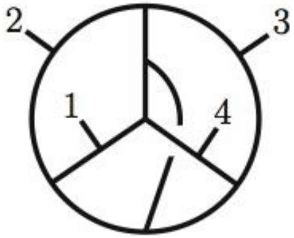
Gravity as a Double Copy of Gauge Theory

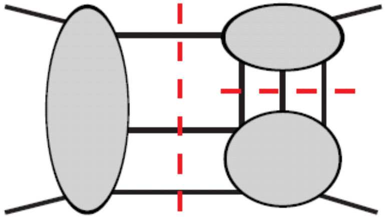
June 21, 2012

IHES

Zvi Bern, UCLA

Based on papers with John Joseph Carrasco, Scott Davies, Tristan Dennen, Lance Dixon, Yu-tin Huang, Harald Ita, Henrik Johansson and Radu Roiban.





Outline

1) **A hidden structure in gauge and gravity amplitudes**
   - — **a duality between color and kinematics.**
   - — **gravity as a double copy of gauge theory.**

2) **Examples of theories where duality and double copy holds.**

3) **Application: Reexamination of sugra UV divergences**
   - — **Lightning review of sugra UV properties.**
   - — **A four-loop surprise in *N* = 8 supergravity.**
   - — **A three-loop surprise in *N* = 4 supergravity.**
   - — **Consequences and prospects for future.**

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

flat metric

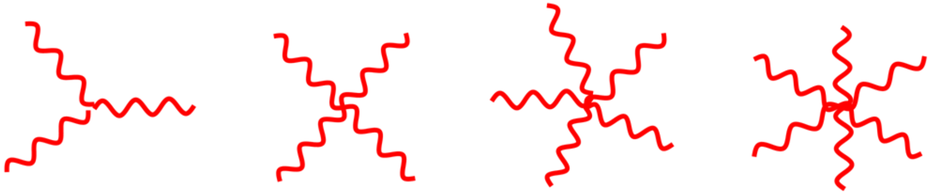
graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions



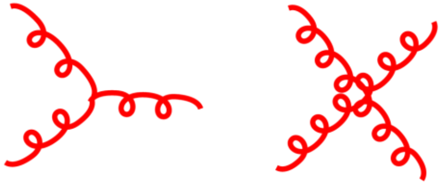
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

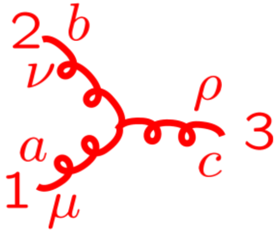


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



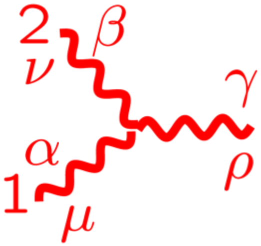
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$\begin{aligned} G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\ \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

Definitely not a good approach.

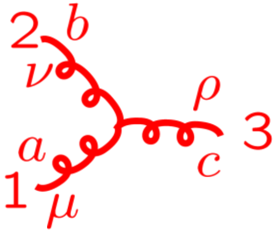
Simplicity of Gravity Amplitudes

On-shell viewpoint much more powerful.

On-shell **three vertices contains all information:**

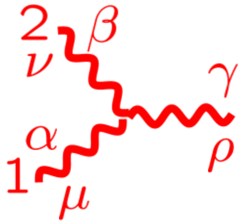
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \\ \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

double copy of Yang-Mills vertex.

- **Using modern on-shell methods, any gravity scattering amplitude constructible solely from *on-shell* 3 vertices. BCFW recursion for trees, BDDK unitarity method for loops.**
- **Higher-point vertices irrelevant!**

Gravity vs Gauge Theory

Consider the gravity Lagrangian

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flat metric

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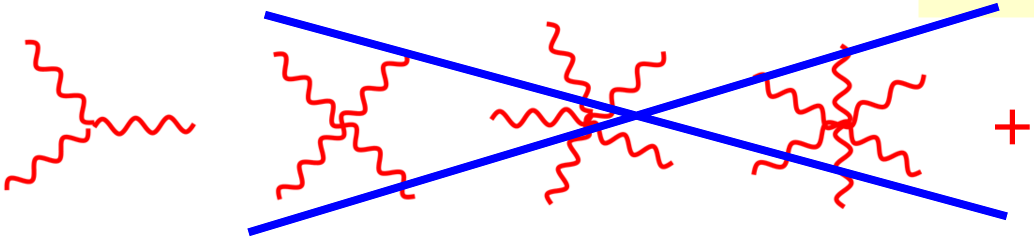
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

Infinite number of irrelevant interactions!

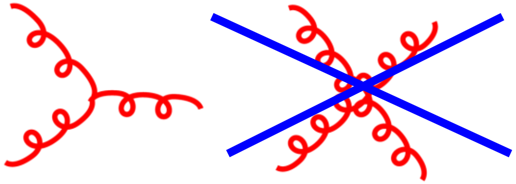


$+ \cdots$

Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

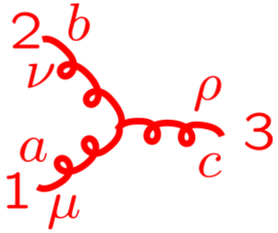


Only three-point Interactions needed

Gravity seems ~~so much~~ no more complicated than gauge theory.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)



coupling constant

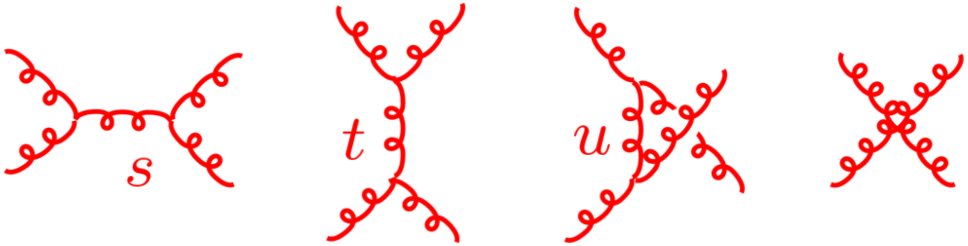
color factor

momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$

Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_sc_s}{s} + \frac{n_tc_t}{t} + \frac{n_uc_u}{u}\right)$$

$s = (k_1+k_2)^2$
$t = (k_1+k_4)^2$
$u = (k_1+k_3)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

Consider five-point tree amplitude:

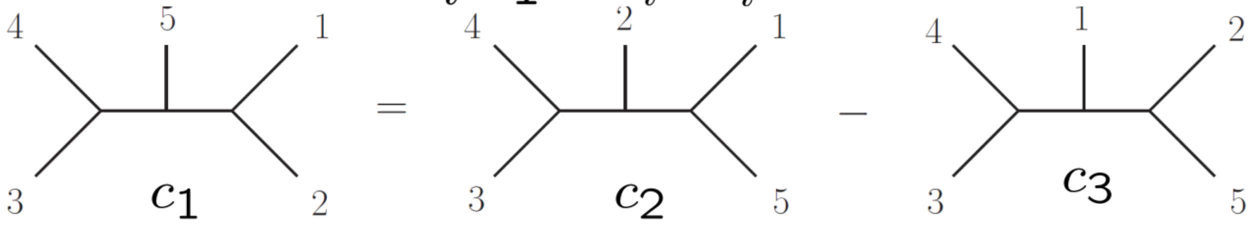
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Claim: At n-points we can always find a rearrangement so color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng,Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

kinematic numerator

color factor

gauge theory:

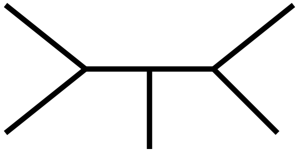
$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity:

$$\boxed{-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}}$$

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!

Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n)=\sum_i\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p^2_{\alpha_i}}$$

	n		$\tilde{n}$
N = 8 sugra:	**(N = 4 sYM)**	×	**(N = 4 sYM)**
N = 4 sugra:	**(N = 4 sYM)**	×	**(N = 0 sYM)**
N = 0 sugra:	**(N = 0 sYM)**	×	**(N = 0 sYM)**

N = 0 sugra: graviton + antisym tensor + dilaton

Duality for BLG Theory

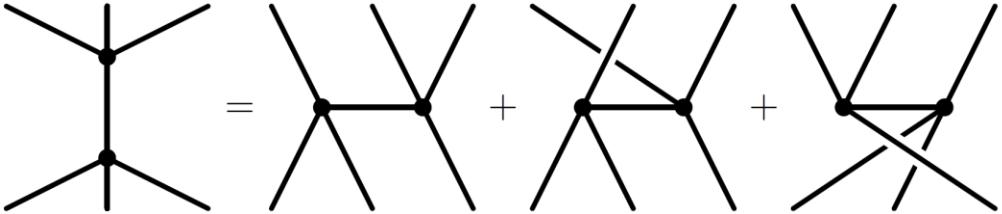
BLG based on a 3 algebra

Bagger,Lambert,Gustavsson (BLG)

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

***D* = 3 Chern-Simons gauge theory**

Four-term color identity:



color factors

kinematic numerators

$$\boxed{c_s = c_t + c_u + c_v \Rightarrow n_s = n_t + n_u + n_v}$$

Such numerators explicitly found at 6 points.

Bargheer, He, and McLoughlin

What is the double copy?

Explicit check at 4 and 6 points shows it is the $E_{8(8)}$ *N* = 16 supergravity of Marcus and Schwarz. Very non-trivial!

A hidden 3 algebra structure exists in this supergravity.

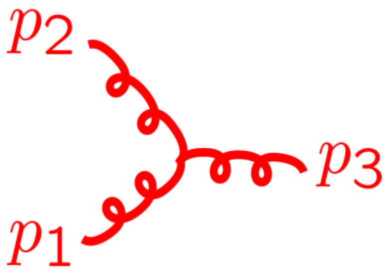
Symmetries

Monteiro and O'Connell

Consider self dual YM. Work in space-cone gauge of Chalmers and Siegel

$$u = t - z$$
$$w = x + iy$$

Generators of diffeomorphism invariance:



$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

The Lie Algebra:

YM vertex

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1 p_2}{}^{k} L_k$$

The $X(p_1, p_2)$ are YM vertices, valid for self-dual configurations.

Explains why numerators satisfy Jacobi Identity

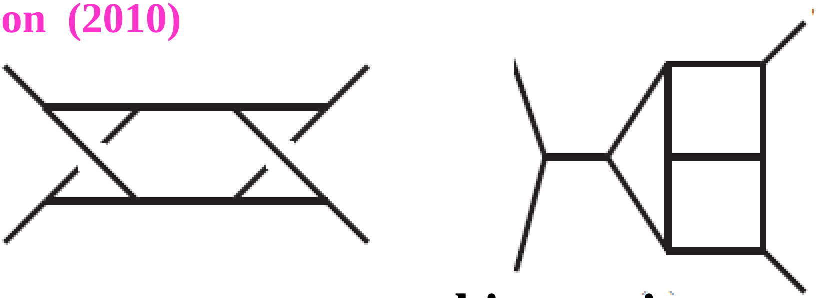
YM inherits diffeomorphism symmetry of gravity!

We need to go beyond self dual.

Recent progress from Bjerrum-Bohr, Damgaard, Monteiro and O'Connell

Loop-Level Conjecture

ZB, Carrasco, Johansson (2010)



$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

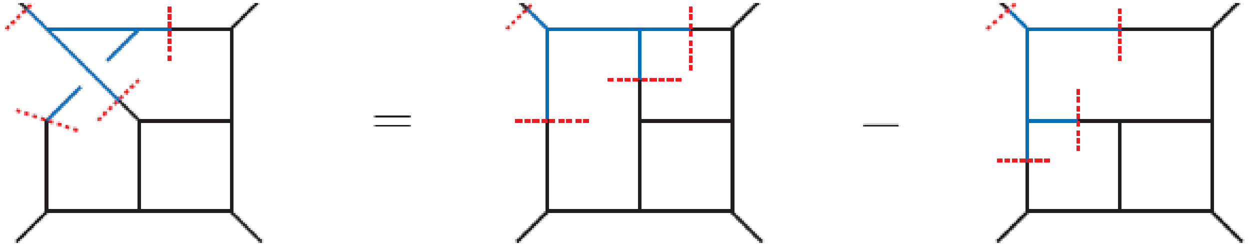
gravity

symmetry factor

Loop-level is identical to tree-level one except for symmetry factors and loop integration.
Double copy works if numerator satisfies duality.

Nonplanar from Planar

BCJ



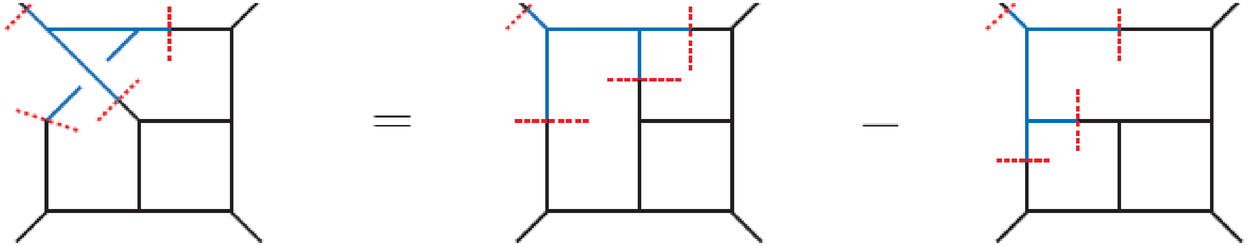
$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

Planar determines nonplanar

- **We can carry advances from planar sector to the nonplanar sector.**
- **Yangian symmetry and integrability clearly has echo in nonplanar sector because it is built directly from planar.**
- **Only at level of the integrands, so far, but bodes well for the future.**

