$]0, A] \cup ]0, M]$, 2 equal eigenvalues, little group U(2); one particular orbit, A, is that of the pure states.

The "interior of the triangle $\cup\, ]M, A[$ (open side)" is the generic stratum: 3 differents eigenvalues; on $]M, A[$, $\gamma = 0$, i.e. rank $\rho = 2$; interior of the triangle, rank $\rho = 3$.

Note that this classification of spin-j polarization states into strata of SU(2j + 1) has some physical interest, but only the orbits and strata of O(3) are completely physical. We leave to the reader the study of this group action.

d) Action of SU(3) on $R^8$, the octet space.

We call "octet" the (real) vector space of the Lie algebra of SU(3). It can be realized as the vector space of 3 × 3 Hermitean matrices with zero trace

$$x = x^*, \quad \text{tr } x = 0 \qquad (3.5)$$

and the action of $u \in SU(3)$ on x is as in (3.3)

$$x \rightarrow u\, x\, u^{-1} \qquad (3.5')$$

So similarly the orbit space is characterized by the eigenvalues $\alpha$, $\beta$, $\gamma$ of x

$$\alpha + \beta + \gamma = 0 \quad \alpha \geq \beta \geq \gamma \qquad (3.6)$$

It is the 60° sector $\gamma'0\alpha$ of Fig. 1 ($\frac{1}{6}$ of the plane) where now 0 represents the zero matrix. There are three strata again.

The octet space is a Euclidean space, with the scalar product

$$(x, y) = \frac{1}{2} \operatorname{tr} xy \qquad (3.7)$$

(indeed $(x,x) = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) \geq 0$, $(x,x) = 0 \implies x = 0$) which is SU(3)-invariant: $(uxu^{-1}, uyu^{-1}) = (x, y)$.

The characteristic equation of the matrix can be written

$$x^3 - (x,x)x - I \det(x) = 0 \qquad (3.8)$$

The reality of the x eigenvalues requires

$$4\,(x,x)^3 \geq 27(\det x)^2 \qquad (3.9)$$

This gives another (diffeomorphic) realization of the orbit space (Figure 2).

We will be also interested in the

e) Action of Aut SU(3) on $S_7$.

Where $S_7$ is the sphere of the vectors of the octet $R^8$ with unit length: $(x, x) = 1$ (see Fig. 2).

The action of SU(3) on $S_7$ yields two strata whose image in the orbit space are:

$]QQ'[$, open segment. The orbits are $[SU(3):U(1) \times U(1)]$. Indeed the matrices u which commute (eq.(3.5')) with a diagonal matrix x with three distinct eigenvalues are diagonal and they form the subgroup U(1) × U(1) of SU(3). It belongs to the conjugacy class of "Cartan subgroups".

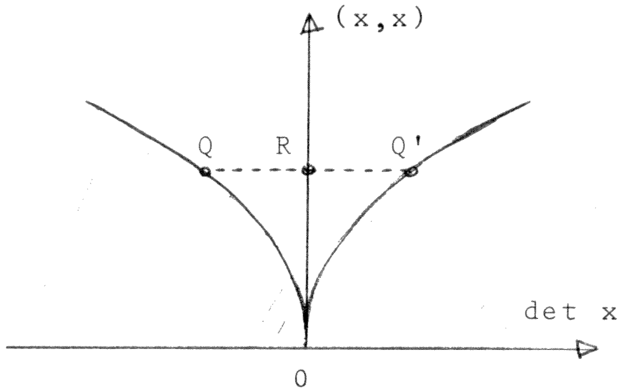


Fig. 2

Q, Q', two points which represent the two orbits: det x = $\pm \frac{2}{3\sqrt{3}}$, which are diffeomorphic to $[SU(3): U(2)] \approx$ $\approx P(2, ¢)$.

The group SU(3) has only one class of outer automorphisms, which can be represented by the complex conjugation $u \to \bar{u} = (u^{-1})^T = (u^T)^{-1}$. Since $x \in R^8 \Longrightarrow e^{ix} \in SU(3)$, the corresponding action on the Lie algebra is

$$x \xrightarrow{\varepsilon} -x^T = -\bar{x} \qquad (3.10)$$

This does not change (x, x) but it changes the sign of det x, so the orbit space $S_7$/Aut SU(3) is the segment QR, with:
$]QR[$, image of the generic stratum, open dense in $S_7$.

