Discrete Wheeler DeWitt Equations

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Based on:

H. Hamber and R. Williams :*"Discrete Wheeler DeWitt Equations"* arxiv: 1109.2530
Phys. Rev. D.84. 104033(2012)

H. Hamber, R. Toriumi, and R. Williams: *"Wheeler DeWitt Equation in 2+1 Dimensions"* arxiv: 1207.3759
Phys. Rev. D. 86, 084010 (2012)

H. Hamber, R. Toriumi, and R. Williams: *"Wheeler DeWitt Equation in 3+1 Dimensions"* arxiv: 1207.3759

Motivation for Discretization

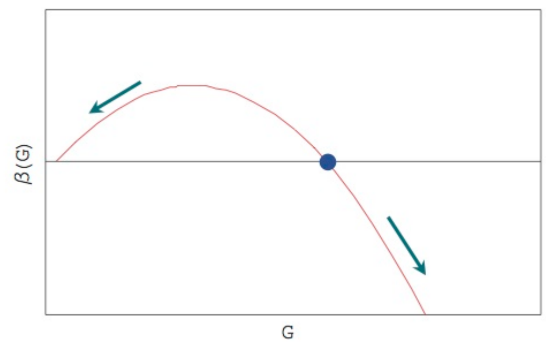
Take Canonical Quantization here instead of Covariant Path Integral Formulation.

No reason not to try quantizing discrete spacetime.
May even be certain advantages
e.g., natural cut off (minimum length)
finite number of variables

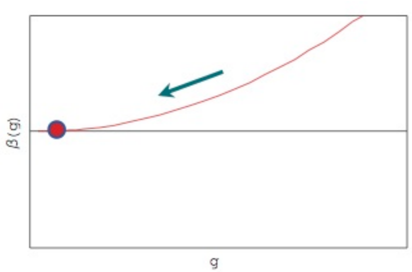
At the Planck scale, spacetime may be discrete anyway, with rapidly changing topologies
(spacetime foam, Wheeler)

Background: Nontrivial Ultraviolet Fixed Point

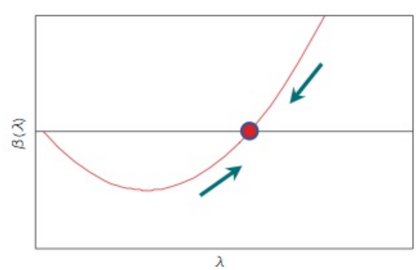
Conjectured by Weinberg (1979),
Evidences from
2+ ε dimensional gravity (Kawai, Ninomiya, Kitazawa 1992)
the functional renormalization group approaches, (Reuter, Litim, Saueressig),
Regge calculus (Hamber, Williams)



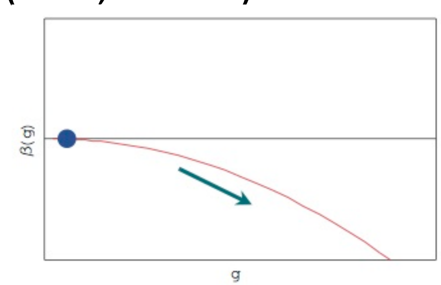
Ising model, nonlinear sigma model, Gravity (2+ ε, lattice)



ϕ^4 theory, QED



Wilson-Fisher fixed point



Yang Mills (asymptotic freedom)

Background:
Physical Phase of Quantum Gravity

Hamber and Williams 1984

Unphysical

Physical

$G < G_c$ Rough phase

collapses to d~2
(into branched polymer)

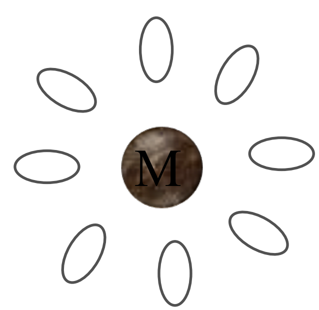
$G > G_c$ Smooth phase R ~ 0

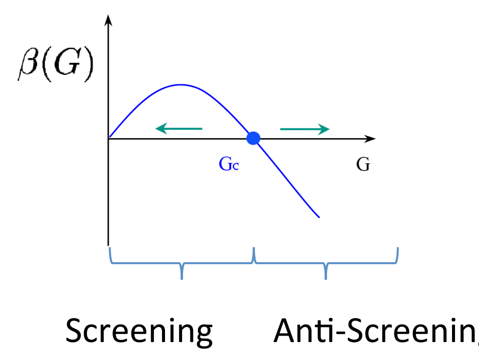
$$\langle g_{\mu\nu} \rangle \approx c\, \eta_{\mu\nu}$$

In gravity we do not have opposite charges.
Expect: larger the cloud is, the stronger the gravitational force is.

→ Antiscreening . (β *(g) < 0*).

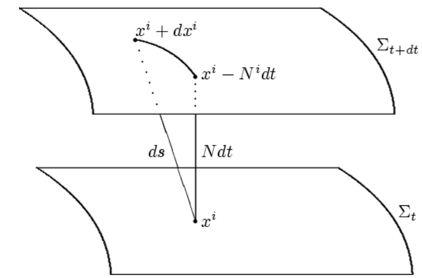
→ Strong coupling phase ($G_c < G$)





Canonical Formulations

ADM (Arnowitt, Deser, Misner) Formalism
(*continuum* classical canonical formalism)



Wheeler DeWitt Equation (*continuum* quantum canonical)

$$\hat{g}_{ij}(\mathbf{x}) \longrightarrow g_{ij}(\mathbf{x})$$
$$\hat{\pi}^{ij}(\mathbf{x}) \longrightarrow -i\hbar \cdot 16\pi G \cdot \frac{\delta}{\delta g_{ij}(\mathbf{x})}$$

Discrete Wheeler DeWitt Equation (*discrete* quantum canonical)

The lack of covariance of the canonical ADM approach has not gone away, and is therefore still part of the present formalism.

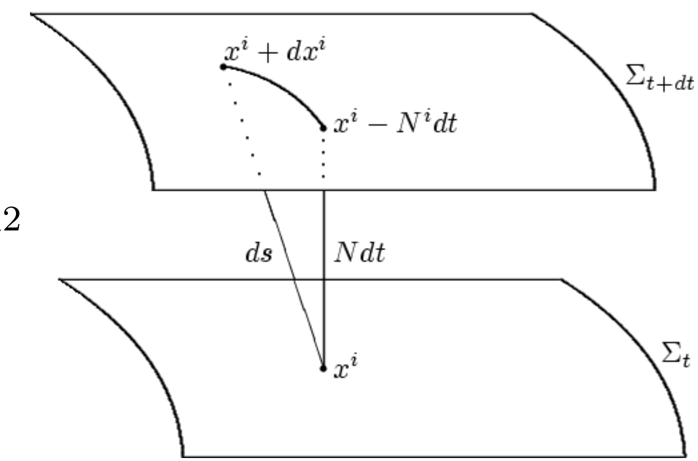
ADM Formalism
(*continuum* classical)

Introducing a time-slicing of space-time, by introducing a sequence of space like hypersurfaces.
ADM decomposition is based on the Lorentzian version of the Pythagoras theorem.

Just a choice of gauge

N :lapse (of proper time between two infinitesimally close hypersurfaces)

N^i :shift (in spatial coordinate)



$$
\begin{aligned}
ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\
&= g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) - N^2\, dt^2
\end{aligned}
$$

ADM gives $\mathcal{H} = 0$.