Gravity from Gauge Theory Amplitudes

BCJ



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

If you have a set of duality satisfying numerators. To get:

gauge theory ⟶ gravity theory

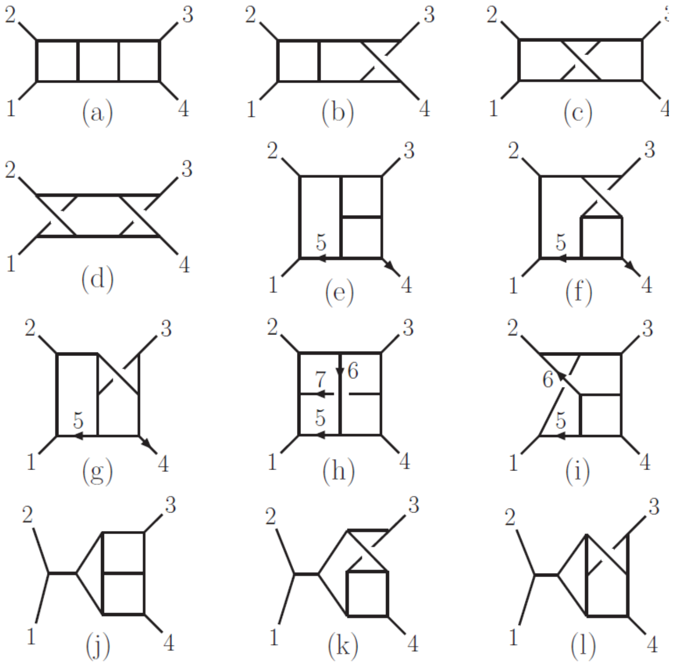
simply take

color factor ⟶ kinematic numerator

Gravity loop integrands are free!

Explicit Three-Loop Construction

ZB, Carrasco, Johansson (2010)



$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k$$

For *N*=4 sYM we have the abililty to go to high loop orders. Go to 3 loops. (1 & 2 loops works.)

Calculation very similar to earlier one with Dixon and Roiban, except now make the duality manifest.

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t)-t\,(\tau_{25}+\tau_{45})+u\,(\tau_{25}+\tau_{35})-s^2\right)/3$
(h)	$\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\big)/3$
(i)	$\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\big)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

- **Duality works!**
- **Double copy works!**

One diagram to rule them all

ZB, Carrasco, Johansson (2010)

N = 4 super-Yang-Mills integrand

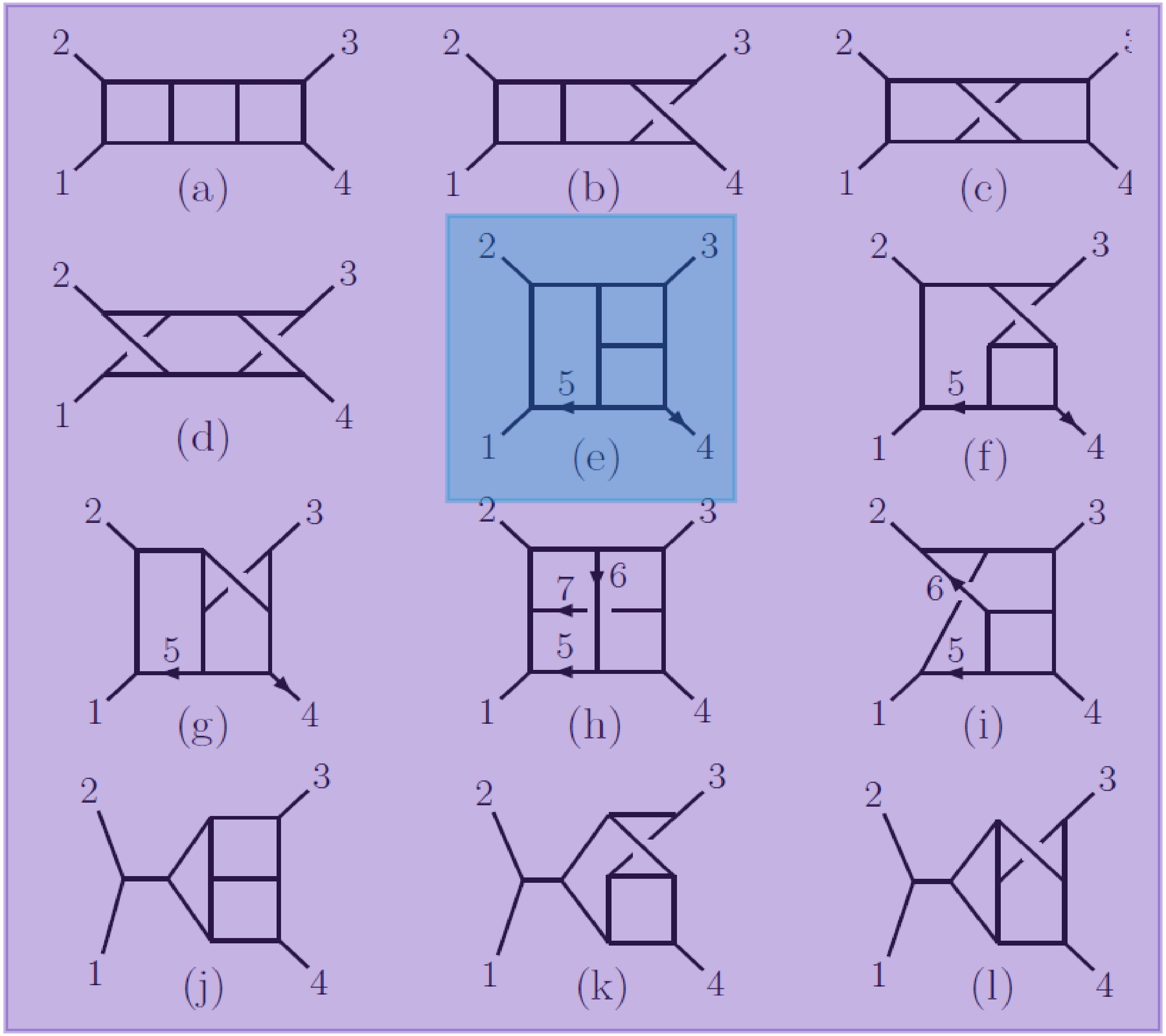


Diagram (e) is the master diagram.

Determine the master integrand in proper form and duality gives all others.