Q, image of stratum of one orbit, $|\det x| = \frac{2}{3\sqrt{3}}$, $(x,x)=1$

R, image of stratum of one orbit, det x = 0, (x,x) = 1; as we shall see later (IV.2) this is the orbit of the roots of the SU(3)-Lie algebra.

## III.2 General properties of smooth actions of compact Lie groups

For a fixed $g \in G$, we denote $\phi(g,.)$ by $\phi_g$. They are diffeomorphisms $M \xrightarrow{\phi_g} M$. For a fixed $m \in M$, we denote $\phi(.,m)$ by $\psi_m$. They are smooth maps $G \xrightarrow{\psi_m} M$ with $\mathrm{Im}\psi_m = G(m)$ the orbit of M.

The orbits G(m) are closed compact submanifolds of M. The layers are also manifolds. They are the strata of the stratification of M obtained by the action of G. The little groups $G_m$ are closed. A good introduction to the study of the compact Lie group smooth actions is R.C. Palais, The classification of G-spaces, Memoirs Amer. Math. Soc. n° 36 (1960). The following properties can be proven:

The canonical map (see I.3)

$$M \xrightarrow{\pi} M/G$$

on the orbit space is open, closed, proper#.

-------

# For a continuous map, inverse image $f^{-1}$ (open set) = open set; $f^{-1}$ (closed set) = closed set and f (compact set) = = compact set. If f (open) = open, the map f is open; f(closed) = closed, then f is closed, and f proper $\Longleftrightarrow$ $\Longleftrightarrow$ $f^{-1}$ (compact) = compact.

Let $M^g$ the set of points of M invariant under $g \in G$, i.e. $M^g = \{m \in M, g.m = m\}$. It is closed; so is $M^H = \bigcap_{g \in H} M^g$, and $GM^H = \bigcup_{g \in G} g.M^H$. If H is a maximal stabilizer in the G-action, $GM^H$ is the corresponding minimal stratum: it is closed.

The differential $d(\phi_g)_m$ at m of the diffeomorphism $\phi_g$ is an inversible linear map between the tangent planes:

$$T_m(M) \xrightarrow{d(\phi_g)_m} T_{g.m}(M) \qquad (3.11)$$

(the matrix of $d(\phi_g)_m$ is the Jacobian). For $g \in G_m$, then $g \longrightarrow d(\phi_g)_m$ is a linear representation of $G_m$ in $T_m(M)$.

The group G has a finite Haar measure $d\mu(g)$ which can be normalized to one: $\int_G d\mu(g) = 1$. Given a Riemannian metric $\Delta(x, y)$ on M, its average by G:

$$\Delta(x, y) = \int_G \Delta(g.x, g.y)d\mu(g) \qquad (3.12)$$

is G-invariant: $\Delta(g.x, g.y) = \Delta(x, y)$, so G acts on M by isometries.

At each point m, $G_m$ transforms the geodesics passing through m into themselves so the choice of a chart on a neighbourhood of m with geodesic coordinates gives a locally <u>linear action</u> of $G_m$ (which is equivalent to the linear representation $g \to d(\phi_g)_m$).

We shall study this local action in III.4. We first give some results on:

## III.3 Global properties of compact Lie group smooth actions

A review of these global properties is given in D. Montgomery, "Compact groups of transformations", p. 43 of Differential Analysis, Bombay Colloquium (1964).

a) If M is compact, the number of strata is finite.

b) This is also the case for a linear representation $g \rightarrow D(g)$ of G on a complex space $\mathbb{C}^n$ of finite dimension n. This can be considered as a real action on $R^{2n}$. Indeed, since G is compact, D(g) is equivalent to a unitary representation# U(g) and $U(g) \oplus \bar{U}(g)$ is equivalent to a real orthogonal representation which acts on $R^{2n}$ and transforms the sphere $S_{2n}$ into itself.

It is very remarkable that the converse theorem is true (G.W. Mostow, also R.C. Palais):

c) If (G, M, $\phi$), a smooth action of a compact Lie group, has a finite number of strata, there is an equivariant injective smooth map $\theta$ into a finite dimensional $R^n$ carrying an orthogonal representation of G (and $\theta(M)$ is a submanifold of $S_n$ since the spheres are the orbits of the orthogonal action).

-----

# Indeed $U(g) = SD(g)S^{-1}$ with S, the positive square root of $S^2 = \int_G D(g)D(g)^* \, d\mu(g)$. Furthermore one proves that $\int_G U(g^2) \, d\mu(g) = \pm 1$ or zero; when it is 1, D(g) is equivalent to an orthogonal representation.