Continuum Quantum Canonical Formulation

Energy constraint $\hat{H}\,|\Psi\rangle \;=\; 0$

Wheeler De Witt Eq. in *d+1* dim.

$$\left\{-\,(16\pi G)^2\, G_{ij,kl}\,(x)\,\frac{\delta^2}{\delta g_{ij}\,(x)\;\delta g_{kl}\,(x)}\,-\,\sqrt{g\,(x)}\,\left({}^{(d)}R\,(x)\,-\,2\,\lambda\right)\right\}\Psi\left[g_{ij}\,(x)\right]=0$$

Supermetric: $G_{ij,kl}=\frac{1}{2}\,g^{-1/2}\;\left(g_{jk}\;g_{jl}+g_{il}\;g_{jk}-\;\alpha\;g_{ij}\;g_{kl}\right)$

Momentum constraint $\hat{H}_i\,|\Psi\rangle \;=\; 0$

$$\left\{2\,i\,g_{ij}(\mathbf{x})\,\nabla_k(\mathbf{x})\,\frac{\delta}{\delta g_{jk}(\mathbf{x})}\right\}\;\Psi[g_{ij}(\mathbf{x})]\;=\;0$$

This constraint implies that the gradient of Ψ on the superspace of g_{ij}'s is zero along those directions that correspond to gauge transformations, *i.e.*, diffeomorphisms on the *d*- space dimensional manifold, whose points are labeled by the coordinates x.

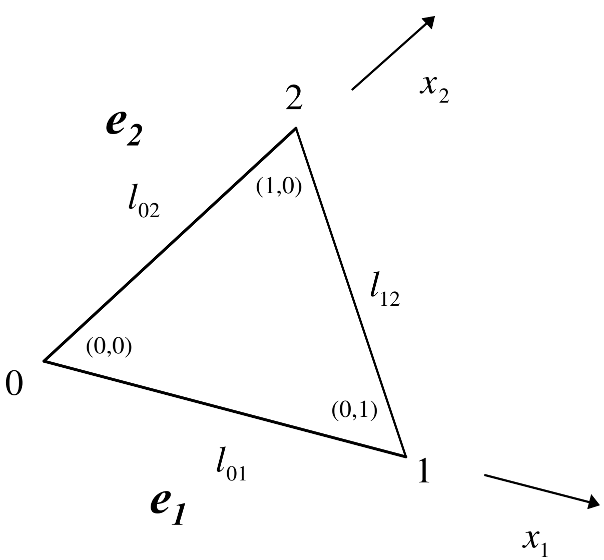
Regge Lattice *Discretization*:

Regge 1961

In constructing a *discrete* Hamiltonian for gravity one has to decide what degrees of freedom one should retain on the lattice.

→Use geometric *Regge lattice discretization* for gravity, with *edge lengths* suitably defined on a random lattice as *the primary dynamical variables*.

Degrees of freedom for edges and metric tensor are both *D(D+1)/2* in *D* dimensions.



$$g_{i\,j}\,(\sigma) = e_i \cdot e_j \qquad \boldsymbol{l_{ij} = |e_i - e_j|}$$

$$g_{ij}(\sigma) \;=\; \tfrac{1}{2}\,\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$$

simplex

Regge Formulation: Constituents

Flat building blocks are D-dim. Simplices $^{(D)}$

Point (0 –simplex) in 0-dim
Line (1 –simplex) in 1-dim
Triangle (2 –simplex) in 2-dim
Tetrahedron (3 –simplex) in 3-dim
all "flat"

Deficit angle → Curvature (defined at a hinge$^{(D-2)}$
at a vertex for 2-dim,
at an edge for 3-dim,
at a triangle for 4-dim)

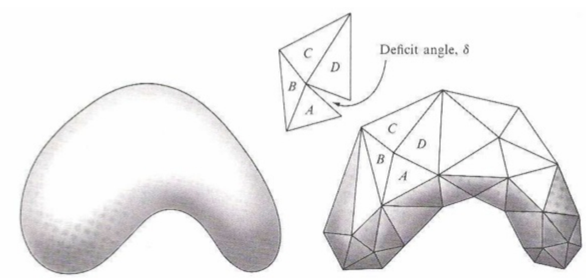
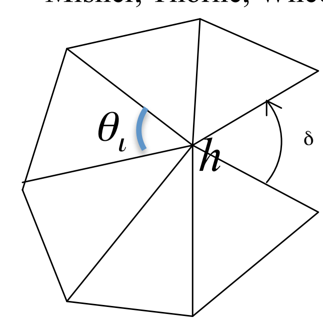


Figure 42.1.

Misner, Thorne, Wheeler

$$\delta(h) = 2\pi - \sum_{\sigma \supset h} \theta\left(\sigma, h\right)$$

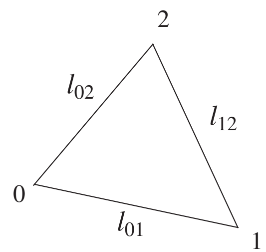
sum over dihedral angles θ extends over all simplices σ meeting on hinge h.

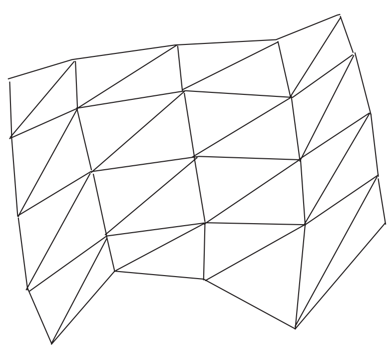


Regge Formulation: Visualization

2 dimensions

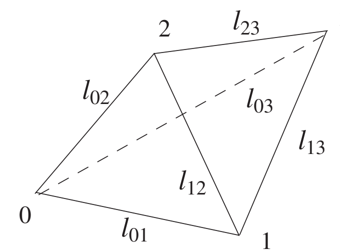
Triangle

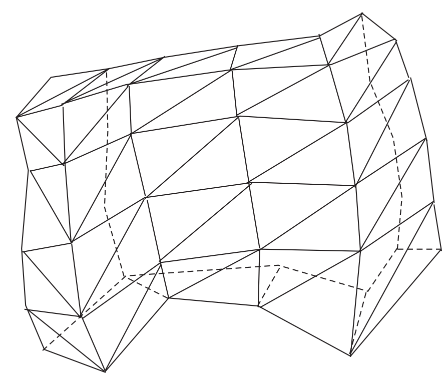




3 dimensions

Tetrahedron





Curved space(time)s are piece-wise linear.

$d+1$- Dimensional Discrete Wheeler DeWitt equation

$$\left\{-(16\pi G)^2\, G_{ij,kl}(x)\,\frac{\delta^2}{\delta g_{ij}(x)\,\delta g_{kl}(x)} - \sqrt{g(x)}\left({}^{(d)}R(x) - 2\lambda\right)\right\}\Psi\left[g_{ij}(x)\right] = 0$$ *Continuum*

$$g_{ij}(x)$$

$$\Downarrow$$

$$g_{ij}(\sigma) = \frac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$$

$$\left\{-(16\pi G)^2\, \underbrace{G_{ij}(l^2)\,\frac{\partial^2}{\partial l_i^2\,\partial l_j^2}}_{\text{Kinetic term}} - \sqrt{g(l^2)}\left(\underbrace{{}^{(d)}R(l^2)}_{\text{Curvature term}} - \underbrace{2\lambda}_{\text{Cosmological constant term}}\right)\right\}\Psi\left[l^2\right] = 0$$ *Discrete*

Both equations are defined at each “point” in space.

