These currents define the following unit vectors in the Lie algebra $\mathcal{G}$ of G:

$$j^{\mu}(x) = -\sqrt{\frac{2}{3}}\, h^{\mu}(x, \tilde{q})$$

$$v^{\mu}_{\pm}(x) = \frac{1}{2}\, h^{\mu}(x, \tilde{c}'_{\pm}) \qquad (4.38)$$

$$a^{\mu}_{\pm}(x) = \frac{1}{2}\, h^{\mu}(x, \tilde{c}''_{\pm})$$

We will assume that the Cabibbo angles θ_V and θ_A are equal, as the experimental data suggest. So we can also assume that the total weak current is an image of the same tensor operator

$$w^{\mu}_{\pm}(x) \equiv v^{\mu}_{\pm}(x) - a^{\mu}_{\pm}(x) = h^{\mu}(x, \tilde{c}_{\pm}) \qquad (4.39)$$

Equations (4.38) and (4.39) define the following directions in the octet: a pseudo root q, and two orthogonal roots c_1, c_2 so that

$$c_{\pm} = \frac{1}{2}\,(c_1 \pm i\, c_2) \qquad (4.40)$$

That the weak currents are electromagnetically charged impose that $c_{\pm}$ are eigenvectors of $q_{\wedge}$, i.e.

$$q_{\wedge}\, c_{\pm} = \pm\, i\, \frac{\sqrt{3}}{2}\, c_{\pm} \qquad (4.40')$$

Then $\tilde{q}$ (as all directions of vector currents) is a unit vector of the diagonal subalgebra $SU(3)^d$, while $\tilde{c}_\pm$ are in the SU(3) algebra of (-)chirality

$$\tilde{q} = \frac{1}{\sqrt{2}} (q \oplus q) \qquad (4.41)$$

$$\tilde{c}_\pm = \frac{1}{2} (\tilde{c}_1 \pm i \tilde{c}_2) \text{ with } \tilde{c}_i = (0 \oplus c_i) \qquad (4.42)$$

There was some similarity between the two strikingly so different electromagnetic interactions and weak interactions as their Hamiltonians show

$$H_{em} = e \int (j^\mu_{em}(x) + \ell^\mu_{em}(x))A_\mu(x)\, d^3\vec{x} \qquad (4.42')$$

$$H_w = \sum_{\varepsilon=\pm} \frac{G}{\sqrt{2}} \int (w^\mu_\varepsilon(x) + \ell^\mu_\varepsilon(x))(w_{-\varepsilon,\,\mu}(x) + \ell_{-\varepsilon,\,\mu}(x))d^3\,\vec{x}$$

$$(4.42'')$$

where ℓ^μ_{em}, $\ell^\mu_\pm$ are the electromagnetic and weak leptonic currents and A_μ is the electromagnetic field.

However, the non leptonic part H_{NL} of H_{weak} contains terms which do not appear in Nature because they would violate the $\Delta\vec{I} = \frac{1}{2}$ rule for $|\Delta Y| = 1$ transitions. This disagreeable feature does not occur in the variant for H_{NL} proposed by L. Radicati (in "Old and New Problems in Elementary

Particle Physics", p. 272 (Ac. Press, New York, 1968)); it keeps only the adjoint representation part $(8,1) \oplus (1,8)$ of H_{NL}, i.e.

$$\text{Radicati} \quad H_{NL} \equiv H'_{NL} = \frac{G}{\sqrt{2}} \sum_{\varepsilon} \int h^{\mu}(x, \tilde{c}_{\pm}) \vee h_{\mu}(x, \tilde{c}_{\pm}) d^3\vec{x} \qquad (4.43)$$

where we have defined the symbol $\vee$ between tensor operators elsewhere (see ref. at the end of IV.4). So H'_{NL} is a tensor operator in the direction:

$$\tilde{z} = (0 \oplus z)$$

with (compare with equations 4.25-28)

$$z = 2c_+ \vee c_- = c_1 \vee c_1 = c_2 \vee c_2 = c_3 \vee c_3 \qquad (4.44)$$

$$c_3 = c_1 \wedge c_2 \qquad (4.45)$$

So z is a pseudo-root. We call z the weak hypercharge direction. The strong hypercharge direction y in the octet is given by the hadron strong mass breaking. It is also a pseudo-root.

We remark that if q_i $(i = 1, 2)$ are two pseudo-roots

$$q_i \vee q_i + q_i = 0; \quad q_1 \wedge q_2 = 0 \quad \Longleftrightarrow \quad (q_1, q_2) = -\frac{1}{2} \text{ or } 1 \qquad (4.46)$$

Electromagnetic charge conservation by strong and weak non leptonic interactions implies

$$q \wedge y = 0, \quad q \wedge z = 0 \tag{4.47}$$

or equivalently

$$q \vee y = q + y, \quad q \vee z = q + z \tag{4.47'}$$

so

$$(q, y) = -\frac{1}{2} = (q, z) \tag{4.47''}$$

But since weak interactions violate hypercharge, (y, z) has a different value:

$$-\frac{1}{2} < (y, z) = 1 - \frac{3}{2}\sin^2\theta < 1 \tag{4.48}$$

where θ is the famous Cabibbo angle.