Hence, in this case the action of G on M is globally equivariant to a linear action, but n can be very large.

d) Palais has also shown that for n large enough there is a countable infinity of inequivalent actions of G on $S_n$.

e) To prove c) one first uses that the regular representation of G, i.e. the infinite dimensional representation R of G on the Hilbert space $\mathcal{L}_2(G)$ of square integrable functions[#] on G, contains every type of orbit $[G:H]$, with H closed, and that each orbit is contained into a finite dimensional subspace of $\mathcal{L}_2(G)$ (containing also the origin 0).

f) Montgomery and Yang have proved that there is a stratum which is open dense. Indeed we remarked this property in all examples III.1.

-----

[#] $R = \bigoplus_i n_i D_i$, the direct sum over all irreducible representations of G, each appearing $n_i$ times with $n_i$ = dimension of $D_i$. The matrix elements of all $D_i$ form an orthogonal basis of $\mathcal{L}_2(G)$.

## III. 4 Local properties of the action $(G, M, \phi)$

### a) The equivariant retraction

Let Q be a G-invariant submanifold of M and $\Delta(x,y)$ a G-invariant Riemann metric. There exists a neighbourhood $V_Q$ such that, given $x \in V_Q$, $\operatorname{Min}_{y \in Q} \Delta(x, y)$ is unique and is obtained for a unique# $y \in Q$ that we denote by $r_Q(x)$. Then $V_Q \xrightarrow{r_Q} Q$ is a smooth retraction which is equivariant for the (isometric) action of G. If $g \in G_x$, the equivariance of the map yields

$$r(x) = r(gx) = g.r(x), \text{ so } g \in G_{r(x)}, \quad \text{i.e.} \quad G_x \subset G_{r(x)} \tag{3.13}$$

and when Q is the orbit G.m, this implies

$$(G_x) \leq (G_m), \ \forall x \in V_{G.m} \tag{3.13'}$$

### b) The local representation of $G_m$ on the orbit

If H is a subgroup of G, the normalizer $\mathcal{N}_G(H)$ is the largest subgroup of G which has H as invariant subgroup. So the points† $\psi_m(\mathcal{N}_G(G_m))$ of the orbit G.m have also $G_m$ as

-----

# It is the foot of the geodesic from x, orthogonal to Q.

† $\psi_m$ has been defined in the beginning of III.2.

little group, i.e.

$$\psi_m\ (\mathcal{N}_G(G_m)) = G.m \cap M^{G_m} \qquad (3.14)$$

We shall call $(G.m)^o$ the connected component of $G.m \cap M^{G_m}$.

c) <u>The local representation of $G_m$ on $T_m(G.m)$</u>

We denote by $\mathcal{G}$ the vector space of the Lie algebra of G. The group G acts on $\mathcal{G}$ by an orthogonal linear representation, called the adjoint representation. It leaves invariant a Euclidean scalar product and we shall denote by $\mathcal{H}^{\perp}$ the subspace of $\mathcal{G}$ orthogonal to $\mathcal{H}$, the space of the Lie algebra of the subgroup H of G. Let (G, G, $\phi$) be the action of G on itself by left translation: $\phi(g_1, g_2) = g_1g_2$. Consider now a G-manifold (G, M, $\phi$). Then $\psi_m$ is an equivariant map from (G, G, $\phi$) to the orbit G.m, and its differential at the unit $e \in G$

$$T_e(G) = \mathcal{G} \xrightarrow{d(\psi_m)_e} T_m(G.m) \qquad (3.15)$$

maps $\mathcal{G}$ on the tangent space to the orbit at m. Ker $d(\psi_m)_e = \mathcal{G}_m$ so Im $d(\psi_m)_e$ is isomorphic to $(\mathcal{G}_m)^{\perp}$. And one sees that the local representation of $G_m$ on $T_m(G.m)$ is equivalent to that of $G_m$ on $(\mathcal{G}_m)^{\perp}$ obtained by the adjoint representation of G. This representation is generally reducible. And if $(G.m)^o$ contains some point besides m, $T_m(G.m)^o$ is the subspace of the trivial component of the representation of $G_m$ on $T_m(G.m)$.

d) The slice N(m) at m. Action of $G_m$ on N(m)

The slice N(m) is the inverse image of m by the equivariant retraction on G.m. Locally:

$$N(m) = r^{-1}_{G.m}(m) \qquad (3.16)$$

In the geodesic coordinates, N(m) is a linear manifold so it is locally the subspace of $T_m(M)$ normal to the orbit#. For points of N(m) equation (3.13) reads

$$x \in N(m) \implies G_x \subset G_m \qquad (3.17)$$

and furthermore

$$\text{for } x \in N(m), \quad G_x = G_m \iff x \in S(m) \qquad (3.18)$$

the stratum of m.

