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GEOMETRICAL ASPECTS OF SYMMETRY BREAKING

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0. INTRODUCTION

Broken symmetry is a fascinating subject of physics. Essentially it occurs in this fashion: for a physical problem we have a group of invariance G and the solution of the problem is invariant only under a subgroup $H \subset G$. For instance the interactions among the ions or atoms of a crystal are invariant by translations and rotations and therefore under the group G they generate, namely E(3), the Euclidean group in three dimensions. However the crystal itself is invariant only under its crystallographic group H which is a subgroup of G = E(3).

Given a solution with broken symmetry of a G-invariant problem, this solution can be transformed by G and yields a family (we shall say a G-orbit) of solutions. For instance, by rotating and translating a crystal we obtain a family of states of the same crystal.

To know which state of the family (= which point of the G-orbit) occurs is sometimes not specially interesting; the answer may invoke extraneous conditions: the presence of impurities, inhomogeneities, crystal seeds decide that the crystal is formed by a growth from this point and not from a neighbouring point. For other problems, as you will see in the other lecturers courses, the choice of a particular point in the orbit of solutions is physically very important. In this set of lectures we will mainly study another aspect of symmetry breaking: which subgroups H of G can be groups of invariance for states with broken G-symmetry?.

In the first lecture we will discuss this problem when G is the Euclidean group E(3) (see II). We will have to use some general (geometrical) concepts for the study of group actions (I). The second lecture will be devoted to the special, but very interesting case of differentiable action of a compact Lie group on a manifold (III). This will give us some hints as to why the symmetry G may prefer to break into a given subgroup H.

The last two lectures (IV) will be devoted to the geometrical study of the breaking of the symmetry of the hadrons. The approach will be quite tentative because it does not correspond to the study of the breaking for a given state, but rather one wants to consider the breaking of symmetry of the physical laws themselves. We shall show that all directions of breaking of the hadronic symmetry $G=(SU(3)\times SU(3))$ (1, P, C, PC) are very special, with exceptional mathematical properties that we will have to explain in these lectures. The two main properties of these directions are:

1) they lie on critical orbits for some G-action,

2) they are idempotents or nilpotents of a G-invariant algebra.

Note.- Notation for groups:

Cyclic group of n elements: Z_n; Z_∞ is also denoted by Z.

Additive group of integers: Z, of real numbers: R.

Multiplicative group of positive real numbers R_+^x .

General linear group of nxn real (resp. complex) inversible matrices GL(n, R) (resp.GL(n, ¢)). Their subgroups of matrices with determinant one: SL(n, R), SL(n, ¢).

Group of nxn unitary matrices U(n); then SU(n) = = U(n) $\cap$ SL(n, ¢).

Group of nxn real orthogonal matrices O(n) = U(n) $\cap$ GL(n,R); then SO(n) = O(n) $\cap$ SL(n, R).

E(n) is the Euclidean group (= O(n) - inhomogeneous group) in n (real) dimensions and DE(n) the corresponding group of "similitude" (Euclidean transformations and dilatations).

The pseudo-orthogonal groups are denoted by O(p, q).

I. FUNDAMENTAL NOTIONS FOR THE STUDY OF GROUP ACTIONS

We just give here the essential definitions and results#.

I.1 Definitions

Consider a set $\mathcal{E}$ carrying a mathematical structure (e.g. set, vector space, manifold).We denote by Aut $\mathcal{E}$ its automorphism group; e.g.:

for $\mathcal{E}$ a set, Aut $\mathcal{E}$ is the group $\mathcal{P}(\mathcal{E})$ of permutations of elements of $\mathcal{E}$.

for $\mathcal{E}$ a vector space, Aut $\mathcal{E}$ = GL($\mathcal{E}$), the linear group on $\mathcal{E}$ whose elements are the inversible linear operators on $\mathcal{E}$.

for $\mathcal{E}$ a manifold, Aut $\mathcal{E}$ = Diff $\mathcal{E}$, the group of diffeomorphisms of $\mathcal{E}$.

The participants to the GIFT III Seminar were given the text of two lectures given elsewhere; they contain proofs and more details.

Def. 1.- A group action of G on $\mathcal{E}$ is a homomorphism $G \xrightarrow{f} \mathrm{Aut}\,\mathcal{E}$

G acts effectively on $\mathcal{E}$ if Ker f, the kernel of f, is trivial. e.g. $\mathcal{E}$, vector space, f is a linear representation of G on $\mathcal{E}$, which is faithful when Ker f = {e}.