Discrete WDW eq.: one eq. for each simplex

Kinetic Term

$$\left\{ \boxed{-(16\pi G)^2 \, G_{ij}\left(l^2\right) \frac{\partial^2}{\partial l_i^2 \, \partial l_j^2}} - \sqrt{g\left(l^2\right)} \left({}^{(d)}R\left(l^2\right) - 2\,\lambda \right) \right\} \Psi\left[l^2\right] = 0$$

Two ways to implement kinetic term:

1. Use relation between metric and edge length^2 *i.e.,* $g_{ij}(\sigma) = \frac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$

Starting with continuum kinetic term $G_{ij,kl}\left(x\right) \dfrac{\delta^2}{\delta g_{ij}\left(x\right) \, \delta g_{kl}\left(x\right)}$

where $G_{ij,kl}(\sigma) = \frac{1}{2}\, g^{-1/2}(\sigma)\left[g_{ik}(\sigma)g_{jl}(\sigma) + g_{il}(\sigma)g_{jk}(\sigma) - \alpha g_{ij}(\sigma)g_{kl}(\sigma)\right]$

$$\frac{\partial}{\partial g_{ij}(l^2)} = \sum_m \frac{\partial l_m^2}{\partial g_{ij}} \frac{\partial}{\partial l_m^2}$$

$\alpha = 2$ for 2 +1 dimensions

2. Use result of Regge and Lund

$$G^{ij}(l^2) = -\, d! \sum_\sigma \frac{1}{V(\sigma)} \frac{\partial^2 \, V^2(\sigma)}{\partial l_i^2 \, \partial l_j^2}$$

with $V^2(\sigma) = \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(\sigma))$

Only involves the variables within one simplex.

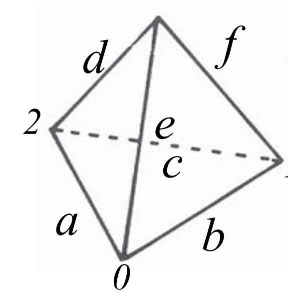
Curvature Term

$$\left\{-(16\pi G)^2 G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2\,\partial l_j^2}\boxed{-\sqrt{g(l^2)}\left({}^{(d)}R(l^2)\right.}\left.-2\lambda\right)\right\}\Psi\left[l^2\right]=0$$

Curvature term in WDW eq. in d+1 dim. $\sqrt{g}\;{}^{(d)}R = \dfrac{2}{q}\displaystyle\sum_{h\subset\sigma}\delta_h\;{}^{(d-2)}V_h$

q : how many simplices are meeting at a vertex

e.g., tetrahedron configuration in $2+1$ dim. for the triangle abc,



$$\sum_{h\subset\sigma}\delta_h\;{}^{(d-2)}V_h \;\xrightarrow[d=2]{}\; \sum_{h\subset\triangle}\delta_h \;\xrightarrow[\text{triangle }abc]{}\; \delta_0+\delta_1+\delta_2$$

$$\delta_0 = 2\pi - \cos^{-1}\left(\frac{a+c-b}{2\sqrt{a}\sqrt{c}}\right) - \cos^{-1}\left(\frac{a+d-e}{2\sqrt{a}\sqrt{d}}\right) - \cos^{-1}\left(\frac{c+d-f}{2\sqrt{c}\sqrt{d}}\right)$$

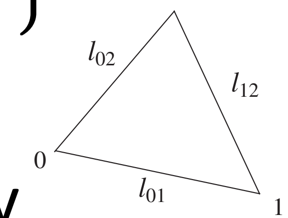
$$\delta_1 = 2\pi - \cos^{-1}\left(\frac{a+b-c}{2\sqrt{a}\sqrt{b}}\right) - \cos^{-1}\left(\frac{a+e-d}{2\sqrt{a}\sqrt{e}}\right) - \cos^{-1}\left(\frac{b+e-f}{2\sqrt{b}\sqrt{e}}\right)$$

$$\delta_2 = 2\pi - \cos^{-1}\left(\frac{b+c-a}{2\sqrt{b}\sqrt{c}}\right) - \cos^{-1}\left(\frac{c+f-d}{2\sqrt{c}\sqrt{f}}\right) - \cos^{-1}\left(\frac{b+f-e}{2\sqrt{b}\sqrt{f}}\right)$$

Involves the variables of the neighbor simplices of a simplex.

Discrete WDW eqns in *2+1 dim.*

$$\left\{-(16\pi G)^2\,(-\,4\;A_\triangle)\left(\frac{\partial^2}{\partial l_{01}\;\partial l_{02}}+\frac{\partial^2}{\partial l_{02}\;\partial l_{12}}+\frac{\partial^2}{\partial l_{12}\;\partial l_{01}}\right)-\frac{2}{q}\sum_{i\subset\triangle}\delta_i+2\lambda\;A_\triangle\right\}\Psi\left[l^2\right]_2=0$$



$\Psi[l^2]$ is a function of the whole simplicial geometry (overall *geometry* of the manifold), due to the built-in *diffeomorphism invariance*.

$\Psi[l^2]$ depends collectively on all the edge lengths in the lattice.

Therefore even though we have one equation for each simplex, there should be *one wave function that satisfies all the equations* for each simplices in one configuration.

Scaled Units

(Like a field redefinition)

Recall Euclidean Path Integral $P(V) \propto \exp(-\lambda_0 V)$

where $V = \int \sqrt{g}$

To compare different lattice configurations,
the choice $\lambda_0 = 1$ fixes the overall scale once and for all.

Bare cosmological constant and scaled cosmological constant $\lambda_0 = 2\lambda / 16\pi G$

So $\lambda = 8\pi G$ in units of λ_0

Strong Coupling Limit

Schematically, in $d+1$ dim., discrete WdW eq. looks like in units of λ_0

$$\left\{ -G\,\mathbf{\Delta} - \frac{1}{G}\,{}^{(d)}R + 1 \right\} \Psi = 0$$

Note that for $1/G \to 0$, the coupling term between different simplices, which is due to the curvature term, disappears.

One ends up with a *decoupled* problem, where the edge lengths in each simplex fluctuate independently.