To explain its value $\theta = .23$, is a challenge of hadronic physics (see V and Cabibbo's lectures).

However we do note the vector relation in the octet

$$2y \vee z + y + z + 3q\sin^2\theta = 0 \tag{4.49}$$

IV.6 The algebra of currents

The action of SU(2) or $U_y(2)$ on the Hilbert space of hadrons is obvious (and it is given in the Rosenfeld tables!); but how do we know the action of G = (SU(3) x SU(3)) $\square$ (1, P, C, PC)?. The answer is that Nature tells us and a second sentence (the first one was given in IV.5) summarizes the main progress in hadron physics:

"The generators $F(\tilde{a})$ of the Lie algebra representation of G on the hadron space $\mathcal{H}$ are given by the integral of the currents on a space-like surface:

$$F(t, \tilde{a}) = \int_{x^o=t} h^o(x; \tilde{a})\, d^3\vec{x} \text{ "} \qquad (4.50)$$

If the symmetry G were exact, we would have

$$\partial_\mu h^\mu(x; \tilde{a}) = 0 \qquad (4.51)$$

and $F(\tilde{a})$ would be independent of t.

Since this is not the case, the action of G on $\mathcal{H}$ depends on t according to (4.50). There is a drawback. If (4.51) is not satisfied, Coleman (J. Math. Phys., 7, 787, 1966) has shown that $F(t, \tilde{a})$ is not selfadjoint so $\exp\{iF(t, \tilde{a})\}$, for all $\tilde{a}$, do not yield a unitary representation of G. There is a mathematical difficulty here to be solved, but meanwhile physics should continue. Gell-Mann has proposed a richer formulation of the algebra of currents:

$$[h^{\circ}(t, \vec{x}; \tilde{a}), h^{\circ}(t, \vec{x}'; \tilde{b})] = \delta(\vec{x} - \vec{x}')\, h^{\circ}(t, \vec{x}; \tilde{a} \wedge \tilde{b}) \tag{4.52}$$

If (4.50) is well defined, by double integration (4.52)yields (4.34').

Another interesting question, since (4.51) is not satisfied, is to find the value of the divergence of the currents. It is often assumed that the G non-invariant part of the total Hamiltonian is the integral on space of a Lorentz invariant density $\mathcal{H}$ (which has to be the image of a tensor operator in a direction $\underset{\approx}{m}'$):

$$\mathcal{H}(x, \underset{\approx}{m}') = \mathcal{H}(x, \underset{\sim}{m} \oplus \tilde{q} \oplus \tilde{c}_1 \oplus \tilde{c}_2 \oplus \tilde{z}), \tag{4.53}$$

when one takes into account the mass breaking term of strong interaction, the electromagnetic interaction, the weak leptonic and non leptonic interaction). With such a hypothesis and some assumption of good mathematical behaviour of the physical quantities and use of equation (4.35), one proves that

$$\partial_{\mu} h^{\mu}(x, \tilde{a}) = \mathcal{H}(x, L(\tilde{a})\, \underset{\approx}{m}') \tag{4.54}$$

The relevance to hadronic physics of the different hypothesis (4.50) and (4.34') or (4.52) or (4.53) will be mainly discussed in B. Renner's lectures.

IV.7 Why critical orbits and idempotents of the algebra appear in the breaking of hadron symmetry

We have to emphasize that by a large variety of experiments observing for instance parity and/or hypercharge violating or conserving effects, the octet directions y,q,z, c_1, c_2 which Nature gives us enable us to orient ourselves completely in the octet space (indeed with the operations $\wedge$ and $\vee$ and linear combination y, q, z, c_1, c_2 generate the full octet#). This is not completely true for the $SU(3)^+ \oplus SU(3)^-$; some directions in $SU(3)^+$ seem out of reach if one does not add the parity operator which exchanges $SU(3)^+$ and $SU(3)^-$.

Although the tensor operators depend linearly on their argument, it is only their value on the sphere of unit vectors which is physically relevant: e.g. $\vec{J}.\vec{n}$ is the angular momentum component only when $\vec{n}$ is a unit vector; e.g. see the quoted Cabibbo's paper and how he had to insist that the current was of "unit length" (his quotation marks).

We have already noticed that the primordial directions of the octet are on the two critical orbits (y,q,z, on one, c_1, c_2 on the other) of the action of Aut SU(3) = $= SU(3) \square (1, C)$ on S_7.

More precisely, the smallest subspace of the octet which is both a $\wedge$ - and a $\vee$-subalgebra and which contains y, q, z, c_1, c_2 is the full octet.

The action of G on its Lie algebra is richer, (see Michel and Radicati quoted above). It has twelve strata and five critical orbits, one per minimal stratum. $\tilde{q}, \tilde{y}$ are on one, $\tilde{z}$ is on another, $\tilde{c}_1, \tilde{c}_2$ on a third one; a fourth one could be a candidate for bearing the CP violation direction. Must the fifth one also bear a primordial physical direction?.