The transform of $m \in \mathcal{E}$ by $g \in G$ is denoted by f(g).m or simply g.m.

Our next task is to compare the actions of G.

Def. 2.- Two actions of G: G, $\mathcal{E}$, f and G, $\mathcal{E}'$, f' are equivariant if there is a morphism[#] $\mathcal{E} \xrightarrow{\theta} \mathcal{E}'$ compatible with the group action:

$$\forall g \in G, \forall m \in \mathcal{E}, \quad f'(g).\theta(m) = \theta(f(g).m) \qquad (1.1)$$

It is very convenient to use commutative diagrams of morphisms:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\theta} & \mathcal{E}' \\ {\scriptstyle f(g)}\downarrow & & \downarrow{\scriptstyle f'(g)} \\ \mathcal{E} & \xrightarrow{\theta} & \mathcal{E}' \end{array} \qquad \text{(Diagram 1)}$$

i.e. $\forall g \in G$, $\theta_{\circ} f(g) = f'(g)_{\circ}\theta$.

One also says that θ is an equivariant morphism.

A morphism is a structure preserving map. For sets, vector spaces, manifolds, morphisms are respectively called maps, linear maps, smooth maps.

Def. 3.- Two G actions are equivalent if there exists between them an equivariant isomorphism. That is, θ is inversible and

$$f'(g) = \theta o f(g) o \theta^{-1} \qquad (1.2)$$

An interesting problem is to class the inequivalent actions of G on $\mathcal{E}$.

I.2 Classification of orbits

Def. 4.- The orbit of $m \in \mathcal{E}$ is the set $G.m = \{g.m,\ g \in G\}$ of the transforms of m by G.

Def. 5.- The little group (= stabilizer = isotropy group) of $m \in \mathcal{E}$ is the set $G_m = \{g \in G,\ g.m = m\}$. It is a subgroup of G.

If $n \in G.m$, $\exists g \in G$ such that $n = g.m$; therefore

$$G_n = G_{g.m} = g G_m g^{-1} \qquad (1.3)$$

One says that G_n and G_m are conjugated in G.

The abstract set of all orbits equivalent (see Def. 3) to a given one $\mathcal{O}$ is called an orbit type, and it is denoted by $(\mathcal{O})$. The set of all subgroups of G conjugated to a given subgroup H is called the conjugacy class of H, and it is denoted by (H). The fundamental theorem for the classification of orbits is:

Theorem 1.- To each type of G-orbits $(\mathcal{O})$ there corresponds a conjugacy class (H), and conversely, to each (H) there corresponds an orbit type $(\mathcal{O})$. The groups H of (H) are the little groups G_m of the points m of any orbit of $(\mathcal{O})$.

The subgroups of a group G form a lattice, with the partial order: $H \leq H' \iff H \subset H'$.

The conjugacy classes of the subgroups of G form also a lattice: $(H) \leq (H') \iff \forall H \in (H), \forall H' \in (H'), \exists g \in G, gHg^{-1} \subset H'$.

With theorem 1, this defines a partial order on the set of G-orbit types. It is customary to take the reverse order: the larger the little group, the smaller the orbit. Indeed for a Lie group G and its closed subgroups H we have:

dim G = (dim of little group $H = G_m$) +(dimension of orbit G.m)

We shall often denote a G-orbit of type (H) by $[G:H]$. A prototype of such an orbit is the set of left cosets gH of H in G, with the left multiplication g.aH = gaH for the action of G.

I.3 Orbit space. Layers

Consider an action of G on $\mathcal{E}$. It partitions $\mathcal{E}$ into (disjoint) orbits.

Def. 6.- The set of G-orbits of $\mathcal{E}$ will be denoted by $\mathcal{E}/G$ and it is called orbit space. We shall denote by π the canonical map $\mathcal{E} \xrightarrow{\pi} \mathcal{E}/G$. The set of all elements of $\mathcal{E}$ whose little group is conjugated to G_m is called the layer of m and we shall denote it by S(m). Hence $\mathcal{E}$, by the action of G, is partitioned into layers; each layer is the union of all orbits of the same type. Given an action of G on $\mathcal{E}$ we shall denote by $\mathcal{E}// G$ the set of layers. It is a subset of the set $\{(\mathcal{O})\}$ of orbit types. It can also be identified with a subset of $\{(H)\}$ (the set of conjugacy classes of subgroups of G), that subset which contains the conjugacy classes of the little groups which appear in the action of G on $\mathcal{E}$. Then one can speak of the minimal (resp. maximal) little groups of the action of G on $\mathcal{E}$ or, equivalently of the maximal (resp. minimal) layers of that action.