***N* = 8 sugra given by double copy.**

One diagram to rule them all

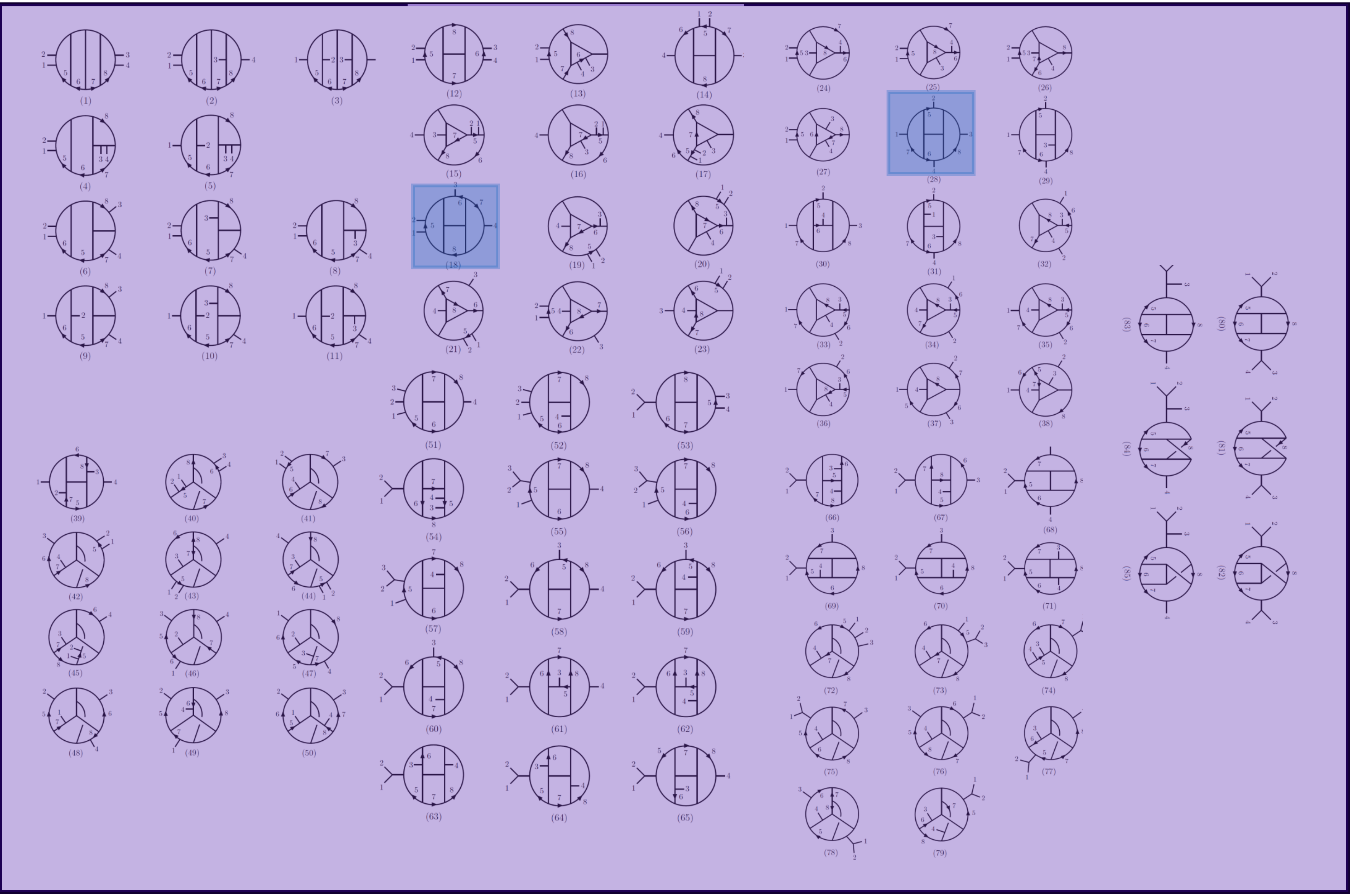
$$
\begin{aligned}
N^{(\mathrm{a})} &= N^{(\mathrm{b})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{b})} &= N^{(\mathrm{d})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{c})} &= N^{(\mathrm{a})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{d})} &= N^{(\mathrm{h})}(k_3,k_1,k_2,l_7,l_6,k_{1,3}-l_5+l_6-l_7) + N^{(\mathrm{h})}(k_3,k_2,k_1,l_7,l_6,k_{2,3}+l_5-l_7)\,,\\
N^{(\mathrm{f})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{g})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{h})} &= -N^{(\mathrm{g})}(k_1,k_2,k_3,l_5,l_6,k_{1,2}-l_5-l_7) - N^{(\mathrm{i})}(k_4,k_3,k_2,l_6-l_5,l_5-l_6+l_7-k_{1,2},l_6)\,,\\
N^{(\mathrm{i})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_7,l_6) - N^{(\mathrm{e})}(k_3,k_2,k_1,-k_4-l_5-l_6,-l_6-l_7,l_6)\,,\\
N^{(\mathrm{j})} &= N^{(\mathrm{e})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{e})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{k})} &= N^{(\mathrm{f})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{f})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(\mathrm{l})} &= N^{(\mathrm{g})}(k_1,k_2,k_3,l_5,l_6,l_7) - N^{(\mathrm{g})}(k_2,k_1,k_3,l_5,l_6,l_7)\,,
\end{aligned}
$$

triangle subdiagrams
vanish in *N* = 4 sYM

All numerators solved in terms of numerator (e)

N = 4 sYM Four Loops

ZB, Carrasco, Dixon, Johansson, Roiban (to appear)



Generalized Gauge Invariance

BCJ
Bern, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL} p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

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- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**
- **Used to find expressions for $N \geq 4$ supergravity amplitudes at 1, 2 loops.**

ZB, Boucher-Veronneau and Johansson; Boucher-Veroneau and Dixon

Application: UV Properties of Gravity

Finiteness of *N* = 8 Supergravity?

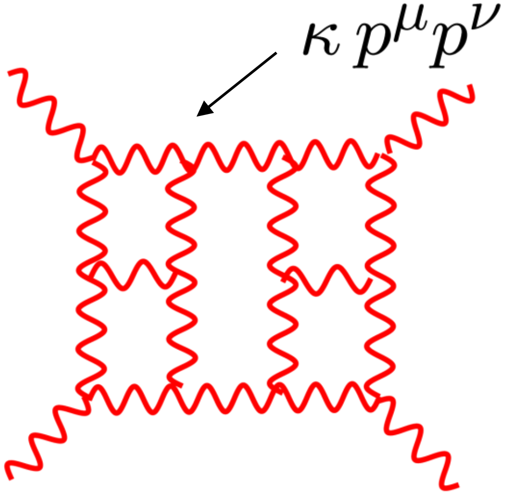
We are interested in UV finiteness of *N* = 8 supergravity because it would imply a new symmetry or non-trivial dynamical mechanism. No known symmetry can render a UV theory finite.

The discovery of either would have a fundamental impact on our understanding of gravity.

Note: Perturbative finiteness is not the only issue for consistent gravity: Nonperturbative completions? High energy behavior of theory? Realistic models?

Power Counting at High Loop Orders

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu) \cdots}{\text{propagators}}$

Gauge theory: $\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu) \cdots}{\text{propagators}}$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Reasons to focus on *N* = 8 supegravity:

- **With more susy expect better UV properties.**
- **High symmetry implies technical simplicity.**

Where is First Potential UV Divergence in *D*= 4 *N* = 8 Sugra?