Exact Solution for A Single Triangle *(2+1 dim.)*

A single triangle:

- Curvature term is absent in this configuration.
- as a starting point for the strong coupling expansion in *1/G*.
- should show the physical nature of the wavefunction solution deep in the strong coupling regime.

$$(16\pi G \to G\ )$$

WDW eq: $$\left\{ G^2\, 4A_\triangle \left(\frac{\partial^2}{\partial a\, \partial b} + \frac{\partial^2}{\partial b\, \partial c} + \frac{\partial^2}{\partial c\, \partial a} \right) + 2\lambda A_\triangle \right\} \Psi[\, a,\ b,\ c\,] = 0$$

$$\Psi\,[\,a,\ b,\ c\,] = \mathcal{N}\, \frac{J_{1/2}\left(\frac{2\sqrt{2}\,\lambda}{G} \boxed{A_\Delta} \right)}{\left(\frac{2\sqrt{2}\,\lambda}{G} \boxed{A_\Delta} \right)^{1/2}} \qquad \left(= \tilde{\mathcal{N}}\, \frac{\sin\left(\frac{2\sqrt{2}\,\lambda}{G}\, A_\triangle \right)}{A_\triangle} \right)$$

Normalization constant fixed by the standard rule of quantum mechanics: $$\int_0^\infty dA_\Delta\, |\, \Psi(A_\Delta)\, |^2 = 1$$

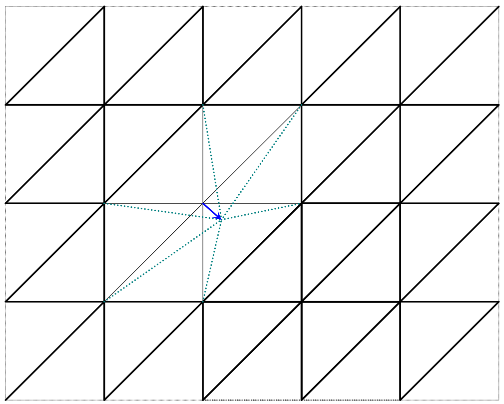
Significance of Single Triangle Solution *(2+1 dim.)*

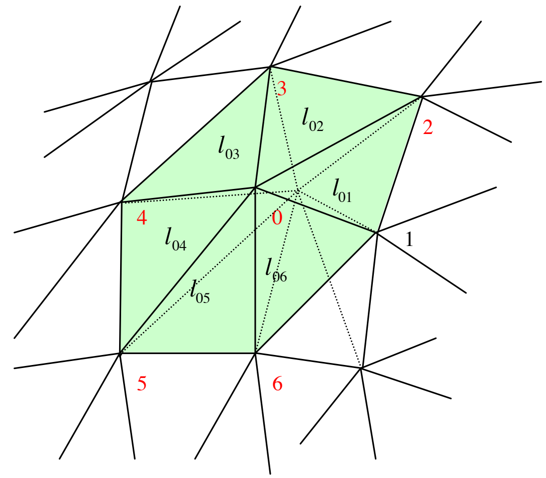
nontrivial result $\qquad \Psi\left[\,a,\ b,\ c\,\right] = \Psi\left[\boxed{A_\triangle}\right]$

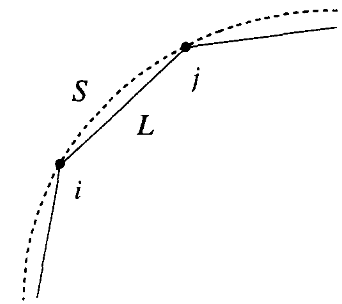
Since a discretization of space-time breaks the diffeomorphism invariance, it raises the question of whether and in what form part of the diffeomorphism symmetry can still be realized at the discrete level.

The solution only depends on geometry
i.e., spatial diffeomorphism is retained.

Lattice Diffeomorphism







Consider the two-dimensional skeleton. One can move around a point on the surface, keeping all the neighbors fixed, without violating the triangle inequalities and leave all curvature invariants unchanged. (no preferred directions)

It is a local deformation of the edges corresponding to the local gauge transform.

In d dimensions this transformation has d parameters and is an exact invariance of the action. (HWH, RMW 1998, Hartle 1985, Regge 1961, Wheeler 1964)

Problem Set-up *(2+1 dim.)*

In principle, any solution of the Wheeler-DeWitt equation corresponds to a possible quantum state of the universe.

The boundary conditions on the wavefunction will act to restrict the class of possible solutions;

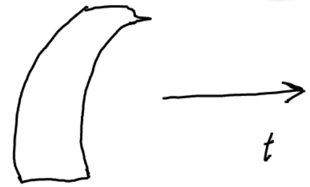
in ordinary quantum mechanics, they are determined by the physical context of the problem and some set of external conditions.

In our analytical calculations, we used *spherical boundary conditions for the spatial manifold*,
further, regular polyhedra approximations to a 2-sphere.

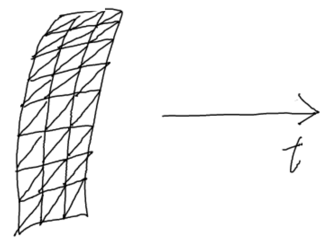
Problem Set-up *(2+1 dim.)*

The idea:

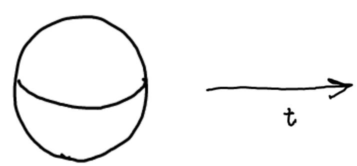
$$R \bigotimes S^2$$

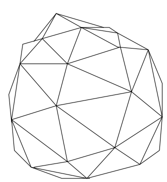


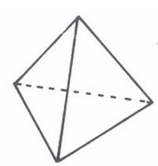
Latticize



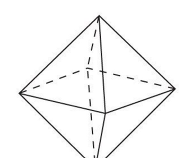
Boundary condition is such that:



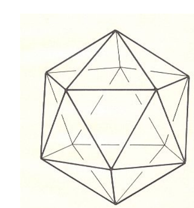
(More precisely  , further more



tetrahedron



octahedron



icosahedron)

With Curvature, and Equilateral *(2+1 dim.)*

Equilateral (edges fluctuating together)

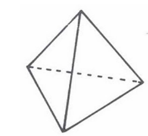
WDW eq. $$\Psi'' + \frac{2n+1}{x}\,\Psi' \boxed{-\frac{2\beta}{x}\,\Psi} + \Psi = 0$$

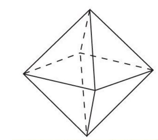
$$x = \frac{\sqrt{2\,\lambda}}{G}\,A_{tot}$$

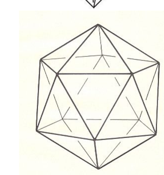
$$n = \tfrac{1}{4}\,(N_\Delta - 2)$$

$$\beta = \frac{2\sqrt{2}\,\pi\,G}{\sqrt{\lambda}}$$

$$\left\{\begin{array}{l} n_{tetra} = \frac{1}{2} \\ n_{octa} = \frac{3}{2} \\ n_{icosa} = \frac{9}{2} \end{array}\right.$$







Regular solution: $$\Psi(x) \simeq e^{-i\,x}\,{}_1F_1\left(n + \tfrac{1}{2} \boxed{- i\,\beta,} \; 2\,n+1,\, 2\,i\,x\right)$$