Example: The set of fixed points $\{m \in \mathcal{E} : \forall g \in G,\ g.m. = m\}$ usually denoted by $\mathcal{E}^G$, when it is not empty, is the minimal layer of $(G, \mathcal{E}, f)$.

I.4 Transfer of group action

Given an action of G on $\mathcal{E}_1, \mathcal{E}_2 \ldots$, there is a "natural" action of G on the mathematical structures that one builts "canonically" with $\mathcal{E}_1, \mathcal{E}_2 \ldots$. For instance if $(G, \mathcal{E}_1, f_1)$, $(G, \mathcal{E}_2, f_2)$ are linear representation of G,one defines the linear representation $f_1 \otimes f_2$ on the tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$. As another example the natural action of G on $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$, the vector space of linear maps from $\mathcal{E}_1$ to $\mathcal{E}_2$, is defined by

$$g \in G, \quad \mathcal{E}_1 \xrightarrow{\theta} \mathcal{E}_2,$$

$$g.\theta = f_2(g)_\circ\ \theta_\circ f_1(g)^{-1} \qquad (1.4)$$

So: θ equivariant $\Longleftrightarrow$ $\theta \in \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)^G$

In particular, if $\mathcal{E}$ is a complex vector space, $\mathrm{Hom}(\mathcal{E}, \mathbb{C}) = \mathcal{E}^*$ is the dual of $\mathcal{E}$. The elements of $\mathcal{E}^*$ are the linear forms on $\mathcal{E}$. The action of G on $\mathcal{E}^*$ is $g \to f(g^{-1})^T$ (T = transposed); it is the contragredient representation of f.

I.5 Example

Action of the Lorentz group: O(3,1) on the space-time with a chosen origin.

There are four layers that we can label by the conjugacy class of their little groups:

(O(3,1)): the origin; one orbit.

(E(2)): the light cone minus the origin; one orbit.

O(2,1): the outside of the light cone; one parameter family of orbits, the one-sheet hyperboloids of the space-like vectors of fixed length.

(O(3)): the inside of the light cone; one parameter family of orbits, the two-sheet hyperboloids of the time-like vectors of fixed length.

Of these four strata, the first one is minimal, the three other are maximal.

Note.- Sections III, IV, V are independent of the next section (II), although some examples in III use the vocabulary introduced in II.

II. BROKEN EUCLIDEAN SYMMETRY

To an audience of high energy physicists, it might be useful to recall that broken symmetries appear elsewhere in physics, and to explain how the breaking is described.

II.1 C^*-algebra of observables and states

In the most general present formulation of quantum mechanics, the physical observables of an infinite physical system# form a C^*-algebra, $\mathcal{A}$, i.e. a Banach space on the complex numbers, carrying an algebra structure with a * operation: $(\lambda a)^* = \bar{\lambda} a^*$, $(ab)^* = b^* a^*$ such that the norm $\|a\|$ satisfies: $\|a\|^2 = \|a^* a\|$. (Of course the algebra $\mathcal{L}_n$ formed by the nxn complex matrices is a C^*-algebra when a^* is the Hermitean conjugate of a, i.e. $a^* = \bar{a}^T$ and $\|a\|$ = square root of the largest eigenvalue of the matrix $a^* a$).

A linear form on $\mathcal{A}$ is a continuous linear map $\mathcal{A} \xrightarrow{\phi} \mathbb{C}$. (For $\mathcal{L}_n$, to any linear form ϕ there corresponds a matrix $\rho_\phi \in \mathcal{L}_n$ such that $\phi(a) = \mathrm{tr}\, \rho_\phi a$). Any state of the system is re

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It is a mathematical idealization to replace the Avogadro number 6×10^{23} by infinity.

presented by a norm - one positive linear form, i.e.

ϕ is a linear form such that

$$\phi(I) = 1, \ \forall a \in \mathcal{A}, \ \phi(a^* a) \geq 0 \tag{2.1}$$

(where I is the unit of the algebra $\mathcal{A}$, $Ia = aI = a$).