Various opinions, pointing to divergences over the years:

3 loops	Superspace power counting	Deser, Kay (1978) Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985), etc
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N}$ = 7 harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* offshell $\mathcal{N}$ = 8 superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments; $E_{7(7)}$ symmetry.	Grisaru and Siegel (1982); Howe, Stelle and Bossard (2009) Vanhove; Bjornsson, Green (2010) Kiermaier, Elvang, Freedman(2010) Ramond, Kallosh (2010); Biesert, et al (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy and duality.	Kallosh; Howe and Lindström (1981)
9 loops	Assumes Berkovits' superstring non-renormalization theorems carries over to *D*=4 $\mathcal{N}$ = 8 supergravity and extrapolates to 9 loops.	Green, Russo, Vanhove (2006)

No divergences demonstrated above. Arguments based on lack of symmetry protection. An unaccounted symmetry can make the theory finite.

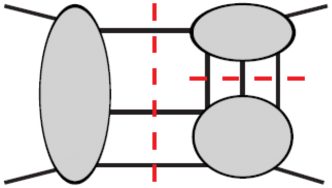
To end debate we need solid calculations.

Constructing Multiloop Amplitudes

We do have powerful tools for complete calculations including nonplanar contributions:

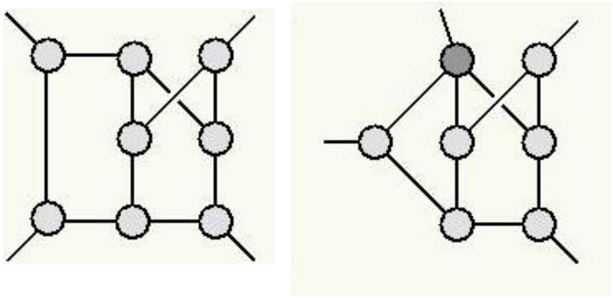
- **Unitarity Method**

  ZB, Dixon, Dunbar, Kosower



- **Method of Maximal cuts**

  ZB, Carrasco, Johansson , Kosower



- **Duality between color and kinematics**

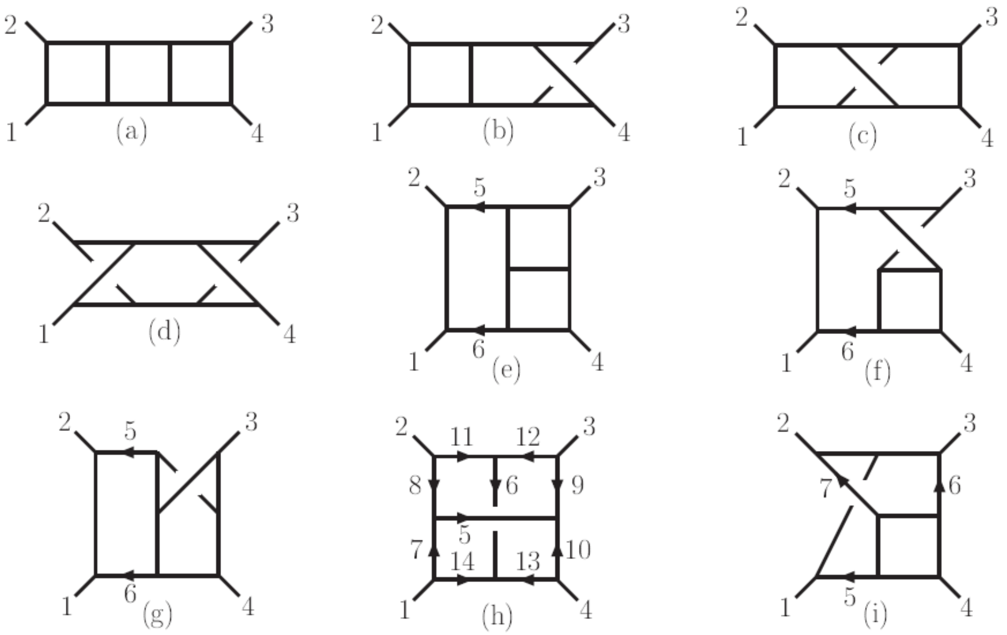
  ZB, Carrasco and Johansson

In this talk we will explain how $N = 4$ sYM (including non-planar) helps us study supergravity theories.

Complete Three-Loop Result

Analysis of unitarity cuts shows highly nontrivial all-loop cancellations. **ZB, Dixon and Roiban (2006); ZB, Carrasco, Forde, Ita, Johansson (2007)**

To test completeness of cancellations, we decided to directly calculate potential three-loop divergence.



ZB, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

Three loops is not only ultraviolet finite it is “superfinite”— finite for $D < 6$.

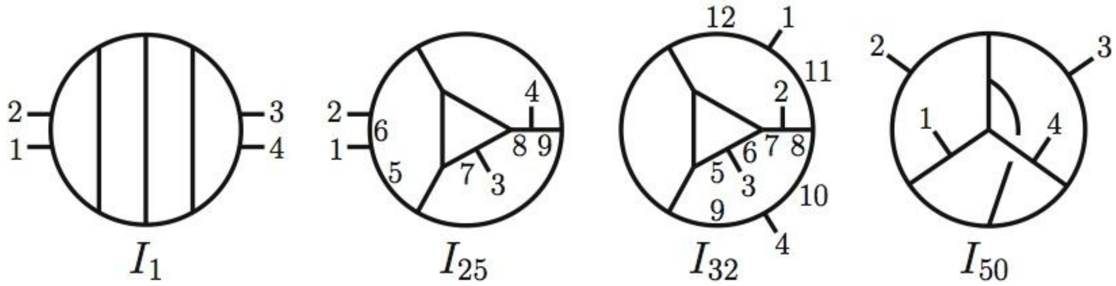
It is very finite!

Obtained via on-shell unitarity method:

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

UV finite for *D* < 11/2

It's very finite!

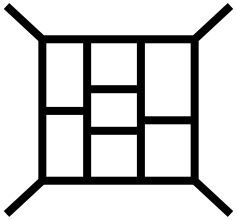
Originally took more than a year.

Today with the double copy we can reproduce it in a couple of days! Another non-trivial example.

Recent Status of Divergences

Consensus that in *N* = 8 supergravity trouble starts at 5 loops and by 7 loops we have valid UV counterterm in *D* = 4 under all known symmetries (suggesting divergences) .