I hope that the role of critical orbits is clear to all physicists. In any physical theory blending a group invariance and a variational principle, symmetry breaking solutions will appear on the critical orbits (other solutions may also appear elsewhere). So the results of many models of symmetry breaking in the physics literature are absolutely <u>independent</u> of the function to be varied in the model (generally the Lagrangian); they just verify the general mathematical theorem.

We also verify on the complexified octet:

$$y \vee y = -y, \quad q \vee q = -q, \quad z \vee z = -z, \quad c_{\pm} \vee c_{\pm} = 0 \qquad (4.55)$$

and on $\mathbb{C}^{16}$

$$\tilde{y} \vee \tilde{y} = \frac{1}{2}\tilde{y}, \quad \tilde{q} \vee \tilde{q} = \frac{1}{2}\tilde{q}, \quad \tilde{z} \vee \tilde{z} = -\tilde{z}, \quad \tilde{c}_{\pm} \vee \tilde{c}_{\pm} = 0 \qquad (4.56)$$

and we understand why idempotents and nilpotents of symmetric algebras appear in symmetry breaking.

The hadronic physical quantities are at each time tensor operators of G and equations of hadronic physics are covariant for G. A covariant relation between tensor operators implies a geometrical equation for their arguments. A non linear equation for a tensor operator (e.g. a bootstrap equation) will imply a non linear equation on its argument; the simplest such equation is

$$\tilde{x} \vee \tilde{x} = \lambda \tilde{x} \qquad (4.57)$$

with $\lambda \neq 0$ (idempotent) or $\lambda = 0$ (nilpotent).

IV.8 Real or complex algebra? SU(3) or SL(3, ¢)?.

Since weak currents carry an electromagnetic charge, they cannot be Hermitean. It is true that we can write the Hermitean Hamiltonian only in terms of Hermitean currents and so only the directions $\tilde{c}_1$, $\tilde{c}_2$ appear in equation (4.53) for instance#. But the algebra generated by the physical currents is a complex algebra: it is $S\mathcal{L}(3, ¢) \oplus S\mathcal{L}(3, ¢)$.

There have been historical arguments for quantum physicists to avoid non-compact Lie groups for particle multiplets because their finite representations are not unitary. However the isospin currents given to us by Nature T_+,

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We leave as an exercise to write the explicit Hamiltonian (4.42") and (4.43) only in terms of c_1 and c_2: hint, replace $\tilde{c}_{\pm}$ by its value in (4.42).

T_-, T_3 form a real non compact algebra $T_3 \wedge T = T_{\pm}$, $T_+ \wedge T_- = 2\, T_3$, namely SL(2, R).

There might be a deep argument for considering the complexified of the invariance group G in hadronic physics. Analytic properties are essential and for instance, complexification of the Lorentz group by analytic continuation yields important results in axiomatic theory (e.g. the CTP theorem). The introduction of the complexified# $\bar{G}$ of G in hadronic physics was considered by L. Abellanas (J. Math. Phys., 13, No.7, 1064 (1972)). The adjoint action of semi-simple Lie groups on their Lie algebra space has beautiful properties. Any finite dimensional Lie algebra can be faithfully represented by an algebra of matrices $\mathcal{G}$. Let $\mathcal{S}$ (for semi-simple) be the set of diagonalizable matrices and $\mathcal{N}$ the set of nilpotent matrices (i.e. $\exists\, k > 0$, $n^k = 0$). Then every $x \in \bar{\mathcal{G}}$ can be written in a unique way:

$$x = s + n, \quad s \wedge n = 0, \quad s \in \mathcal{S}, \quad n \in \mathcal{N} \qquad (4.58)$$

Let $\mathcal{R}$ be the set of regular elements of $\bar{\mathcal{G}}$ i.e those elements x whose little group G_x has minimal dimension ℓ (equal to the rank of the semi-simple Lie group). In the action of the complex semi-simple Lie group $\bar{G}$ on $\bar{\mathcal{G}}$, one finds that:

$\mathcal{R} \cap \mathcal{S}$ is an open dense stratum in $\mathcal{S}$, and

$\mathcal{R} \cap \mathcal{N}$ is an open dense orbit in $\mathcal{N}$

It may be amazing to recall here a general theorem: $\bar{\bar{G}} = \bar{G} \times \bar{G}$ where $\bar{\bar{G}}$ is the complexified of $\bar{G}$ considered as real group again.

For SL(3, ¢), the complement# in $\mathcal{S}$ and $\mathcal{N}$ of this stratum and this orbit is respectively one stratum and one orbit that we shall label as "exceptional". Then y, q, z are in the exceptional strata in $\mathcal{S}$, $c_{\pm}$ are in the exceptional orbit in $\mathcal{N}$.