If $\mathcal{L}_n$ is the algebra of observables, every state ϕ is represented by a positive matrix ρ_ϕ (i.e. ρ_ϕ is Hermitean and its eigenvalues are ≥ 0) which is called the density matrix of the state; it is normalized by $\mathrm{tr}\ \rho_\phi I = \mathrm{tr}\ \rho_\phi = 1$. Denoting by H_n the n-dimensional Hilbert space on which $\mathcal{L}_n$ is the set of operators, the normalized vectors $x> \in H_n$, $<x,x> = 1$, represent only the "pure" states and their density matrices are rank one Hermitean projectors $\rho = x><x$ characterized by $\rho^* = \rho = \rho^2$, $\mathrm{tr}\ \rho = 1$. The density matrices form a convex set (i.e.if ρ_1, ρ_2 are density matrices, $\alpha_1\rho_1 + \alpha_2\rho_2$, with $0 < \alpha_1$, $0 < \alpha_2$, $\alpha_1 + \alpha_2 = 1$, is a density matrix) whose extremal elements (i.e. elements ρ such that $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$, $\alpha_1\alpha_2 \neq 0 \implies \rho_1 = \rho_2$) are the pure states.

Similarly, for any C^*-algebra $\mathcal{A}$, the states ϕ form a convex domain in $\mathcal{A}^*$ (the dual of $\mathcal{A}$) whose extremal points are the pure states (for a general C^*-algebra they may not be representable by a state vector).

The positivity of $\phi((a + \lambda b)^* (a + \lambda b))$ for any λ implies the Cauchy-Schwarz inequality:

$$|\phi(b^* a)|^2 \leq \phi(b^* b)\ \phi(a^* a) \tag{2.2}$$

so if $\phi(a^*a) = 0$, then for every $b \in \mathcal{A}$, $\phi(b^*a) = 0$; i.e. the elements $c \in \mathcal{A}$ such that $\phi(c) = 0$ form a left ideal Ω_ϕ of $\mathcal{A}$. On the quotient vector space $\mathcal{A}/\Omega_\phi$ (i.e. the space whose elements are the subsets of $\mathcal{A}$ of the form $\dot{a} = a\Omega_\phi$) ϕ is a genuine Hermitean scalar product:

$$\langle \dot{a}, \dot{b} \rangle = \phi(a^*b), \quad \langle \dot{a}, \dot{a} \rangle = 0 \Longrightarrow \dot{a} = 0 \in \mathcal{A}/\Omega_\phi, \text{ i.e. } \dot{a} = \Omega_\phi \qquad (2.3)$$

We denote by H_ϕ the topological completion of $\mathcal{A}/\Omega_\phi$. With the Hermitean scalar product (2.3) H_ϕ is a Hilbert space. We can also obtain a linear representation π_ϕ of $\mathcal{A}$ on H_ϕ by

$$\pi_\phi(a)\dot{b} = \dot{a}b\Omega_\phi = (\dot{ab}) \qquad (2.4)$$

It satisfies for the expectation value $\phi(a)$ of the observable a on the state ϕ

$$\phi(a) = \langle \Omega_\phi | \pi_\phi(a) | \Omega_\phi \rangle \qquad (2.5)$$

Moreover ϕ a pure state $\Longleftrightarrow$ π_ϕ is an irreducible representation[#].

[#] For high energy physicists who had never seen that, this is the well-known Gelfand-Naïmark-Segal construction, learned by all your young students in their undergraduated math courses (I.E. Segal, Bull. Am. Math. Soc. 53, 73 (1947), I.M. Gelfand and M.A. Naîmark, Izvest. Sec. Math. 12, 445 (1948)).

II.2 G-invariant states

Consider the physical action of the invariance group G (e.g. $G = E(3)$) on $\mathcal{A}$, the algebra of observables

$$G \xrightarrow{\alpha} \text{Aut} \tag{2.6}$$

We assume that $(g, a) \to \alpha(g).a$, also denoted by $\alpha_g(a)$, is a continuous function from $G \times \mathcal{A}$ onto $\mathcal{A}$. The action α defines also an action $g \to \alpha^t_{g^{-1}}$ of G on $\mathcal{A}^*$, the dual of $\mathcal{A}$. This action transforms the convex set of states into itself.