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



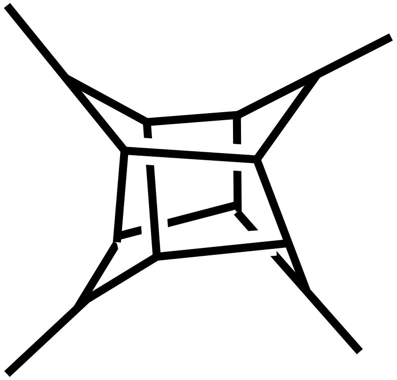
For *N* = 8 sugra in *D* = 4:

- **All counterterms ruled out until 7 loops!**
- **Candidate 7 loop superspace volume counterterm vanishes.**
- **But D^8R^4 apparently available at 7 loops (1/8 BPS) under all known symmetries. (No known nonrenormalization theorem)**

Bossard, Howe, Stelle and Vanhove

N = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Place your bets:

- **At 5 loops in *D* = 24/5 does *N* = 8 supergravity diverge?**
- **At 7 loops in *D* = 4 does *N* = 8 supergravity diverge?**

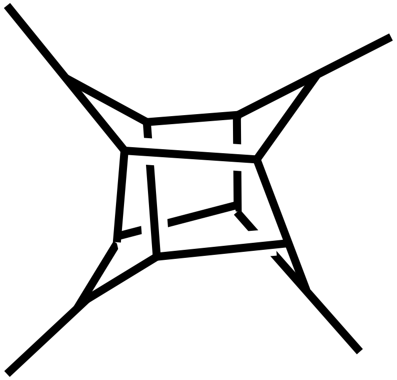
5 loops

Kelly Stelle:
British wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

N = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Dixon, Johannson, Roiban



~500 such diagrams with ~100s terms each

Place your bets:

- **At 5 loops in *D* = 24/5 does *N* = 8 supergravity diverge?**
- **At 7 loops in *D* = 4 does *N* = 8 supergravity diverge?**

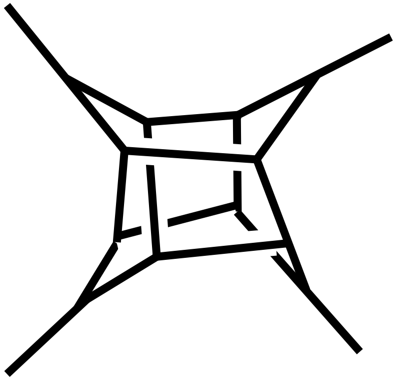
7 loops

David Gross:
California wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

Calculation of *N* = 4 sYM 5 Loop Amplitude

ZB, Carrasco, Dixon, Johansson, Roiban



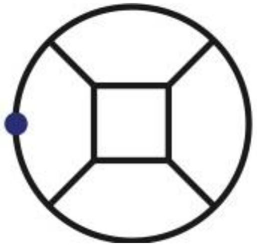
Key step for *N* = 8 supergravity is construction of complete nonplanar 5 loop integrand of *N* = 4 sYM theory. (Still need to find BCJ form).

450 such diagrams with ~100s terms each

Leading color part:

Expanding planar in small external momenta and simplifying via ibp relations: 1 vaccum diagram

$$N_c^5\,\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]\ \frac{114}{5}u$$



Diverges in *D* = 26/5

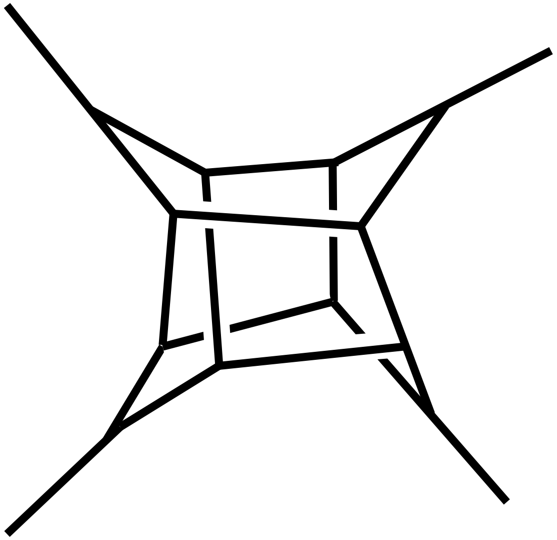
Proves known finiteness bound is saturated

ZB, Carrasco, Dixon, Johansson, Roiban

Stay tuned.

We now have key information needed to calculate the UV properties of *N* = 8 supergravity at five loops.

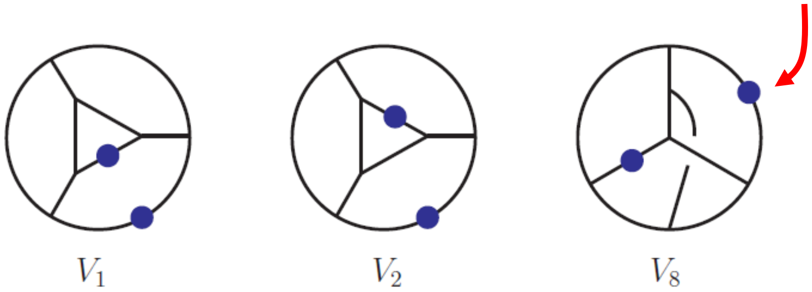
Unclear when this will be finished.



New Four-Loop Surprise

ZB, Carrasco, Dixon, Johansson, Roiban (2012)

Critical dimension *D* =11/2.
Express UV divergences
in terms of vacuum like integrals.



doubled propagator

gauge theory

$$\mathcal{A}_4^{(4)}(1,2,3,4)\Big|_{\rm pole}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2\, V_1 + 12\, \boxed{(V_1 + 2\,V_2 + V_8)}\Big)$$

$$\times \Big(s\,(\mathrm{Tr}_{1324}+\mathrm{Tr}_{1423}) + t\,(\mathrm{Tr}_{1243}+\mathrm{Tr}_{1342}) + u\,(\mathrm{Tr}_{1234}+\mathrm{Tr}_{1432})\Big)$$

same divergence

gravity

$$\mathcal{M}_4^{(4)}\Big|_{\rm pole} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\rm tree}\, \boxed{(V_1+2V_2+V_8)}$$

- Gravity UV divergence is directly proportional to subleading color single-trace divergence of *N* = 4 super-Yang-Mills theory.
- Same happens at 1-3 loops.

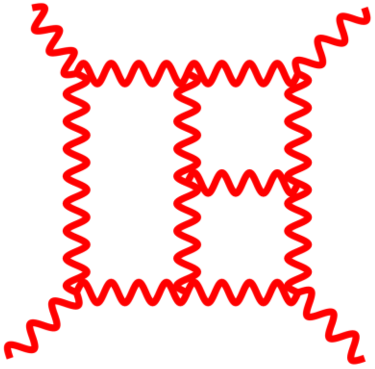
Fine, but do we have any example where a divergence vanishes but for which there is no accepted symmetry explanation?

$N = 4$ supergravity at 3 loops

$N = 4$ Supergravity

$N = 4$ sugra at 3 loops ideal test case to study.

ZB, Davies, Dennen, Huang



Consensus had it that a valid R^4 counterterm exists for this theory in $D = 4$. Analogous to 7 loop counterterm of $N = 8$.