$$= \frac{F_l(\boxed{\beta,}\, x)}{x^{n+\frac{1}{2}}}$$ ← Coulomb wave function

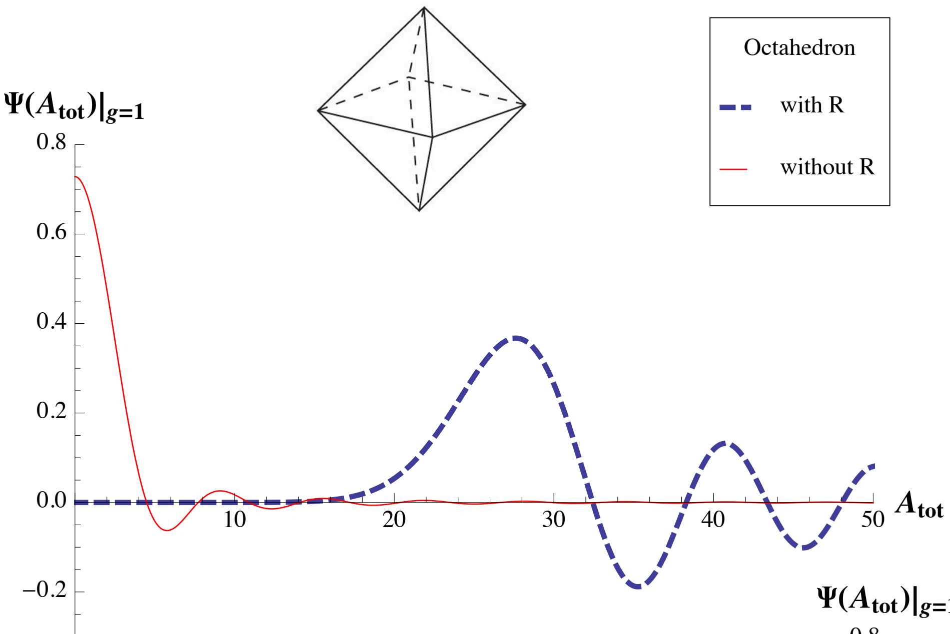
$$l = n - \tfrac{1}{2} = N_\Delta/4 - 1$$

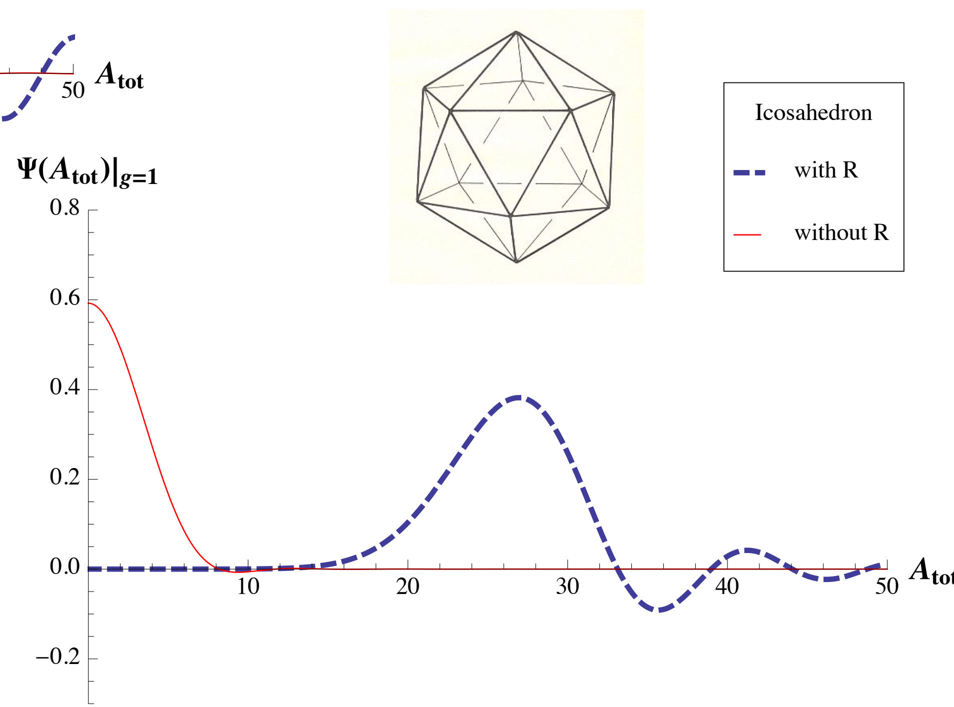
→ $\Psi(x) = \mathcal{N}\,\frac{J_n(x)}{x^n}$ (without curvature, for strong coupling)

Wave function with/without R
(2+1 dim.)

$$\Psi(x) \;\simeq\; e^{-i\,x}\;{}_1F_1\left(n+\tfrac{1}{2}-i\,\beta,\;2\,n+1,\;2\,i\,x\right)$$

$$=\;\frac{F_l(\beta,x)}{x^{n+\frac{1}{2}}}$$





Curvature and Euler Characteristics *(2+1 dim.)*

in *2* dimensions $\int d^2x \sqrt{g}\, R \,=\, 4\pi\,\chi$ (Gauss Bonnet theorem)

χ : Euler characteristics of the manifold $\chi \,=\, 2 - 2g$

$$\begin{cases} \chi = 2 & \text{(sphere)} \\ \chi = 0 & \text{(torus)} \end{cases}$$

where *g* is genus

On a discrete manifold in two dimensions

$$\chi = N_0 - N_1 + N_2$$

N_i : number of simplices of dimension i
N_0 : sites (vertices)
N_1 : edges
N_2 : faces

$$\beta = \frac{\sqrt{2}\,\pi\,\chi}{\sqrt{\lambda}\,G}$$

β's dependence on boundary conditions becomes explicit.

Key Results *(2+1 dim.)* so far from tetra, octa, and icosahedra

-The solution is in totally a generalized form, $\Psi(x) \;=\; \frac{F_l(\beta,x)}{x^{n+\frac{1}{2}}}$

$i.e.$, Ψ [topology, total area, number of triangles]

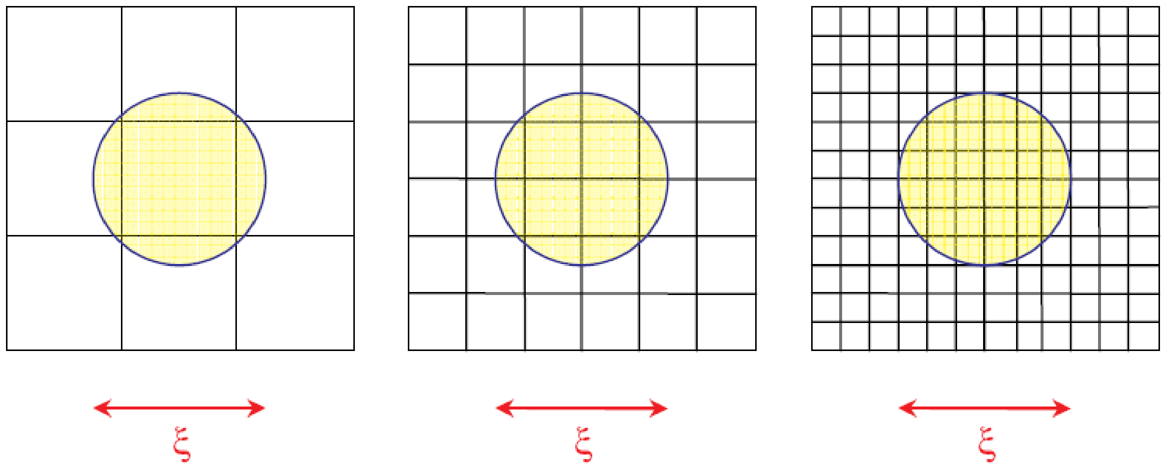
- The solution only depends on the geometric quantities such as total areas in *2 + 1* dimensions.

  (does not just depend on quantities like edge lengths which are not diffeomorphism invariant)

i.e., confirming that this Regge based discretization formulation preserves diffeomorphism.

Lattice Continuum Limit

$$a << \xi$$



The lattice quantum continuum limit is gradually approached by considering sequences of lattices with increasingly larger correlation length in lattice units.
Such a limit requires the existence of an ultraviolet fixed point, where quantum field correlations extend over many lattice spacing, a.