Consider a G-invariant state ϕ, i.e.

$$\forall a \in \mathcal{A}, \forall g \in G, \quad (\alpha^t_{g^{-1}}\phi)(a) = \phi(\alpha_g(a)) = \phi\ (a) \tag{2.7}$$

It defines a unitary representation U_ϕ of G on H_ϕ (I.E. Segal, Duke Math. J. 18, 221, (1951)) such that

$$\pi_\phi(\alpha_g(a)) = U_\phi(g)\ \pi_\phi(a)\ U_\phi(g)^{-1} \tag{2.8}$$

$$U_\phi(g)\Omega_\phi = \Omega_\phi \tag{2.8'}$$

When ϕ is not a pure state we will want to reduce the representation π_ϕ of the algebra of observables. This may introduce the breaking of symmetry.

To explain it we follow the forthcoming paper "Central decomposition of invariant states. Applications to the groups of time translations and of Euclidean transformations in algebraic field theory" by D. Kastler, G. Loupias, M. Mebkhout and L. Michel, to appear in Comm. Math. Phys.#.

II.3 The decomposition of a G-invariant state into pure states. Broken G-symmetry.

If the G-invariant state ϕ is not a pure state, we can decompose it as a sum (which is an integral) of pure states $\psi \in \mathcal{O}$ where $\mathcal{O}$ is a set of pure states transformed into itself by G:

$$\phi = \int_{\mathcal{O}} \psi d\mu(\psi), \qquad \int_{\mathcal{O}} d\mu(\psi) = 1 \tag{2.9}$$

With the following physical assumption of asymptotic abelianness (introduced by S. Doplicher, D. Kastler and D.W. Robinson, Comm. Math. Phys. 3, 1, (1966))

$$\text{i.e.} \forall a, b \in \mathcal{A}, \forall \phi \in \mathcal{A}^*, \ \phi([a, \alpha_t(b)]) \to 0 \text{ as } t(\in G) \to \infty \tag{2.10}$$

(for instance when G = E(3) the Euclidean group, t are translations going to infinity), one can prove that:

$$d\mu \text{ is G-invariant, i.e. } \forall g \in G, \ d\mu(g\psi) = d\mu(\psi) \tag{2.11}$$

and that the only subsets of $\mathcal{O}$ transformed into themselves by G are either of μ-measure one or of μ-measure zero.

The first part of this paper, corresponding to II.3 was circulated as a Marseille preprint of R.Haag, D.Kastler, L.Michel in 1969. See also D.Ruelle, J. Functional Analysis 6, 116 (1970) and A. Guichardet and D. Kastler, J. Math. Pures et Appliquées 49, 349 (1970).

Such sets $\mathcal{O}$ are called ergodic transitive. As a particular case $\mathcal{O}$ can be transitive, i.e. $\mathcal{O}$ is an orbit of G.

The descomposition (2.9) is carried simultaneously with the decomposition of π_ϕ into irreducible representations. To give more details we need the following concepts:

If R is a set of bounded operators on H, the commutant R' of R is the set of bounded operators which commute with every element of R. Then R' is a von-Neumann algebra and $(R')' = R''$ is the von-Neumann algebra generated by R. For instance if R is a representation of an algebra,

$$R \text{ irreducible} \iff R' = \{\lambda I\} \tag{2.12}$$

By definition a <u>factorial</u> representation is a direct sum or direct integral of equivalent irreducible representations. Then

$$R \text{ factorial} \iff R \cap R' = \{\lambda I\} \tag{2.13}$$

We denote by $\pi_{\mathcal{Q}}$, U_G the sets $\{\pi(a),\ a \in \mathcal{Q}\}$, $\{U(g),\ g \in G\}$ and by $R = \pi_{\mathcal{Q}} \cup U_G$. Then for G-invariant asymptotic abelian non pure states, R is factorial. The decomposition of R into irreducible representations yields a decomposition of the Hilbert space H_ϕ into a direct integral

$$H_\phi = \int_{\mathcal{O}} H_\psi \, d\mu(\psi) \tag{2.14}$$

Each H_ψ carries a unitary representation of the subgroup G_ψ of G, where G_ψ is the little group of ψ. Elements $g \in G$, $g \notin G_\psi$ although they are automorphisms of $\mathcal{A}$ and therefore of $\pi(\mathcal{A})$ are not "unitarily implementable" (i.e. they cannot be represented by unitary operators). Of course they can be represented by an isometric operator between different Hilbert spaces:

$$H_\psi \xrightarrow{V_g} H_{\alpha^t_{g^{-1}}(\psi)}$$

$$(H_1 \xrightarrow{V} H_2 \text{ is isometric } \Longleftrightarrow V^*V = I_1,\ VV^* = I_2 \Longleftrightarrow$$

$$\Longleftrightarrow \forall x > \in H_1,\ \ <Vx,\ Vx> = <x,x> \Longleftrightarrow \forall y > \in H_2,\ <V^*y,\ V^*y> =$$

$$= <y,y>$$

To summarize: the G-invariant transitive state ϕ is a mixture of an orbit $\mathcal{O}$ of pure states. Each state ψ is invariant under a subgroup G_ψ of G. For each state ψ the symmetry G is broken. This is exactly the situation we described in the introduction (with the example of a crystal). On the Hilbert space H_ψ which carries the irreducible representation π_ψ of the algebra of observables $\mathcal{A}$, the automorphisms $g \in G$, $g \notin G_\psi$ of $\mathcal{A}$ are not implementable.

II.4 The symmetry groups H of transitive Euclidean states

The classification of ergodic transitive actions of E(3) is still to be done. I will present here the classification of transitive Euclidean states. The list of orbits $\mathcal{O} = [E(3):H]$ of the Euclidean group which can appear in the de composition (2.9) of Euclidean invariant states is completely known. Each such orbit is characterized by a conjugacy class (H) where H is a closed subgroup of E(3) such that $[E(3):H]$ carries an E(3) invariant measure $d\mu$ which is finite($\int d\mu=1$)#. Then one proves that $[E(3):H]$ is compact.

The list below gives the subgroups H in which the E(3) symmetry can be broken for the equilibrium state of an infinite physical system.

Any locally compact group G carries a left invariant positive measure μ(the Haar measure): $\forall g_1, g_2 \in G$, $d\mu(g_1 g_2) = d\mu(g_2)$ which is unique up to a constant factor. But μ might not be right invariant, i.e. $d\mu(g_1 g_2) = \Delta d\mu(g_1)$ where Δ is a positive number ($\Delta \in R_+^x$) which depends on g_2. One shows that $\Delta(g)$ is a continuous representation of G which is called the modular function Δ_G of G. Or course if G abelian $\Delta_G = 1$. If G compact $\Delta_G(G)$ must be a compact subgroup of R_+^x, so $\Delta_G = 1$. One calls $g_1 g_2 g_1^{-1} g_2^{-1}$ "the commutator of g_1 and g_2" ($\in G$) and G' the derived group of G, the group generated by all the commutators of all pairs of elements of G. Then G' is the smallest invariant subgroup of G such that the quotient group G/G' is abelian. Of course if G = G', $\Delta_G = 1$. This is the case of Euclidean group E(n), n > 2. An orbit $[G:H]$ carries a G-invariant measure iff(= if and only if) $\forall h \in H$, $\Delta_G(h) = \Delta_H(h)$. A compact orbit $[G:H]$ may not carry a G-invariant measure(ex: $[SO(3,1):DE(2)]$, isomorphic to S_2, the two-dimensional sphere = the set of light-like directions) but if it does, this measure is finite. There are non compact orbits carrying finite measures, e.g. $[SL(2,R): SL(2,Z)]$.

Note that H is defined up to a conjugation in E(3). However the broken symmetry classes are defined traditionally by crystallographers by subgroups $H \subset E(3)$ up to a conjugation in $GL_+(3, R)$, the identity component of $GL(3, R)$ (for instance two differents crystals which can be transformed into each other by a dilation belong to the same symmetry class). We have found five families of symmetry classes:

1. The crystallographic groups in 3 dimensions. They form 230 symmetry classes.

2. The groups generated by the group R of translations along an axis $\vec{u}$ and the Euclidean transformations which induce a 2-dimensional crystallographic group (17 classes) on the plane $R^2 \perp \vec{u}$; such transformations contain a discrete group Z^2 of translations in R^2, and eventually rotations around $\vec{u}$ (of angle multiple of $\frac{\pi}{3}$ or $\frac{\pi}{2}$), rotation of π around an axis in R^2, symmetry through the R^2 plane or through a plane containing $\vec{u}$. This family contains a finite number of classes.

3. The groups generated by

   i) the translations in a plane R^2.

   ii) a translation $\lambda\vec{u}$ ($\lambda \neq 0$) along the direction $\vec{u} \perp R^2$.

   iii) and zero, one, two or three of the following generators α, β, γ:

   α) a rotation around $\vec{u}$ of angle rational to π.