Bossard, Howe, Stelle; Bossard, Howe, Stelle, Vanhove

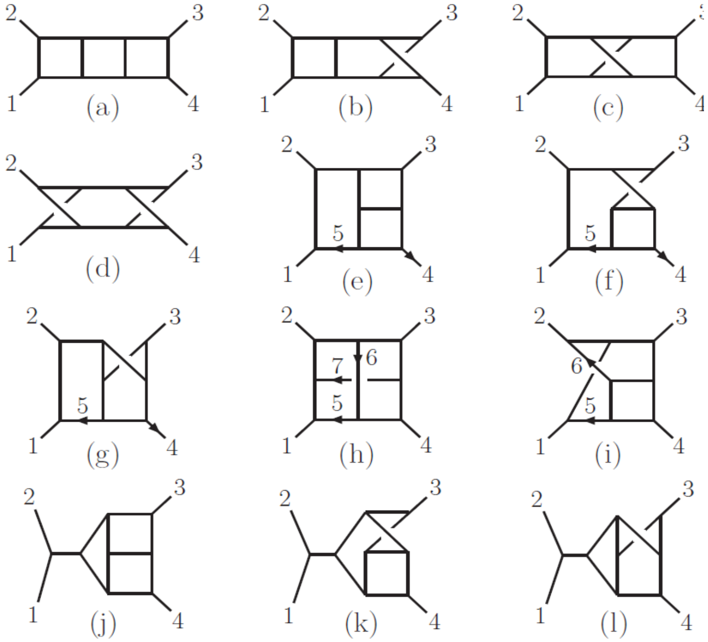
A no lose calculation:
Either we find first example of a $D = 4$ divergence or once again we show an expected divergence is not present.

Duality between color and kinematics gives us the ability to do the calculation.

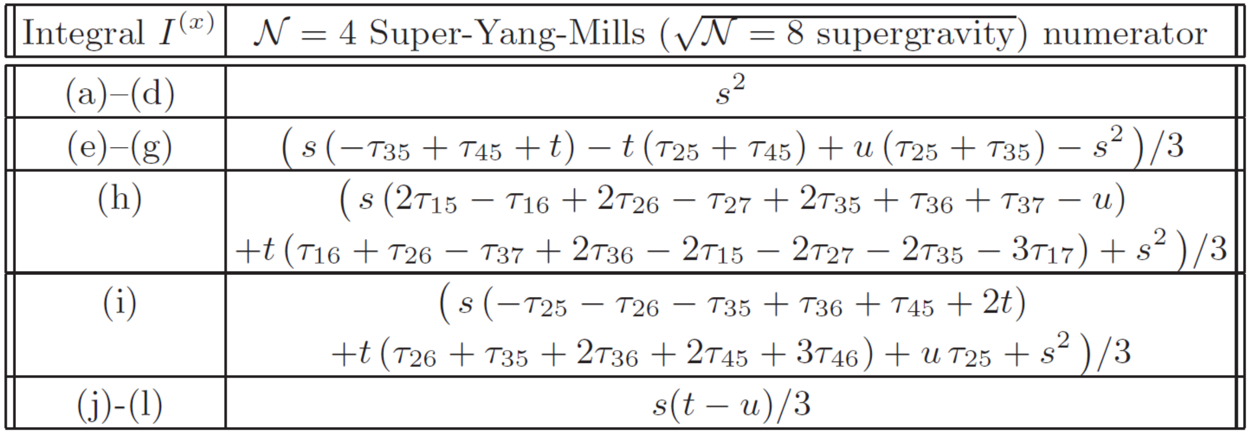
Three-loop Construction

ZB, Davies, Dennen, Huang

N = 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)



- **For *N* = 4 sYM copy use known BCJ representation.**
- **What representation should we use for pure YM side?**



Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(\, s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \,\big)/3$
(h)	$\big(\, s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \,\big)/3$
(i)	$\big(\, s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \,\big)/3$
(j)-(l)	$s(t-u)/3$

BCJ form of the *N* = 4 sYM integrand

Three-loop $N = 4$ supergravity

What is a convenient representation for pure YM copy?

Answer: Feynman diagrams.

Yes, I did say Feynman diagrams!

However, this case is very special.

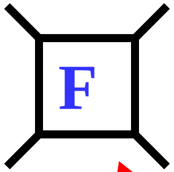
- **We can drop all Feynman diagrams where corresponding $N = 4$ numerators vanish.**
- **We need only the leading UV parts (though we initially keep all pieces).**
- **Completely straightforward. Faster to just do it than to argue about which way might be better.**

Multiloop *N* = 4 supergravity

Does it work? Test at 1, 2 loops

All pure supergravities finite at 1,2 loops

One-loop: keep only box Feynman diagrams

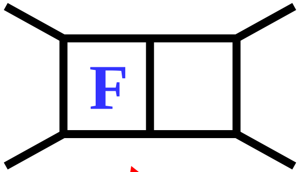
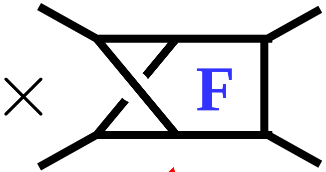
$stA_4^{\text{tree}} \times$  + perms $= \frac{0}{\epsilon} +$ finite

_N_ = 4 sYM box numerator

_N_ = 0 Feynman diagram, including ghosts

Becomes gauge invariant after permutation sum.

Two-loop: keep only double box Feynman diagrams

$s^2tA_4^{\text{tree}} \times$  $+ s^2tA_4^{\text{tree}} \times$  + perms $= \frac{0}{\epsilon} +$ finite

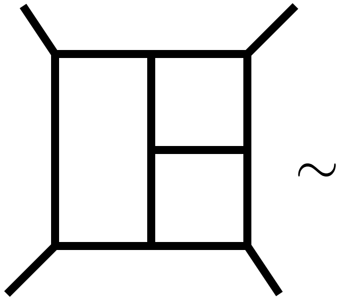
Feynman diagram including ghosts

Get correct results. Who would have imagined it is this simple to get gravity results?

Three-Loop Construction

***N* = 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)**

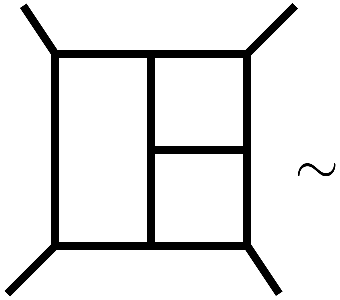
***N* = 4 sYM**



$\sim l \cdot k\, s^2 t A_4^{\text{tree}}$

BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4 \; l^4$

Feynman representation

***N* = 4 sugra linear divergent**

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for *N*=5,6 sugra

Pure YM 4 point amplitude has never been done at three loops.

Integrals have subdivergences which causes complications.
But this was well understood 30 years ago by Marcus and Sagnotti.

Dealing With Subdivergences

Marcus, Sagnotti (1984)

The problem was solve nearly 30 years ago.

Recursively subtract all subdivergences.

regulator dependent

reparametrize subintegral

$$\mathcal{S}\left[\int \prod_{i=1}^{L} dp_i\, I\right] \equiv \text{Div}\left[\int \prod_{i=1}^{L} dp_i\, I\right] - \sum_{l=1}^{L-1} \sum_{\substack{l-\text{loop}\\ \text{subloops}}} \text{Div}\left[\int \prod_{j=l+1}^{L} dp'_j\, \mathcal{S}\left[\int \prod_{i=1}^{l} dp'_i\, I\right]\right]$$

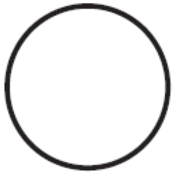
Regulator Independent

Nice consistency check: all log(*m*) terms must cancel

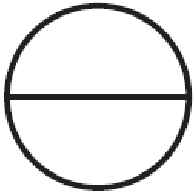
Extracting UV divergence in the presence of UV subdivergences and IR divergences is a well understood problem.