Continuum limit requires the existence of an UV fixed point.

Key Quantities Associated with Phase Transitions

-Universal Exponent ν $\nu^{-1} = - \beta\,'\,(G_c)$ *cutoff-independent* quantity

-Averages $< V > \sim \dfrac{\partial}{\partial \lambda_0} \ln Z_{latt}$

-Fluctuations $\chi_V \sim \dfrac{\partial^2}{\partial \lambda_0^2} \ln Z_{latt}$

A divergence or non-analyticity in Z, as caused for example by a phase transition, is expected to show up in these *local averages* as well.

Scaling Assumption

Correlation length is given by $\xi \sim |g - g_c|^{-\nu}$

e.g., Parisi, Cardy

A divergence of correlation length signals the presence of transition and leads to the appearance of singularity in free energy. $F \sim \frac{1}{V} lnZ$

Scaling Assumption: $F_{sing} \sim \xi^{-d}$ therefore $F_{sing} \sim |g - g_c|^{d\,\nu}$

$$\chi \sim \frac{1}{V}\frac{\partial^2}{\partial g^2} lnZ \sim |g - g_c|^{d\,\nu - 2} \sim \xi^{\frac{2 - d\,\nu}{\nu}}$$

For g close to the critical point g_c, the correlation length saturates to its maximum value $\xi \sim L$.

knowing $L \sim \sqrt{N_\triangle} \sim \sqrt{n}$

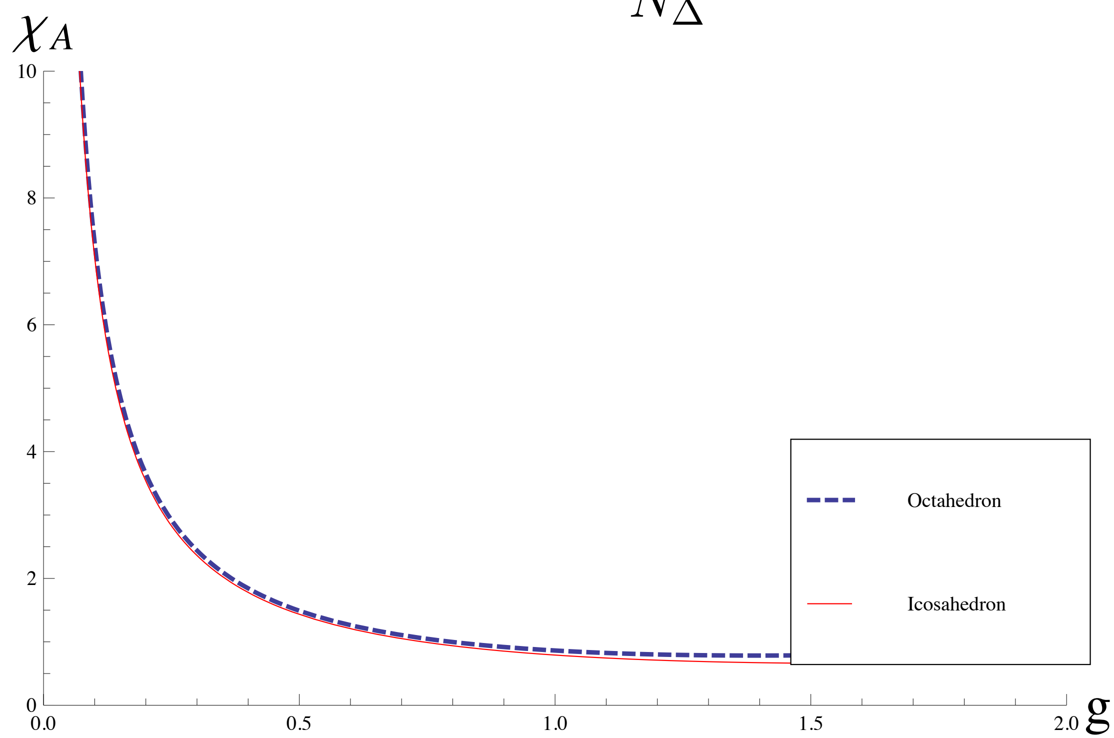
$$\therefore \boxed{\chi_A \underset{g \to g_c}{\sim} \xi^{\frac{2 - 3\,\nu}{\nu}} \sim n^{\frac{1}{\nu} - \frac{3}{2}}} \quad \text{(Fluctuation)}$$

n-dependence of χ provides a way to estimate the exponent ν directly.

Numerically Computed χ_A *(2+1 dim.)*

Using $$\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}}$$

Numerically computed $$\chi_A = \frac{1}{N_\Delta}\{< (A_{tot})^2 > - < A_{tot} >^2\}$$



$(g \equiv \sqrt{G})$

χ_A diverges as $g \to 0$ signaling phase transition

Analytical Expression *(2+1 dim.)*

$$\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}} \qquad l = n - \frac{1}{2}$$

Coulomb wave function

$$F_l\left(\beta, x\right) = \frac{2^{l+1}}{\sqrt{\pi}}\,\Gamma\left(l+\frac{3}{2}\right)\, C_l\left(\beta\right)\, x\sqrt{\frac{\pi}{2\,x}}\,\left\{\sum_{k=l}^{\infty} b_k\left(\beta\right)\, J_{k+\frac{1}{2}}\left(x\right)\right\}$$

with {...} ~ $J_{l+\frac{1}{2}}\left(x\right) + \frac{2\,l+3}{l+1}\,\boxed{\beta}\, J_{l+\frac{3}{2}}\left(x\right) + \frac{2\,l+5}{l+1}\,\boxed{\beta^2}\, J_{l+\frac{5}{2}}\left(x\right)$

$$+\left\{-\frac{\left(2\,l+7\right)}{3\,\left(l+2\right)}\,\boxed{\beta} + \frac{\left(2\,l+7\right)\left(2\,l+5\right)}{3\,\left(l+2\right)\left(l+1\right)}\,\boxed{\beta^3}\right\} J_{l+\frac{7}{2}}\left(x\right)\cdots$$

with more terms linear in β appearing in higher orders of J

Including infinite orders of β means including infinitely many orders of Bessel functions in the expansion, therefore means obtaining exact coulomb wave function.

Analytical Asymptotic Result
Critical Exponent ν

β^m : Send $m \longrightarrow \infty$

Require $< A_\Delta > \sim \dfrac{1}{g^{3m-1}\, n^{\frac{m+1}{2}}}$ to be finite as n is large (thermodynamic limit)

$$\therefore \quad g(n) \sim \frac{1}{n^{\frac{m+1}{2(3m-1)}}}$$

Then in turn $\chi_A \sim \dfrac{1}{g^{3m-2}\, n^{\frac{m}{2}}} \longrightarrow \boxed{\chi_A \underset{m\to\infty}{\sim} n^{\frac{1}{3}}}$

But know from scaling argument $\chi_A \sim n^{\frac{1}{\nu} - \frac{3}{2}}$

$$\therefore \quad \boxed{\nu = \frac{6}{11} = 0.5454...}$$

Conclusions *(2+1 dim.)*

$$\nu = \frac{6}{11} = 0.5454\ldots$$ for *2 + 1* dimension, Lorenzian

- Does not seem to depend on Euler characteristic χ, and therefore on the boundary conditions.