So the linear orthogonal representation of $G_m$ on $T_m(N(m))$ decomposes into a direct sum of the trivial representation on the subspace

$$F(m) = T_m(N(m)) \cap T_m(S(m)) \qquad (3.19)$$

----

# In what follows, however, we will not use explicitly local geodesic coordinates. We need only to know that the slice N(m) is a submanifold defined by (3.16) and its tangent space $T_m(N(m))$ is the subspace of $T_m(M)$ orthogonal to $T_m(G.m)$.

intersection of the tangent spaces to the slice and to the stratum at m. We will call

$$K(m) = F(m)^{\perp} \cap T_m(N(m))$$

To summarize: We have decomposed $T_m(M)$ into the direct sum of 3 orthogonal subspaces invariant for the local (orthogonal) linear representation of $G_m$.

$$T_m(M) = \overbrace{T_m(G.m) \oplus F(m)}^{T_m(S(m))} \oplus K(m) \qquad (3.20)$$

$G_m$ acts trivially only on F(m) and on $T_m(G.m)^o$. If $F(m) = 0$, we say that the <u>orbit is isolated in its stratum</u>; there is a neighbourhood $V_{G.m}$ of G.m which contains no other orbits of the same type.

Note that for m' and m" of the same connected component of a stratum, the local actions of $G_{m'} \sim G_{m''}$ are equivalent#.

For more details for III.4 and III.5 see Louis Michel, C. R. Acad. Sc. Paris, <u>272</u>, 433 (1971).

## III.5 <u>G-invariant vector fields on M</u>

Every physicist knows what is a vector field. For the sake of the argument let us give a formal definition.

-----

# Indeed, the local linear action of G at m is a continuons function R of m valued in the set of linear representations of G. Since the latter has a discrete topology when G is compact, the function R is constant on each connected component of a stratum.

The tangent vector bundle

$$T(M) = \bigcup_{m \in M} T_m(M) \tag{3.21}$$

is a manifold with an action of G, obtained by transfer(see I.4) of the action of G on M and the canonical projection

$$T(M) \xrightarrow{t} M, \; t^{-1}(m) = T_m(M) \tag{3.22}$$

is an equivariant smooth map for the two G-manifolds M and T(M).

A vector field is a smooth section s

$$M \xrightarrow{s} T(M), \; t_o s = \text{Identity on } M \tag{3.23}$$

For a G-invariant vector field, the vector s(m) is invariant under $G_m$; therefore

$$s(m) \in T_m(G.m)^o \oplus F(m) \subset T_m(S(m)) \subset T_m(M) \tag{3.24}$$

Hence, <u>G-invariant vector fields are tangent to the stratum</u>

Furthermore all G-invariant vector fields vanish on the orbits G.m such that

$$(G.m)^o = m, \quad F(m) = 0 \tag{3.25}$$

G-invariant gradient vector fields are the gradient of G-invariant smooth functions $M \xrightarrow{f} R$. Such functions are constant on each orbit so their gradient is orthogonal to the or bit, i.e.

$$(\text{grad } f)_m \in F(m) \qquad (3.26)$$

If we call critical orbits of (G, M, $\phi$) the orbits which are orbits of critical points (i.e. grad f = 0) for all G-invariant smooth functions $M \xrightarrow{f} R$, we have established the

Theorem:

$$\text{orbit isolated in its stratum} \Longrightarrow \text{critical orbit} \qquad (3.27)$$

The converse is also true.

In ex. III.1b, c, e, there were strata containing a finite number of orbits. These orbits are critical.

In example III.1e, for instance, the little group U(2) is maximal in SU(3) so it is its own normalizer. Hence all SU(3)-invariant vector fields on $S_7$ have zeros on the two orbits det x = $\pm$ 2/3 $\sqrt{3}$.

As we have seen (e.g (3.24)), if the identity connected component of $\mathcal{N}_G(G_m)$, the normalizer of the little group $G_m$ of a critical orbit is strictly larger than $G_m$, there may exist G-invariant vector fields tangent and non-vanishing at this critical orbit G.m; however if the Euler-Poincaré characteristic of the orbit $\chi(G.m) \neq 0$, then every smooth vector field on G.m must have a zero, and by G-invariance it is everywhere zero in G.m.