V. THE STRONG BREAKING OF SU(3) x SU(3)

V.1 Critical orbits and idempotent candidates for strong breaking

We want the breaking of G = (SU(3) x SU(3)) ◻ ◻ (1, P, C, PC) invariance to be onto a subgroup $H \subset G$ which still contains the discrete invariances P, C, PC since they are preserved by strong interactions, and we want the connected component H_o of H to contain $U_y(2)^d$, the group of isospin and hypercharge conservation. The connected subgroups of SU(3) x SU(3) which contain $U_y(2)$ are (the arrow indicates inclusion, d is for diagonal)

(We exclude the vector 0, which forms the set $\mathcal{S} \cap \mathcal{N}$. It is a stratum, with little group SL(3, ¢).

$$\begin{array}{ccccc} & \nearrow & SU(3)^{d} & & \\ U_y(2)^{d} & \longrightarrow & U_y(2)^{d} \times U_y(1)^{d} & \searrow & \\ & & & & U_y(2)^{+} \times U_y(2)^{-} \\ & \searrow & SU_y(2)^{+} \times SU_y(2)^{-} \times U_y(1)^{d} & \nearrow & \end{array} \qquad (5.1)$$

C. Darzens in a paper submitted to Ann. Phys. gives a list of all critical orbits $[G:H]$, with H_o in (5.1), which appear in irreducible representations of G.

Some of these results were obtained by B. Renner and A. Sudberry, Nucl. Phys. B.13, 27 (1969) which tried to solve a similar physical problem (not phrased however in terms of critical orbits).

With the usual notation (m, n) for the irreducible representations of SU(3), the representations of G are either of the type (n,n), when $n = \bar{n}$; $(n,\bar{n}) \oplus (\bar{n},n)$; $(n,m) \oplus (m,n)$ when $m = \bar{m}$, $n = \bar{n}$; and $(n, m) \oplus (m, n) \oplus (\bar{m}, \bar{n}) \oplus (\bar{n}, \bar{m})$ in the general case.

Critical orbits for

$H_o = SU(3)^d$ appear only in $(n,\bar{n}) \oplus (\bar{n},n)$

$H_o = U(2)^+ \times U(2)^-$ " " " $(1,8) \oplus (8,1)$

$H_o = SU(2)^+ \times SU(2)^- \times U(1)^d$ appear only in $(3,\bar{3}) \oplus (\bar{3},3)$

$H_o = U(2)^d$ " " " $(3,6) \oplus (6,3) \oplus (\bar{3},\bar{6}) \oplus (\bar{6},\bar{3})$

The vectors of the three first families of orbits are idempotents or nilpotents of the unique symmetric algebra one can form in these representations; there is no such algebra for the representation of the last line and Darzens excludes it. Choosing a n-dependent Lagrangian $\mathcal{L}_n$ with the variance

$$\alpha \underset{\sim}{y} \oplus \beta \underset{\sim}{W}_n \,, \quad \alpha^2 + \beta^2 = 1 \qquad (5.2)$$

where $\underset{\sim}{y}$ is the unique (up to a sign) unit vector of the $(3,\bar{3}) \oplus (\bar{3},3)$ which has $H = (SU_y(2)^+ \times SU_y(2)^- \times U_y(1))$ $\square$ (1, C, P, CP) as little group and $\underset{\sim}{W}_n$ is the vector invariant by $SU(3)^d$ in the $(n,\bar{n}) \oplus (\bar{n},n)$, or (n,n) when $n = \bar{n}$, representation. After a fit of the meson mass spectrum Darzens computes for instance the meson-meson scattering lengths. Of course for low n (values as 3,6,8) the results are near the S. Weinberg values (Phys. Rev. 146, 1568,(1968). However it is a nice feature of the Darzen's paper to see the dependence on n of these physical constants! (they go as $n^{2/3}$ for large n).

V.2 Critical orbits of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation

If one assumes that the strong interaction breaking is in an irreducible (over the reals) representation, the $(3,\bar{3}) \oplus (\bar{3},3)$, proposed by M. Gell-Mann, R.J. Oakes, B. Renner (Phys. Rev. 175, 2195 (1968)) is the best candidate (see also the arguments in B. Renner's lectures). Radicati and I have studied in our paper (Ann. Phys. 66, 759 (1971)) the orbits of SU(3) x SU(3) in this representation. The $(3,\bar{3})$ representation space can be realized as the 9 dimensional complex space of the 3 x 3 complex matrices $\underset{\sim}{x}$, on which the SU(3) x SU(3) action is defined by

$$(u_1, u_2) \in SU(3) \times SU(3), \qquad \underset{\sim}{x} \mapsto u_1 \underset{\sim}{x} u_2^{-1} \tag{5.3}$$

This action preserves the Hermitean scalar product:

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \frac{1}{2} \operatorname{tr} \underset{\sim}{x}^* \underset{\sim}{y} \tag{5.4}$$

This space considered as 18-dimensional real space R^{18} is that of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation; it is an orthogonal representation which leaves invariant the Euclidean scalar product:

$$(\underset{\sim}{x}, \underset{\sim}{y}) = \operatorname{Re} \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \frac{1}{4} \operatorname{tr} (\underset{\sim}{x}^* \underset{\sim}{y} + \underset{\sim}{y}^* \underset{\sim}{x}) \tag{5.5}$$