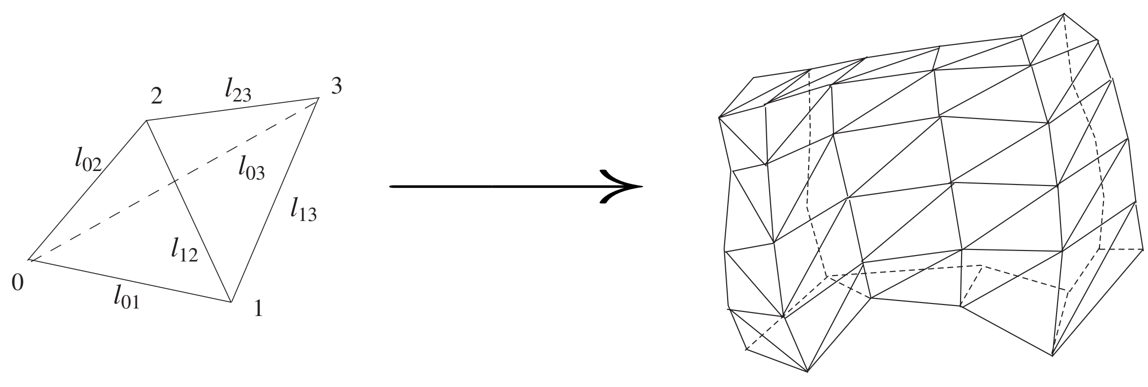
- Compare with the numerically exact Euclidean three-dimensional quantum gravity result obtained in Hamber and Williams Phys. Rev. D47, 510 (1993), $\nu \sim 0.59(2)$. The exponent ν is expected to represent a universal quantity, independent of short distance regularization details. Therefore, it should apply to both the Lorentzian and Euclidean formulation, and our results are consistent with this conclusion.

- $Gc \rightarrow 0$, indicating that weak coupling is not present at all.

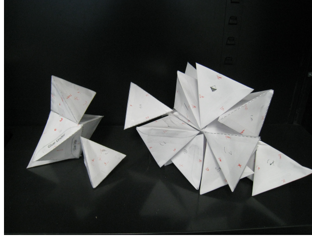
Discrete
Wheeler DeWitt equations
in *3 + 1* dimensions

Discrete Wheeler DeWitt equations in 3 + 1 dimensions

Building blocks are tetrahedra.



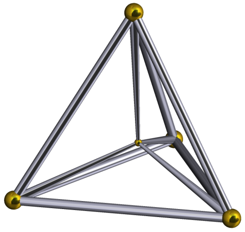
Regular Triangulations
(3 + 1 dim.)

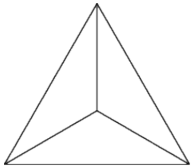


Regular triangulations of 3-sphere

2 sphere analogs

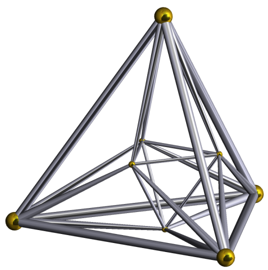
5 cell ($q = 3$)
5 tetrahedra glued together
"Hyper-tetrahedron"

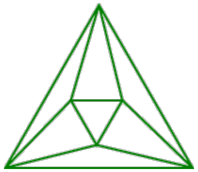




(Tetrahedron)

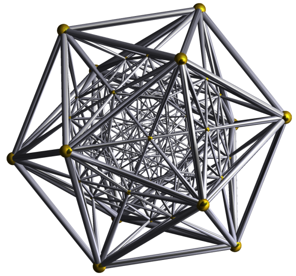
16 cell ($q = 4$)
16 tetrahedra glued together
"Hyper-octahedron"

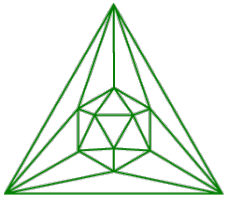




(Octahedron)

600 cell ($q = 5$)
600 tetrahedra glued together
"Hyper-icosahedron"





(Icosahedron)

Schlegel diagrams

Discrete Wheeler DeWitt Equation (*3* + *1* dim.)

$$\left\{ -\,(16\pi G)^2 \sum_{i,j\subset\sigma} G_{ij}\,(\sigma)\,\frac{\partial^2}{\partial l_i^2\,\partial l_j^2} - \frac{2}{q}\sum_{h\subset\sigma} l_h\,\delta_h + 2\,\lambda\,V_\sigma \right\} \Psi\left[l^2\right] = 0$$

$$R_{tot} \equiv\; 2 \sum_{h\subset\sum\sigma} \delta_h\, l_h \;\longleftrightarrow\; \int \sqrt{g}\, R$$

$\rightarrow$ $\boxed{\psi(R_{tot}, V_{tot})}$

Differential eq. ($3+1$ dim.)

WDW: $$\boxed{\frac{\partial^2\psi}{\partial V^2}} + \boxed{c_V\,\frac{\partial\,\psi}{\partial\,V}} + c_R\,\frac{\partial\,\psi}{\partial\,R} + c_{VR}\,\frac{\partial^2\,\psi}{\partial\,V\partial\,R} + c_{RR}\,\frac{\partial^2\,\psi}{\partial\,R^2} + \boxed{c_\lambda\,\psi} + \boxed{c_{curv}\,\psi} = 0$$

$$\left\{\begin{aligned}
c_V &= \frac{11+9\,q}{2\,q^2}\cdot\frac{N_3}{V} = \boxed{\frac{11+9\,q_0}{2\,q_0^2}\cdot\frac{N_3}{V}} + \frac{22+9\,q_0}{48\,\sqrt{2}\,3^{1/3}\,\pi\,q_0}\cdot\frac{N_3^{1/3}\,R}{V^{4/3}} + \mathcal{O}(R^2)\\
c_R &= -\frac{2}{9}\,\frac{R}{V^2} + \frac{11+9\,q_0}{6\,q_0^2}\cdot\frac{N_3\,R}{V^2} + \mathcal{O}(R^2)\\
c_{VR} &= \frac{2}{3}\,\frac{R}{V} + \mathcal{O}(R^2)\\
c_{RR} &= \frac{2}{9}\,\frac{R^2}{V^2}\\
c_\lambda &= \frac{32\,\lambda}{q^2\,G^2} = \boxed{\frac{32}{G^2\,q_0^2} + \frac{4\sqrt{2}\lambda}{3\,3^{1/3}\,\pi\,q_0\,G}\cdot\frac{R}{N_3^{2/3}\,V^{1/3}}} + \mathcal{O}(R^2)\\
c_{curv} &= -\frac{16}{G^2\,q^2}\cdot\frac{R}{V} = \boxed{-\frac{16}{G^2\,q_0^2}\cdot\frac{R}{V}} + \mathcal{O}(R^2)\,.
\end{aligned}\right.$$

q_0: flat (*i.e.*, $R = 0$)

So far we have not been able to find the general solution for the above differential eq. but probably still some type of Bessel function or hypergeometric function.

Simplified Differential eq.
(*3* + *1* dim.)