   β) a rotation of π around an axis in the plane R^2.

   γ) the symmetry through the plane R^2 or through a plane containing $\vec{u}$.

This family contains an infinity of symmetry classes, and among them the 2 classes of 1-dimensional crystal groups.

4. The semi-direct product T □ K' where K' is a closed subgroup of the orthogonal group in 3 dimensions (an infinity of classes).

5. The groups generated by

i) the translations in a plane R^2 (as in 3.i)

ii) a helicoidal transformation $(\vec{a}, r)$ along the axis $\vec{u} \perp R^2$ with an angle $\theta(r)$ irrational to π, i.e. $\vec{a} = \lambda\vec{u}$, $\lambda \neq 0$, $\vec{n}(r) = \vec{u}$, $\theta(r)/\pi$ irrational.

iii) as in 3.iii).

This family is generally forgotten in such a classification. It also contains an infinity of symmetry classes.

When the angle θ of the generator in ii)becomes rational, the groups go from family 5 to family 3.

Some symmetries only in each family are realized in nature#. Examples of symmetries in family 5 or 3 are the cholesteric liquids and the matter in a helimagnetic state,while ferromagnetism is an example of a broken Euclidean symmetry in family 4.

Some other realizations are only due to man, e.g. the 17 classes of two dimensional crystal symmetries are all present in the decorations of the Granada Alhambra (see e.g. H. Weyl, Symmetry, Princeton Univ. Press, 1952, p. 109).

II.5 Broken symmetry in molecules, in phase transitions.

There is a famous theorem (H.A. Jahn and E. Teller, Proc. Roy. Soc. (London), A 161, 220 (1937)) in molecular physics. It applies to molecules which do not possess an axial symmetry# so their symmetry group G is discrete, e.g.

$H_2O \rightarrow$ symmetry group $G = Z_2 \times Z_2$

$CH_4 \rightarrow$ symmetry group $G = \mathcal{P}_4$ (permutation of the 4 H atoms, $\mathcal{P}_4 \subset O(3)$)

$NH_3 \rightarrow$ symmetry group $G = \mathcal{P}_3$

This theorem tells us that the lowest orbital electron state belongs to a one-dimensional linear representation of G. Therefore if this representation is not the trivial one, the symmetry G is broken into a subgroup H such that $G' \subset H \subset G$ where G' is the derived group of G, defined in the footnote of p.69.

How does the symmetry change in a phase transition e.g: liquid → crystal?. In which crystal symmetry classes can go a given crystal under a allomorphic phase transition?. There is a theory by Landau on this problem (see Landau and Lifschitz, Statistical Physics, § 136).

Those better understood examples can help us for the study of symmetry breaking in hadron physics.

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i.e. it does not apply to molecules with all their atoms on a straight line, e.g. CO, CO_2 ...

III. SMOOTH ACTION OF COMPACT LIE GROUPS

We consider in this section only real, C^∞ (i.e infinitely differentiable) manifolds M. An action of an abstract group G on M would be defined by a homomorphism f: $G \xrightarrow{f}$ Diff M. However, since a Lie group G is defined on a manifold, one defines a subfamily of G-actions on M, called the smooth actions.

Def. 1.- A smooth action of the Lie group G on the manifold M is a smooth map

$$G \times M \xrightarrow{\phi} M \tag{3.1}$$

which satisfies

$$\phi(e, m) = m, \quad \phi(g_2, \phi(g_1, m)) = \phi(g_2 g_1, m) \tag{3.2}$$

As before, we shall denote $\phi(g, m)$ simply by g.m.

The smooth actions of compact Lie groups have very rich properties. We list them in III.2, 3. One of them is that M is stratified (in the sense of Thom lectures) by the group action, so the layers we use in that case are Thom's strata.

III.1 Physical examples of smooth compact Lie group action

a) Action of O(3) on the phase space of three particles.

Let M be the phase space of 3 particles, of mass m_i, ($m_1 m_2 m_3 \neq 0$), with energy-momenta p_i, with $p_1 + p_2 + p_3 =$ $= p$ fixed. G is the little group of p in the Lorentz group, so G is isomorphic to O(3) (see example I.5).

There are two strata: A generic stratum, when p_1, p_2, p_3 are linearly independent, the little group being Z_2. This stratum contains a 2 parameter family of 3 dimensional orbits. It is open dense in M. The other stratum, p_1, p_2, p_3 linearly dependent, is a one parameter family of two dimensional orbits [O(3):O(2)] diffeomorphic to S_2, the two-dimensional sphere.