Similarly, if S(m) is compact (which is the case for instance when M is compact), if $G_m$ maximal, so S(m) closed (III.2) and has an infinity of orbits, all G-invariant smooth functions have at least two orbits of critical points; these orbits depend on the function.

## III.6 Structure of the orbit space

We have seen in III.3 f that there is in (G, M, $\phi$) a generic stratum open dense; then equation (3.13')shows that there is a minimal little group, corresponding to this stratum which is therefore the maximal stratum. Montgomery and Yang have given several conditions on orbit and stratum dimensions (see Montgomery's review or the Haïfa lectures); we will give others here. Let $\gamma$, $\mu$ be the dimensions of G and M respectively, $\omega_o$ the dimension of the orbits of the maximal stratum

$$\gamma'_o = \gamma - \omega_o \geq 0 \qquad (3.28)$$

is the dimension of the minimal little group which appears in the action.

As we have also seen in III.3 c, the action (G,M, $\phi$) is equivalent to a linear action through an orthogonal representation. Let us consider more in details this case of linear action.

Then orbits can be labelled by algebraic invariants (we did it in the examples) so the orbit space will be a semi-algebraic set, i.e. a set given by algebraic equations or inequalities. What is the dimension of the orbit space?. In the generic stratum, it is the dimension of the stratum $\mu$, minus the dimension of the orbits, $\omega_o$, so

$$\dim M/G = \mu - \omega_o = \mu - \gamma + \gamma'_o \qquad (3.29)$$

The image of the other (smaller strata) is given by relations between these $\mu - \omega_o$ algebraic invariants so it is of smaller dimension.

As a matter of fact, since $M \xrightarrow{\pi} M/G$ is an open map, the image of the generic stratum $S_o$ is open dense and the image of the other strata is in the boundary $\partial S_o$. The critical orbits, which are the orbits isolated in their strata, are isolated points in the orbit space: they correspond to well defined values of the algebraic invariants (as we have seen in the examples) and we will be able to recognize them easily.

# IV. THE BREAKING OF HADRONIC SYMMETRY BY WEAK AND ELECTROMAGNETIC INTERACTIONS

Except if otherwise stated, the content of IV and V reports on papers made with L. A. Radicati. Previous references are in the last one: Ann. Phys. 66, 758 (1971).

## IV.1 G-invariant algebras

We use here the large definition of algebra on a vector space $\mathcal{E}$. It is a homomorphism:

$$\mathcal{E} \otimes \mathcal{E} \xrightarrow{\phi} \mathcal{E} \qquad (4.1)$$

The tensor product $\mathcal{E} \otimes \mathcal{E}$ can be decomposed into a direct sum

$$\mathcal{E} \otimes \mathcal{E} = (\mathcal{E} \overset{S}{\otimes} \mathcal{E}) \oplus (\mathcal{E} \overset{A}{\otimes} \mathcal{E}) \qquad (4.2)$$

of a symmetric and antisymmetric tensor product. So we can also define symmetric, $\sigma$, and antisymmetric, $\alpha$, algebras by

$$\mathcal{E} \overset{S}{\otimes} \mathcal{E} \xrightarrow{\sigma} \mathcal{E}, \qquad \mathcal{E} \overset{A}{\otimes} \mathcal{E} \xrightarrow{\alpha} \mathcal{E} \tag{4.3}$$

(Of course, there are algebras which are neither symmetric nor antisymmetric!).

Assume now that $\mathcal{E}$ carries a linear representation of G. If the representation on $\mathcal{E} \otimes \mathcal{E}$ contains (by direct sum decomposition) a subrepresentation equivalent to that on $\mathcal{E}$, we can therefore find an equivariant map (as in (4.1))for the group action and we obtain an algebra on $\mathcal{E}$ which has G as automorphism group.

More precisely, the set of algebras one can make in this fashion forms a $\nu$-dimensional vector space where

$$\nu = \dim \mathrm{Hom}\,(\mathcal{E} \otimes \mathcal{E}, \mathcal{E})^G \tag{4.4}$$

It is easy to see that $\mathcal{E} \overset{S}{\otimes} \mathcal{E}$ and $\mathcal{E} \overset{A}{\otimes} \mathcal{E}$ are stable by the G action, so we can define the corresponding $\nu_+$, $\nu_-$

$$\nu_+ = \dim \mathrm{Hom}\,(\mathcal{E} \overset{S}{\otimes} \mathcal{E}, \mathcal{E})^G, \quad \nu_- = \dim \mathrm{Hom}\,(\mathcal{E} \overset{A}{\otimes} \mathcal{E}, \mathcal{E})^G \tag{4.5}$$

$$\nu = \nu_+ + \nu_- \tag{4.5'}$$

For example, for a simple Lie group G and its adjoint representation (i.e. as we have seen, the representation on the vector space of its Lie algebra $\mathcal{G}$), $\nu_- = 1$ and the corresponding algebra is the Lie algebra; $\nu_+ = 0$ except for the simple Lie groups of the series $A_l$ $(l > 1)$ whose compact form is SU(n).

