu] \longrightarrow \sum_{i=1}^{N} S[A_\mu^i]$$

- The generating functional for the replicated theory is:

$$\mathcal{Z}^N = \int [\mathcal{D}A_\mu^1] \ldots [\mathcal{D}A_\mu^N] \; e^{\mathrm{i}\sum_i S[A_\mu^i]} \left[(\Phi_1^{(1)}\Phi_2^{(1)} \ldots \Phi_N^{(1)}) \otimes (\Phi_1^{(2)}\Phi_2^{(2)} \ldots \Phi_N^{(2)}) \otimes \cdots\cdots \otimes (\Phi_1^{(L)}\Phi_2^{(L)} \ldots \Phi_N^{(L)}) \right]$$

- The order of attachments to the Wilson lines is important:

$$\left[\Phi_1{}^{(l)}\Phi_2{}^{(l)} \ldots \Phi_N{}^{(l)}\right]_{a_1 b_1} = \left(\mathcal{P}\exp\left[\mathrm{i}g_s \int dt\, \beta_l^\mu A_\mu^1\right]\right)_{a_1 c_2} \cdots \left(\mathcal{P}\exp\left[\mathrm{i}g_s \int dt\, \beta_l^\mu A_\mu^N\right]\right)_{c_N b_1}$$

- So

$$\left[\Phi_1^{(l)}\Phi_2^{(l)} \ldots \Phi_N^{(l)}\right]_{a_l b_l} \neq \left(\mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^{N} \int dt\, \beta_l^\mu A_\mu^i\right]\right)_{a_l b_l}$$

Non-abelian exponentiation using the replica trick

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- The order of attachments to the Wilson lines is important:

$$\left[\Phi_1{}^{(l)}\Phi_2{}^{(l)} \dots \Phi_N{}^{(l)}\right]_{a_1 b_1} = \left(\mathcal{P}\exp\left[\mathrm{i}g_s \int dt\, \beta_l^\mu A_\mu^1\right]\right)_{a_1 c_2} \dots \left(\mathcal{P}\exp\left[\mathrm{i}g_s \int dt\, \beta_l^\mu A_\mu^N\right]\right)_{c_N b_1}$$

- Instead $$\left[\Phi_1^{(l)}\Phi_2^{(l)} \dots \Phi_N^{(l)}\right]_{a_l b_l} = \left(\mathcal{R}\mathcal{P}\exp\left[\mathrm{i}g_s \sum_{i=1}^{N} \int dt\, \beta_l^\mu A_\mu^i\right]\right)_{a_l b_l}$$

- $\mathcal{R}$ is a replica-ordering operator

Non-abelian exponentiation using the replica trick

- The generating functional for the replicated theory can be written as:

$$\mathcal{Z}^N = \int [\mathcal{D}A_\mu^1] \ldots [\mathcal{D}A_\mu^N]\; e^{\mathrm{i}\sum_i S[A_\mu^i]}\, \mathcal{R} \left\{ \mathcal{P} \exp\left[\mathrm{i} g_s \sum_{i=1}^{N} \int dt\, \beta_1^\mu A_\mu^i \right] \otimes \ldots \otimes \mathcal{P} \exp\left[\mathrm{i} g_s \sum_{i=1}^{N} \int dt\, \beta_L^\mu A_\mu^i \right] \right\}$$

- Diagram D computed in this theory will have kinematic dependence $\mathcal{F}(D)$ as in the original theory, but colour factor $C_N(D)$ which differ from $C(D)$ due to the action of $\mathcal{R}$.

- Now expand in powers of N

$$\mathcal{Z}^N = 1 + N \ln \mathcal{Z} + \mathcal{O}(N^2)$$

- Contribution of a given diagram D to $\ln \mathcal{Z}$ can be readily determined as the coefficient of N^1 in the expansion of $\mathcal{F}(D)\, C_N(D)$.

- That's it! Here is an algorithm to compute the exponent directly: *diagram* D *contributes with a modified colour factor* $\widetilde{C}(D)$*, which is* $\mathcal{O}(N^1)$ *term in the expansion its colour factor in the replicated theory.*