The orbit space is the Dalitz plot. Its inside is the image of the generic stratum, its boundary that of the closed stratum. It is well known that the Dalitz plot has a natural O(3)-invariant measure.

b) Action of O(3) on the states of pure polarization of a spin 1 particle.

It is an error spread in some textbooks that every spin 1 state vector (e.g. p-state of a hydrogen atom), can be transformed to any other one by a rotation. Pure states of a spin j particle with non vanishing mass and given energy-momentum are described by the normalized state vectors, defined up to a phase, in a $2j + 1$ dimensional Hilbert space and by definition of $P(2j, ¢)$, this set is the complex $2j$ dimensional projective space; its real dimension is $4j$. Since dim $O(3)$ is 3, if $4j > 3$, i.e. $j \geq 1$, the group $G = O(3)$ can not act transitively on $P(2j, ¢)$.

For $j = 1$, there are 3 strata: A generic one, (4-dimensional) one parameter family of 3-dimensional orbits, little group Z_2, and two two-dimensional strata of one orbit each: The orbit $[O(3): O(2) \times Z_2] = P(2, R)$ of longitudinaly polarized states (by a suitable choice of quantization axis this state has $j_Z = 0$), the orbit $[O(3): O(2)] = S_2$ of circularly polarized states; by a suitable choice of quantization axis such state has $j_Z = \pm 1$ which is changed into ∓ 1 by a reflexion (the $O(2)$ little group is generated by the $SO(2)$ group of rotations around the quantization axis and by the rotations of π around a perpendicular axis#).

Professor Galindo told me the following parametrization of the orbit space. Take the particle at rest. The polarization vector $\vec{z}$ can be decomposed into real and imaginary part $\vec{z} = \vec{x} + i\vec{y}$. Note that $\vec{z}$ is defined up to a phase $e^{i\phi}$ and its Hermitean product is $\vec{z}^* . \vec{z} = 1$. The orbit space is the set of values of $|\vec{z}.\vec{z}|^{1/2} \equiv \mu(\vec{z})$ with $0 < \mu(\vec{z}) < 1$ for the generic stratum, $\mu(\vec{z}) = 0$ for circular polarization, $\mu(\vec{z}) = 1$ for the longitudinal one.

c) Action of SU(3) on the spin-1 polarization states.

The spin-j polarization states are described by a $(2j + 1) \times (2j + 1)$ Hermitean matrix $\rho = \rho^* \geq 0$ (i.e non negative eigenvalues) with tr $\rho = 1$. The group $SU(2j+1)$ acts on this $M_{(2j+1)^2-1}$ manifold by

$$\rho \rightarrow u\rho u^* = u\rho u^{-1} \tag{3.3}$$

The pure states are defined by $\rho^2 = \rho$ i.e. rank $\rho = 1$; they form one orbit $[SU(2j + 1): U(2j)] \approx P(2j, ¢)$. ($\approx$ diffeomorphic). We treat now the case $j = 1$, so the manifold M is eight dimensional.

By a unitary transformation (3.3) every ρ can be diagonalized and its eigenvalues α, β, γ put in a decreasing order, so the orbit space is labelled by

$$\alpha + \beta + \gamma = 1 \quad \alpha \geq \beta \geq \gamma \geq 0 \tag{3.4}$$

This is the triangle MOA, 1/6 of the equilateral triangle ABC of Fig. 1. There are 3 strata whose images in the orbit space are:

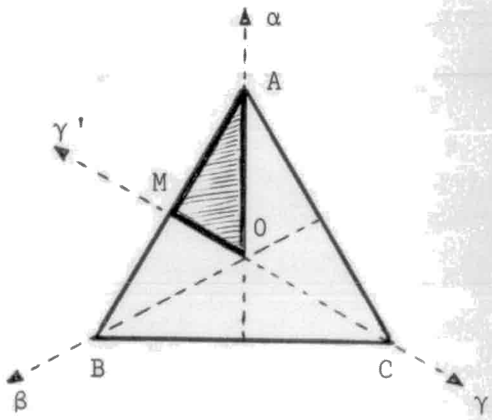

Fig. 1

O, one fixed point, the unpolarized state

$$(\alpha = \beta = \gamma = \tfrac{1}{3});$$