## IV.2 Geometry of the octet

The aforementioned property was well exploited by Gell-Mann in his first paper on SU(3) to study the two algebras on the Octet.

In III.1d eq.(3.5) we have represented the octet space $R^8$ by 3 x 3 matrices

$$x = x^*, \qquad \mathrm{tr}\, x = 0 \tag{4.6}$$

with the Euclidean scalar product

$$(x, y) = \frac{1}{2}\, \mathrm{tr}\, xy \tag{4.7}$$

and the SU(3) action

$$u \in SU(3): x \longrightarrow u\, x\, u^{-1} \tag{4.8}$$

The Lie algebra is represented by

$$x \wedge y = \frac{i}{2}\,(xy - yx) \tag{4.9}$$

and the symmetric algebra by

$$x \vee y = \frac{\sqrt{3}}{2}\left(xy + yx - \frac{4}{3}(x,y)\, I\right) \qquad (4.10)$$

(where I is the unit matrix).

Note that our definition of the symmetrical algebra is $\sqrt{3}$ times that of Gell-Mann, but there are good esthetical arguments to use the $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ coefficients in (4.9) and (4.10).

Gell-Mann called $f_{ijk}$ and $d_{ijk}$ the structure constants of the algebras, i.e. with an orthonormal basis $\lambda_i$

$$(\lambda_i, \lambda_j) = \delta_{ij}, \quad \lambda_i \wedge \lambda_j = f_{ijk}\lambda_k, \quad \lambda_i \vee \lambda_j = \sqrt{3}d_{ijk}\lambda_k \qquad (4.11)$$

The characteristic equation (3.8) becomes

$$x \vee (x \vee x) = (x \vee x) \vee x \equiv x \vee x \vee x = (x,x)x \qquad (4.12)$$

So all elements are idempotents of power 3.

One easily proves from 4.12 that

$$(x \vee x,\ x \vee x) = (x,x)^2 \qquad (4.13)$$

From now on to simplify the writing we shall use only unit vectors, i.e.

$$(x, x) = 1 \qquad (4.14)$$

Every idempotent of power 2 satisfies then:

$$q \vee q \pm q = 0 \qquad (4.15)$$

Since this is a second degree equation, it shows that the q's have only two distinct roots, so they have two equal roots and they form the two critical orbits of the SU(3)-action on S(7) (see III.1e). Given an arbitrary element x, generally x and $x \vee x$ are linearly independent, and they generate a $\vee$-algebra. Because of (4.12) this algebra is two dimensional and it is the maximal abelian $\wedge$-algebra containing x, so by definition it is the Cartan-subalgebra $\mathcal{G}_x$ generated by x; it is the Lie algebra of $G_x$, the little group of x.

The adjoint representation of the Lie algebra of SU(3) on $R^8$ is

$$x \longmapsto f(x), \qquad f(x)y = x \wedge y \qquad (4.16)$$

By definition the roots of the Lie algebra satisfy the following property: given a Cartan subalgebra $\mathcal{G}_x$, the Hermitean operators $i \times f(a)$, $a \in \mathcal{G}_x$ commute and they have as spectrum (= set of eigenvalues):

$a \in \mathcal{G}_x$, Spect.$[i \times f(a)]$ = twice zero and $(a, r_i)$ (4.17)

$r_i (i = 1$ to $6)$ are the 6 roots of $\mathcal{G}_x$.

One shows that the characteristic equations of the roots are given by

$$(r, r) = 1, \ \theta(r) \equiv (r \vee r, r) = 0 \qquad (4.18)$$

So all roots are on a single orbit of SU(3) on $S_7$. This orbit is critical for the action of Aut SU(3) as we have seen in III.1e.

As every high energy physicist knows the 6 roots of a Cartan are at the vertices of a regular hexagon, so if $r_i$ is a root, $- r_i$ is also one and $\sum_{i=1}^{6} r_i = 0$.