Gell-Mann has defined an orthonormal basis, generally used in the literature: $u_o, u_1, \ldots, u_8, v_o, v_1, \ldots, v_8$. The group action leaves also invariant a symplectic (= antisymmetric) bi linear form A in R^{18} defined by:

$$(\underset{\sim}{x},\ A\underset{\sim}{y}) = \mathrm{Im} \langle \underset{\sim}{x},\ \underset{\sim}{y} \rangle = -\frac{1}{4i}\ \mathrm{tr}\ (\underset{\sim}{x}^{*}\underset{\sim}{y} - \underset{\sim}{y}^{*}\underset{\sim}{x}) \qquad (5.6)$$

By the action (5.3) it is possible to diagonalize the matrix $\underset{\sim}{x}$ such that the (diagonal) matrix elements α, β, $\gamma\ e^{i\phi}$ satisfy:

$$\alpha \geq \beta \geq \gamma \geq 0 \qquad (5.7)$$

Note that two invariants of the action are

$$\det \underset{\sim}{x} = \alpha\beta\gamma e^{i\phi},\ (\underset{\sim}{x},\ \underset{\sim}{x}) = \frac{1}{2}\ (\alpha^2 + \beta^2 + \gamma^2) \qquad (5.8)$$

The orbit space of the action of SU(3) x SU(3) on $S_{17} \equiv \{(\underset{\sim}{x},\ \underset{\sim}{x}) = 1\}$ is a cone (Fig. 3) cut by a plane ⊥ to the axis, in a 3 dimensional space.

For this action of SU(3)x x SU(3), the four strata are represented by: The inside of the cone for the generic stratum, (little group# $(U(1) \times U(1))^d$), open dense.

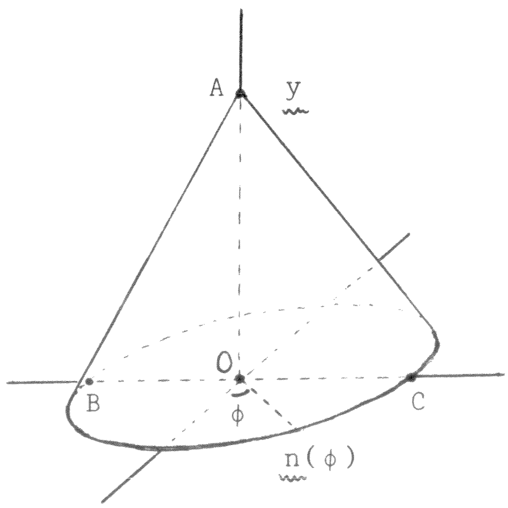

Fig. 3

\# We recall that the little groups are defined up to a conjugation.

The surface of the cone minus the vertex A and the circle; i.e. the two disconnected pieces: lateral side and bottom. Little group $U(2)^d$.

The circle of the base represents a closed stratum of one parameter (ϕ) family of orbits; the little group is $SU(3)^d$ (it is maximal). The vertex of the cone is a stratum of one orbit (which is therefore critical); little group $SU(2)^+ \times SU(2)^- \times U(1)^d$ which is also a maximal subgroup in the lattice of little groups of this action

$$(U(1) \times U(1))^d \longrightarrow U(2)^d \begin{array}{l} \nearrow SU(2)^+ \times SU(2)^- \times U(1)^d \\ \\ \searrow SU(3)^d \end{array} \qquad (5.9)$$

Exercise: Must all $SU(3) \times SU(3)$ invariant vector fields on S_{17} have zeros?. (Solution at the end of this section).

The problem we have to solve is that of the action of the full group $G = (SU(3)^+ \times SU(3)^-)_{\square}(Z_2(P) \times Z_2(C))$. We have solved it for the connected subgroup G_o which is an invariant subgroup. The following general theorem is easy to prove and useful:

Theorem: The quotient group G/G_o acts on the orbit space M/G_o and the diagram 2 of canonical maps on orbit spaces is commutative, so

$$M/G = (M/G_o)/(G/G_o) \qquad (5.10)$$

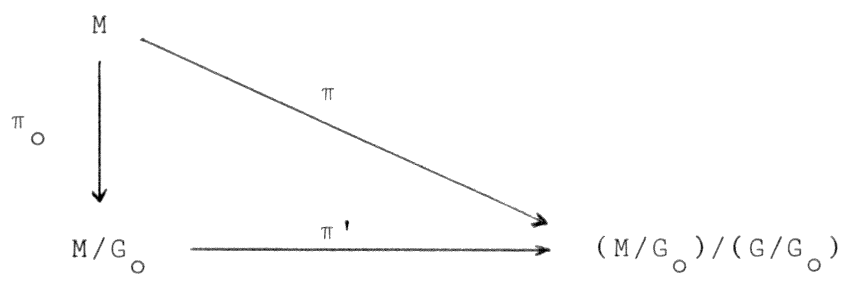

Diagram 2

C acts trivially on $u_o \ldots u_8$, $v_o \ldots v_8$, i.e. C acts trivially on the physical space R^{18} we consider, and the actions of P and CP are identical. The u's (v's) are eigenvectors of P or CP for the eigenvalues + 1 (-1)#. On the orbit space of Fig. 3, this represents the symmetry through the plane of azimuth $\phi = 0$ (or π). Therefore only orbits whose representative points are in the triangle ABC have a little group containing P, CP and C. There are seven strata. Two are closed; one contains one orbit (represented by A) which is critical, with little group $SU_y(2)^+ \times SU_y(2)^- \times U_y(1)^d \,\square\, \square\,(1, P, CP, C)$, where y is an arbitrary pseudo-root of the octet. The other closed stratum contains two orbits, which are therefore critical. These are represented by the points B and C ($\phi = 0$, $\phi = \pi$) with little group $SU(3)^d \,\square\, (1, P, CP, C)$.