In the limit of the *small curvature* and the *large volume*,

Further, set $c_{VR} = 0$ and keeps only the leading term in c_V

$$\frac{\partial^2 \psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_\lambda \psi + c_{curv} \psi = 0$$

$$g = \sqrt{G}$$

$$\psi(V,\, R) \;\simeq\; e^{-\frac{4\,i\,V}{q_0\,g}} \cdot \frac{\Gamma\left(\frac{(11+9\,q_0)\,N_3}{4\,q_0^2} + \frac{2\,i\,R}{q_0\,g^3}\right)}{\Gamma\left(1 - \frac{(11+9\,q_0)\,N_3}{4\,q_0^2} + \frac{2\,i\,R}{q_0\,g^3}\right)}$$

$$ {}_1F_1\left(\frac{(11+9\,q_0)N_3}{4\,q_0^2} - \frac{2\,i\,R}{q_0\,g^3},\; \frac{(11+9\,q_0)N_3}{2\,q_0^2},\; \frac{8\,i\,V}{q_0\,g}\right)$$

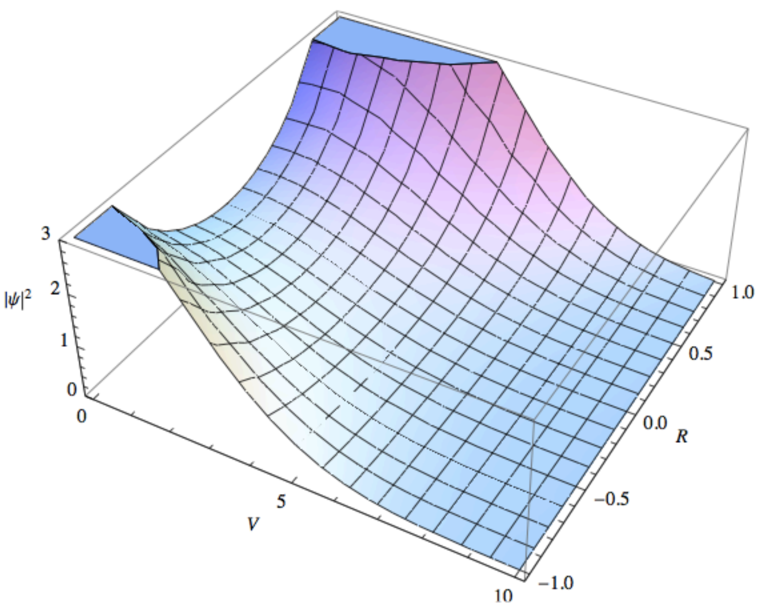
${}_1F_1$: confluent hypergeometric function of first kind

Check: a function of geometric invariants *V* and *R* only.

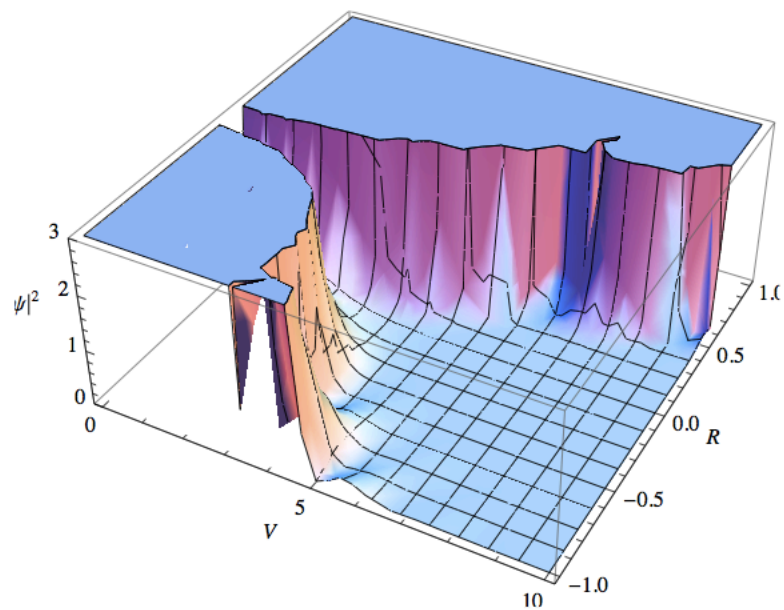
Probability as a function of V and R ($3+1$ dim.)

Not trivial $\quad \langle \mathcal{O}(l^2) \rangle \equiv \frac{\langle \Psi | \mathcal{O} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\int d\mu[l^2] \cdot \mathcal{O}(l^2) \cdot |\Psi[l^2]|^2}{\int d\mu[l^2] \cdot |\Psi[l^2]|^2}$

$g = \sqrt{G}$



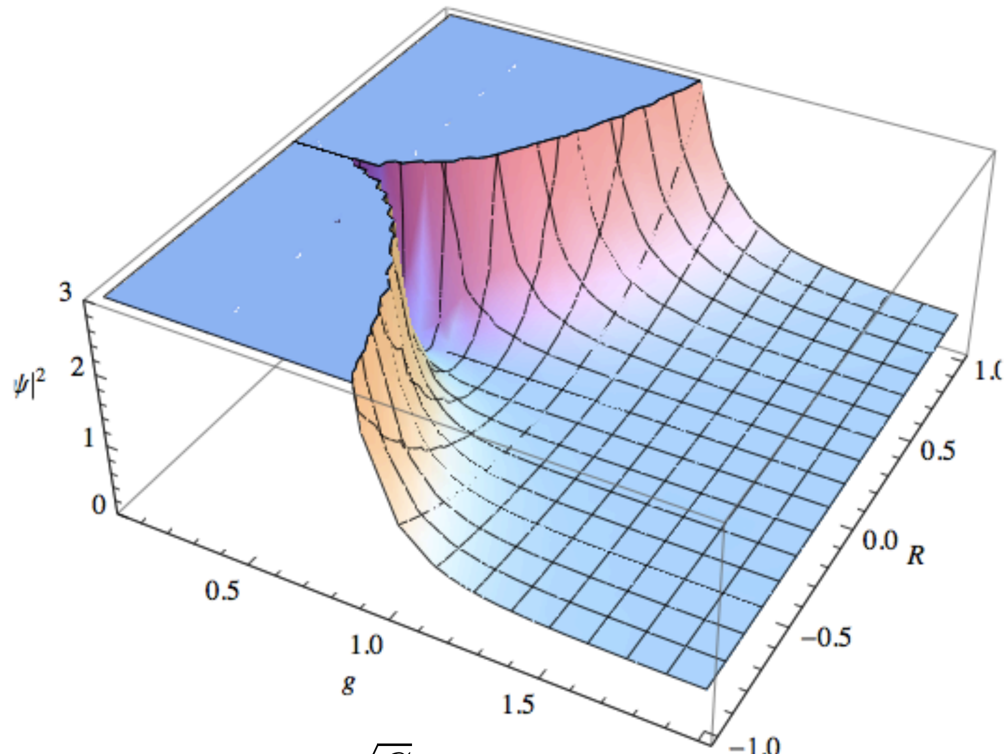
strong coupling $\quad g = \sqrt{G} = 1$



Weak coupling $\quad g = \sqrt{G} = 0.5$

$N_3 = 10$

Probability as a function of G ($3 + 1$ dim.)



$g = \sqrt{G}$

$N_3 = 10$

For strong coupling, different curvature scales are equally important.

Very small probability at $R \sim 0$ for small G, *so no sensible continuum limit.*

Still work in progress

($3 + 1$ dim.)