The vectors q defined for every root r as

$$q = r \vee r \qquad (4.19)$$

are idempotents of the Lie algebra. They satisfy (4.15) with the + sign. Note that there are 3 such vectors in a Cartan. They satisfy

$$\sum_i q_i = 0 \qquad (4.20)$$

We call them pseudo-roots or $\vee$-roots because if one defines d(x) by

$$d(x)y = x_{\vee} y \qquad (4.21)$$

then d(x) is an orthogonal operator on $R^8$, with spectrum:

$$a \in \mathcal{G}_x, \text{ Spect } d(a) = \left\{ \sqrt{\gamma(a)}, \ - \sqrt{\gamma(a)}, \text{twice } (a,q_i), \ i = 1,2,3 \right\} \qquad (4.22)$$

i.e. each d(a) has 3 eigenvalues doubly degenerate.

Since the continuous function $x \mapsto \theta(x)$

$$\theta(x) = (x_{\vee} x, \ x) \qquad (4.23)$$

is odd $$\theta(x) = - \theta(- x) \qquad (4.24)$$

in every 2-plane of the octet this function must have at least two zeros on the unit circle. So every two-plane contains at least two roots. There are 3-planes and 4-planes of the octet which contain only root-vectors (up to a scalar factor).

For instance, given a pseudo-root q, its centralizer, (i.e the Lie algebra of its little group) is the four dimensional space $U_q(2)$. The perpendicular space $U_q(2)^{\perp}$ contains only root vectors. The perpendicular space to q in $U_q(2)$ is $SU_q(2)$ which also contains only root-vectors.

Equation (4.12) shows also that there are no nilpotents: $a \neq 0$, $a \vee a = 0$. However, if we consider the complexified space $\mathbb{C}^8$ of $R^8$, its elements are traceless 3 x 3 complex matrices and we can extend by (4.9) and (4.10) the $\wedge$-and $\vee$-algebras.

Then consider two real roots $r_1$, $r_2$ belonging to the same $SU_q(2)$. One proves

$$r_1,\ r_2 \text{ roots: } r_1 \vee r_1 = r_2 \vee r_2 = q \implies r_1 \vee r_2 = (r_1, r_2)q \tag{4.25}$$

So if

$$(r_1, r_2) = 0, \quad r_\pm = \frac{1}{2}(r_1 \pm i\, r_2) \tag{4.26}$$

we have complex nilpotents

$$r_+ \vee r_+ = 0 \qquad r_- \vee r_- = 0 \tag{4.27}$$

Note that

$$r_3 = r_1 \wedge r_2 \tag{4.28}$$

forms with $r_1$ and $r_2$ an orthonormal basis of $SU_q(2)$.

For more information on the "Geometry of the octet" the reader is referred to the preprint of that name (1969 - Pisa and Tel Aviv) which will appear this year in Ann. Inst. H. Poincaré.

## IV.3 The geometry of the Lie algebra of SU(3) x SU(3)

This Lie algebra is the direct sum

$$SU(3)^{+} \oplus SU(3)^{-}$$

where ± are the sign of the chirality. We know the action of the charge conjugation C on it (see 3.10). The parity operator P is also an outer automorphism of SU(3) x SU(3) which exchanges the two factors. The action of the product CP is then defined.

We can extend the scalar product and algebras defined on $R^8$ to $R^{16}$, the vector space of $SU(3)^+ \oplus SU(3)^-$. We shall use a ~ to indicate elements of $R^{16}$ and no ~ for those of $R^8$. Explicitly,

$$\tilde{a} = a_+ \oplus a_-, \quad (\tilde{a}, \tilde{b}) = (a_+, b_+) + (a_-, b_-) \qquad (4.29)$$

$$(\tilde{a} \wedge \tilde{b}) = (a_+ \wedge b_+) \oplus (a_- \wedge b_-) \qquad (4.30)$$

$$(\tilde{a} \vee \tilde{b}) = (a_+ \vee b_+) \oplus (a_- \vee b_-) \qquad (4.31)$$

We leave to the reader to look for the idempotents of the $\vee$-algebra.