In Michel and Radicati quoted above, C and CP have been mistakenly exchanged; this error does not change the little groups of critical orbits and the conclusions of the paper.

Solution of the exercise

Gradient fields must have a zero on the critical orbit represented by the point A of the orbit space. The little group of this critical orbit is $H = SU_y(2)^+ \times SU_y(2)^- \times U_y(1)^d$ where y is an arbitrary pseudoroot of the octet. The orbit is $[SU(3) \times SU(3): H]$, it has dimension $16 - 7 = 9$. Since it is odd dimensional and compact, its Euler-Poincaré characteristics is zero. So it is possible to have on the orbit a non vanishing vector field. Is it possible to have it SU(3) x SU(3) invariant?. Yes, because the normalizer of H in SU(3) x SU(3) is $U(2)^+ \times U(2)^-$ which has a dimension larger than H by one unit and which is connected. There is therefore no obstruction to the construction of a SU(3) x SU(3) invariant vector field on S_{17}. We can realize one such vector field with the antisymmetric bilinear form A defined in equation (5.6):it is simply $A\underset{\sim}{x} = i\underset{\sim}{x}$. Indeed the vector $A\underset{\sim}{x}$ at $\underset{\sim}{x}$ is in the tangent plane to the unit sphere since $(\underset{\sim}{x}, A\underset{\sim}{x}) = 0$ and $A\underset{\sim}{x}$ is always different from zero when $\underset{\sim}{x} \neq 0$; the invariance of A by the linear action of the $SU(3)^+ \times SU(3)^-$ in the space R^{18} of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation insures that the vector field $A\underset{\sim}{x}$ on S_{17} is SU(3) x SU(3) invariant and has no zero.

V.3 The idempotents and nilpotents of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation

For more details we refer to Michel and Radicati, already quoted in V.2 and also: Evolution of Particle Physics (E. Amaldi Festschrift) p. 191, Academic Press, New York(1970)

The $(3,\bar{3}) \oplus (\bar{3},3)$ representation of SU(3) x SU(3) appears only once in its (symmetrical) tensor product. As we have seen in IV.1, this defines, up to a normalization factor, a symmetrical algebra on R^{18}, the vector space carrying the $(3,\bar{3}) \oplus (\bar{3},3)$ representation, and this algebra has SU(3) x SU(3) as group of automorphisms. We have denoted in our paper the algebra law by the symbol $\top$. With the realization of R^{18} by 3 x 3 complex matrices, the law $\top$ is:

$$\underset{\sim}{x} \top \underset{\sim}{y} = \frac{1}{2}\Big[I(\mathrm{tr}\,\underset{\sim}{x}^*\mathrm{tr}\,\underset{\sim}{y}^* - \mathrm{tr}\,\underset{\sim}{x}^*\underset{\sim}{y}^*) - \underset{\sim}{x}^*\mathrm{tr}\,\underset{\sim}{y}^* - \underset{\sim}{y}^*\mathrm{tr}\,\underset{\sim}{x}^* + \underset{\sim}{x}^*\underset{\sim}{y}^* + \underset{\sim}{y}^*\underset{\sim}{x}^*\Big] \tag{5.11}$$

We leave to the reader who prefers to use basis the pleasure to write in a full page all equations defining the structure constants!

Specific equations of this algebra are:

if det x ≠ 0,
$$\underset{\sim}{x} \top \underset{\sim}{x} = (\underset{\sim}{x}^*)^{-1} \det \underset{\sim}{x}^* \tag{5.12}$$

$$(\underset{\sim}{x} \top \underset{\sim}{x}) \top (\underset{\sim}{x} \top \underset{\sim}{x}) = \underset{\sim}{x} \det \underset{\sim}{x} \tag{5.12'}$$

Note that (5.12) is defined only on the open dense set of invertible matrices, but as an algebraic relation this defines it everywhere (and $(\underset{\sim}{x} \top \underset{\sim}{x})\underset{\sim}{x}^* = I \det \underset{\sim}{x}^*$ is everywhere defined). Equation (5.12') shows that for some fourth power, every element is idempotent.

One finds for unit vectors one orbit of nilpotents

$$\underset{\sim}{y} \top \underset{\sim}{y} = 0 \qquad (5.13)$$

represented by A in the orbit space (Fig. 3) and two orbits of idempotents

$$\underset{\sim}{n}_{\pm} \top \underset{\sim}{n}_{\pm} = \pm \sqrt{\frac{2}{3}}\, \underset{\sim}{n}_{\pm} \qquad (5.13')$$

which are represented respectively by C and B in Fig. 3.