Integral Basis

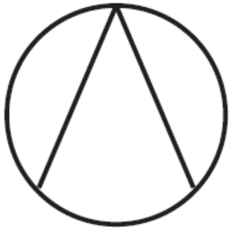
Using FIRE we obtain a basis of integrals:



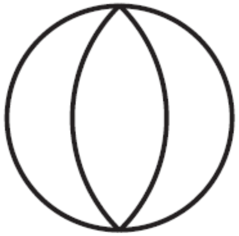
$$(m^2)^{1-\epsilon}\left(\frac{1}{\epsilon}+1+\left(1+\frac{1}{2}\zeta_2\right)\epsilon\right)$$



$$(m^2)^{1-2\epsilon}\left(\frac{3}{2\epsilon^2}+\frac{9}{2\epsilon}+\frac{21}{2}+\frac{3}{2}\zeta_2-2c\right)$$



$$(m^2)^{1-3\epsilon}\left(\frac{1}{\epsilon^3}+\frac{17}{3\epsilon^2}+\left(\frac{67}{3}+\frac{3}{2}\zeta_2-4c\right)\frac{1}{\epsilon}\right)$$



$$(m^2)^{2-3\epsilon}\left(\frac{2}{\epsilon^3}+\frac{23}{3\epsilon^2}+\left(\frac{35}{2}+3\zeta_2\right)\frac{1}{\epsilon}\right)$$

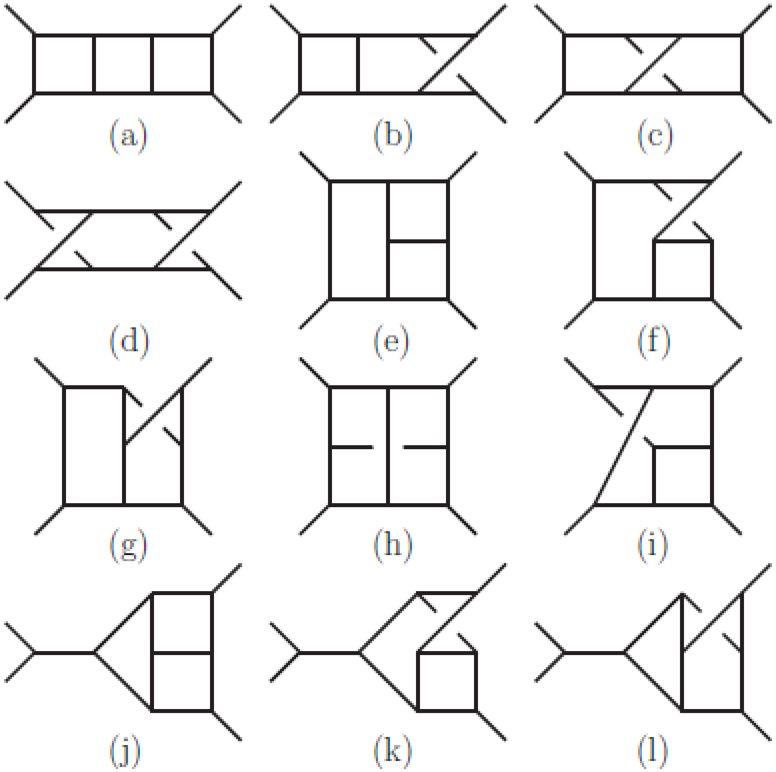
$$(m^2)^{-3\epsilon}\left(\frac{2\zeta_3}{\epsilon}\right)$$

$$c=\sqrt{3}\mathrm{Im}\left(\mathrm{Li}_2(e^{i\pi/3})\right)$$

Use Mellin-Barnes resummation of residues method of Davydychev and Kalmykov on all but last integral. Last one doable by staring at paper from Grozin or Smirnov's book (easy because no subdivergences).

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	(divergence)/$(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity used to clean up

Sum over diagrams is gauge invariant

All divergences cancel completely!

It is UV finite contrary to expectations

Explanations?

Key Question: Is there an ordinary symmetry explanation for this?
Or is something extraordinary happening?

- Non-renormalization understanding from heterotic string.

Tourkine and Vanhove (2012)

- Quantum corrected duality current nonconservation.

Kallosh (2012)

- Do similar cancellations happen for $N = 8$ supergravity, killing potential 7 loop counterterm in $D = 4$?
- Further explicit calculations needed to help settle the UV properties and settle various bets.

Summary

- **A new duality conjectured between color and kinematics. Locks down the multiloop structure of integrands of amplitudes.**
- **When duality between color and kinematics manifest, gravity integrands follow *immediately* from gauge-theory ones.**
- **Duality exists outside ordinary gauge theories. BLG theory.**
- **Double copy gives us a powerful way to explore the UV properties of gravity theories.**
- **$\mathcal{N} = 4$ sugra has no three-loop four-point divergence, contrary to expectations from symmetry considerations.**
- **Power counting using known symmetries and their known consequences can be misleading. Concrete example.**

Duality between color and kinematics offers a powerful new way to explore gauge and gravity theories at high loop orders, in particular their UV properties. Expect many new results in the coming years.

Extra Slides

Lagrangians

ZB, Dennen, Huang, Kiermaier
arXiv:1004.0693

$$L_{\text{YM}} = \frac{1}{g^2} F^2 \qquad L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g}\, R$$

How can one take two copies of the gauge-theory Lagrangian and get a gravity Lagrangian?

Add zero to the YM Lagrangian in a special way:

$$\mathcal{L}'_5 = -\tfrac{1}{2} g^3 \big(f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c} \big) f^{c a_4 a_5} \\ \times \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3 \mu} \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}) \quad = 0$$

Through five points:

- **Feynman diagrams satisfy the color-kinematic duality.**
- **Introduce auxiliary field to convert nonlocal interactions into local three-point interactions.**
- **Take two copies: you get gravity!**

$$A^\mu \tilde{A}^\nu \;\to\; h^{\mu\nu}$$

At each order need to add more and more vanishing terms.

Recent String Theory Studies

1) Nontrivial consequences for *n*-point color-ordered tree amplitudes

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}}$$

BCJ

First proven using monodromy relations in string theory.

Bjerrum-Bohr, Damgaard,Vanhove; Steiberger; Sondergaard

Also proven using on-shell recursion in field theory. Feng, Huang , Jia; Chen, Du, Feng

2) Duality between color and kinematics studied in string theory for five-point trees. Heterotic string especially insightful treatment.

Tye and Zhang; Mafra, Schlotterer Stieberger Bjerrum-Bohr, Damgaard, Vanhove, Sondergaard.

3) Helps organize construction of tree and one-loop amplitudes in string theory. Mafra, Schlotterer and Stieberger; Mafra and Schlotterer

String theory offers an important handle for understanding and applying duality between color and kinematics.