## IV.4 Tensor operators

We need to recall a precise definition of "tensor operators" used in physics: they are not operators on $\mathcal{H}$, the Hilbert space of state-vectors. Assume that we have a symmetry group G acting on $\mathcal{H}$ by the unitary representation U(g) and on a finite dimensional space $\mathcal{E}$ by the representation D(g). We denote by $\mathcal{L}(\mathcal{H})$ = Hom $(\mathcal{H},\mathcal{H})$ the vector space of linear operators on $\mathcal{H}$. Then a $\mathcal{E}$-tensor operator for G is an element of Hom $(\mathcal{E},\mathcal{L}(\mathcal{H}))^G$, i.e. it is an equivariant linear map T:

$$\mathcal{E} \xrightarrow{T} \mathcal{L}(\mathcal{H}) \qquad (4.32)$$

which therefore satisfies

$$\forall \underset{\sim}{m} \in \mathcal{E},\ \forall g \in G:\ \mathcal{U}(g)\, T(\underset{\sim}{m})\, \mathcal{U}(g)^{-1} = T(D(g)\underset{\sim}{m}) \qquad (4.33)$$

Let us call respectively L and iF the representation of the Lie algebra $\mathcal{G}$ obtained by differentiation of D and $\mathcal{U}$, i.e.

$$\forall \tilde{a}, \tilde{b} \in \mathcal{G},\ [L(\tilde{a}), L(\tilde{b})] = L(\tilde{a} \wedge \tilde{b}) \quad \text{on} \quad \mathcal{E} \qquad (4.34)$$

$$[F(\tilde{a}), F(\tilde{b})] = iF(\tilde{a} \wedge \tilde{b}) \quad \text{on} \quad \mathcal{H} \qquad (4.34')$$

(the $F(\tilde{a})$'s are Hermitean).

Then equation (4.33) yields:

$$[F(\tilde{a}),\ T(\underset{\sim}{m})] = iT(L(\tilde{a})\underset{\sim}{m}) \qquad (4.35)$$

We remark that, if $\underset{\sim}{m}_i \in \mathcal{E}_i$ and $T_1$, $T_2$ are two tensor operators, we can define the tensor-operators $T_1 + T_2$, $T_1 \oplus T_2$, $T_1 \otimes T_2$ by sum or product of images, namely

$$\mathcal{E} \ni \underset{\sim}{m} \xmapsto{T_1 + T_2} T_1(\underset{\sim}{m}) + T_2(\underset{\sim}{m})$$

$$\mathcal{E}_1 \oplus \mathcal{E}_2 \ni \underset{\sim}{m}_1 \oplus \underset{\sim}{m}_2 \xmapsto{T_1 \oplus T_2} T_1(\underset{\sim}{m}_1) + T_2(\underset{\sim}{m}_2) \qquad (4.36)$$

$$\mathcal{E}_1 \otimes \mathcal{E}_2 \ni \underset{\sim}{m}_1 \otimes \underset{\sim}{m}_2 \xmapsto{T_1 \otimes T_2} T_1(\underset{\sim}{m}_1) \cdot T_2(\underset{\sim}{m}_2)$$

For the readers not used to our notation, may we remind that for the rotation group they use the notation: $\vec{J}\cdot\vec{n}$ instead of $F(\tilde{n})$ with $[\vec{J}.\vec{n}_1,\ \vec{J}.\vec{n}_2] = i\ \vec{J}.\vec{n}_1\times\vec{n}_2$, which is strictly similar. So for instance the symbolic notation $\vec{J}\times\vec{J} = i\ \vec{J}$ corresponds to $F \wedge F = iF$ and the D operators on $\mathcal{H}$ are defined by $D = F \vee F$ (see also L. Michel. Lecture Notes in Physics 6, 36 (1970) Springer).

## IV.5 The weak and electromagnetic directions of breaking; weak and strong hypercharges

The hadronic symmetry group, started in 1932 by Heisenberg as SU(2) has grown very much since 1960. We shall stabilize it at

$$G = (SU(3) \times SU(3)) \,\Box\, (1, P, C, PC) \qquad (4.37)$$

and in the following, except if otherwise stated, we shall use G for this group. It always describes an approximate symmetry. If G, or even a subgroup H of it, were a group of exact symmetry, there would exist a fundamental degeneracy in nature that we could not explore (it may appear just as parastatistics). Because the G symmetry is approximate we are able to orient ourselves in the hadron internal symmetry space and it was soon recognized that the electromagnetic interaction and also the weak interaction were pointing at some definite directions in this space. However Feynman and Gell-Mann (Phys. Rev. 109, 193 (1958)), Cabibbo (Phys. Rev. Lett. 10, 531 (1963)) and Gell-Mann (Physics 1, 63 (1964)) found a much deeper relation between these two kinds of interaction. This fundamental discovery can be summarized in one sentence:

For hadrons the electromagnetic current $j^\mu(x)$, the vector part $v^\mu(x)$ and the axial vector part $a^\mu(x)$ of the weak current are images of the *same tensor operator $h^\mu(x)$*, which belongs to the adjoint representation of G.