We remark that this representation $(3,\bar{3}) \oplus (\bar{3},3)$ can be extended to $G = (SU(3) \times SU(3)) \,\square\, (1,P,PC,C)$ where C is represented trivially and the algebra $\top$ has still G as group of automorphisms. So elements in G-critical orbits of S_{17} are idempotents or nilpotents of the linear representation, and conversely.

V.4 <u>Is the direction of strong breaking an idempotent</u>?.

As we have seen in V.1, if we choose the direction of strong breaking in a real reducible representation of G (e.g. $\alpha \underset{\sim}{y} \oplus \beta \underset{\sim}{W}_n \in (3,\bar{3}) \oplus (\bar{3},3) \oplus (8,8)$) it is an idempotent. However if we stay only inside the $(3,\bar{3}) \oplus (\bar{3},3)$ it is not. Consider the closed set $(S_{17})^H$, with $H = U_y(2)^d \,\square\, (1,C,CP,P)$ (see III.2 for the definition of M^H). It is a one dimensional submanifold of S_{17} which, with the usual definition of phases of P,C,CP, is mapped in the orbit space on the triangle ABC (of Fig. 3). In the realization of R^{18} by 3 x 3 matrices, it is the set of real, diagonal matrices d satisfying

$$\frac{1}{2} \operatorname{tr} \underset{\sim}{d}^2 = 1 \qquad (5.14)$$

The matrices represented by A,B,C respectively are

$$\underset{\sim}{y} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \sqrt{2} \end{pmatrix}, \quad \underset{\sim}{n}_- = - I \sqrt{\frac{2}{3}}, \quad \underset{\sim}{n}_+ = I \sqrt{\frac{2}{3}} \qquad (5.15)$$

This one parameter family of vectors $\in R^{18}$ has been specially studied by Kuo, and by Okubo and Mathur in several papers. In the u_i, v_j basis

$$\underset{\sim}{y} = \frac{1}{\sqrt{3}} (u_o - \sqrt{2}\, u_8), \quad \underset{\sim}{n}_+ = u_o \qquad (5.16)$$

and, according to Gell-Mann, Oakes and Renner already quoted, the direction of breaking is along the vector

$$\underset{\sim}{m} \sim u_o - 1.25\, u_8 \qquad (5.17)$$

which is nearer to $\underset{\sim}{y}$ than to $\underset{\sim}{n}_+$ i.e. nearer to SU(2) x SU(2)x x U(1) breaking than to SU(3) breaking.

More generally, we defined a vector $\underset{\sim}{n}'$ orthogonal to $\underset{\sim}{y}$ in the plane $\underset{\sim}{y}$, $\underset{\sim}{n}_+$, i.e.

$$\underset{\sim}{n}' = \frac{1}{\sqrt{2}} (\sqrt{3}\, \underset{\sim}{n}_+ - \underset{\sim}{y}), \text{ i.e. } \underset{\sim}{n}' = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad (5.18)$$

(n' is 0 in Fig. 3) and m is fixed by the angle ω

$$m = y \cos \omega + n' \sin \omega \tag{5.19}$$

so that the direction $u_o + cu_8$ corresponds to the angle

$$\operatorname{tg} \omega = \frac{\sqrt{2} + c}{1 - \sqrt{2}\, c} \tag{5.20}$$

and $c = -1.25$ corresponds to $\operatorname{tg} \omega = .058$. To summarize:

$$\begin{aligned} c = -\sqrt{2} \quad &\Longleftrightarrow \quad m = y \quad \Longleftrightarrow \quad \operatorname{tg} \omega = 0 \\ c = -1.25 \quad &\Longrightarrow \quad \operatorname{tg} \omega = .058 \\ c = 0 \quad &\Longleftrightarrow \quad m = n_+ \quad \Longleftrightarrow \quad \operatorname{tg} \omega = \sqrt{2} \end{aligned} \tag{5.21}$$

F. Pegoraro and S. Subba Rao in a preprint: "Effect of weak interactions on the breaking of hadronic internal symmetry", to appear in Nucl. Phys., assume that

$$m = \lambda(u_o + cu_8 + du_3) \tag{5.22}$$

and impose that in the (unique) symmetric (complex) G-invariant algebra on the space of the

$$((3,\bar{3}) \oplus (\bar{3},3)) \oplus (1,8) \tag{5.23}$$

$$m \oplus \tilde{c}_{\pm} \tag{5.23'}$$

must be nilpotent.

They find four families of solutions. The physically interesting one yields

$$d = -\frac{\sqrt{2}+c}{\sqrt{3}} \text{ i.e. } \underset{\sim}{m} = \lambda(u_o + cu_8 - \frac{\sqrt{2}+c}{\sqrt{3}} u_3) \tag{5.24}$$

which was already proposed by Oakes somewhat more arbitrarily but seems to fit better the data than $\underset{\sim}{m}$ of equation (5.17). (R.J. Oakes, Phys. Let. 29B, 683 (1969) and 30B, 262 (1969). More generally we can ask the question of the:

V.5 Relation between strong and weak breaking

This subject is quite open and you will hear more about it in N. Cabibbo's lectures. I do not believe that there is a "cheap", geometrical way to compute the Cabibbo angle θ. But one can hope that θ is not a fundamental constant of physics and that it can be obtained through some computation; or at least, be related to some other variables. For instance, which G-invariant relations there can be between the direction of weak breaking, defined by θ, and that of strong breaking defined by ω in the previous section.

We just give here the remarks made with Radicati (quoted paper). If $\tilde{a} \mapsto L(\tilde{a})$ is the representation of the Lie algebra of G on R^{18}, the space of the $(3,\bar{3}) \oplus (\bar{3},3)$, the lowest degree algebraic invariants one can form with $\tilde{z}$ and $\underset{\sim}{m}$ are:

$$(\underset{\sim}{m}, L(\tilde{z})\underset{\sim}{m}) = 0 \tag{5.25}$$

$$(\underset{\sim}{m}, AL(\tilde{z})\underset{\sim}{m}) = -\frac{1}{\sqrt{3}}\left(1-\frac{3}{2}\sin^2\theta\right)\left(1-\frac{3}{2}\sin^2\omega\right) = f(\theta,\omega) \qquad (5.25')$$

$$(L(\tilde{z})\underset{\sim}{m}, L(\tilde{z})\underset{\sim}{m}) = \frac{1}{2\sqrt{3}}\left(\sqrt{3} - f(\theta, \omega)\right) \qquad (5.25'')$$

Note that the invariants with odd power in $\underset{\sim}{m}$ vanish.

We leave to the reader the search for more invariants. Most of them are symmetrical functions of θ and ω. For such invariants, whatever the fashion they will enter in a physical problem, they will have solutions $\theta = \omega$ (there may have also broken symmetric solutions). This does not correspond to the actual values:

$$\mathrm{tg}\ \theta = 0.25, \qquad \mathrm{tg}\ \omega = .058 \qquad (5.26)$$

It is surely useful to look at all invariants built with all unit vectors of the directions of breaking ($\tilde{q}$, $\tilde{z}$, $\tilde{c}_{\pm}$) including the possible values of $\underset{\sim}{m}$. But it is not enough to do physics.

To know that the directions of breaking possess so many beautiful mathematical properties and are very exceptional directions in their space is fascinating. It is also a physicist's work to find regularity in Nature as we did. Examples of such a useful work was Balmer's discovery of the $(1/n^2) - (1/m^2)$ law for the hydrogen spectral lines, although Balmer could have no idea how the inverse of the square of integers would enter in physics.

It also seems that Niels Bohr did not know Balmer's law when he made his first hydrogen atom model!.

V.6 Relations between critical orbits on the unit sphere and idempotents or nilpotents of symmetrical algebra for the orthogonal actions of compact Lie groups

As Mr. Jourdain was doing prose, physicists found critical orbits and idempotents without knowing their existence. Now that they are conscious of these concepts and want to use them they naturally ask me the relations existing between them. They are not simple. Consider the four statements concerning the orthogonal action of a compact Lie group for a unit length vector m:

a) The little group G_m is maximal in the (finite)set of little groups (up to a conjugation) which appear in the action.

b) The stratum S(m) of the unit sphere is closed.

c) The orbit G(m) of the unit sphere is critical.

d) m is idempotent or nilpotent of the symmetric algebra.

From the examples we gave, we have seen the following non implications

c $\not\Longrightarrow$ d: indeed it is the case for the root orbit of ex III.1.e: Action of Aut SU(3) on S_7, R(the "equator") is a critical orbit but its vectors r satisfy equation (4.19), so they are not idempotent or nilpotent.

d $\not\Longrightarrow$ c: indeed for $\phi = 0$ or π on the closed stratum of the SU(3) x SU(3) action on the unit sphere S_{17} of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation (see Fig.3) the vectors $\underset{\sim}{n}_+$, $\underset{\sim}{n}_-$ of equation 5.13' and 5.15 are idempotent but their orbits are not critical for the action of the connected group SU(3) × SU(3).

These two examples also show that

a $\not\Longrightarrow$ c and

b $\not\Longrightarrow$ c

We have proven (beginning of III.2) a $\Longrightarrow$ b.Could we prove with slightly stronger assumption (a and c) $\Longrightarrow$ d?

The G-invariance of the symmetric algebra $\Longrightarrow$ $\Longrightarrow$ $G_{x_\tau x} \supset G_x$: it might be that $x_\tau x = 0$; if not, and if G_x is maximal $G_{x_\tau x} = G_x$ and $x_\tau x$, x are on the same stratum. If this stratum has a unique orbit (stronger hypothesis c) then $x_\tau x$ and x are on the same orbit. Hence $x_\tau x = h.x$ with $h \in \mathcal{N}_G(G_x)$. Hence the

<u>Theorem</u>: Given an orthogonal action of G which possesses a symmetrical algebra τ , if there is a stratum on the unit sphere with a single orbit $[G:H]$ and if $\mathcal{N}_G(H) = H$, the vectors of this orbit are idempotents or nilpotents: $x_\tau x = \lambda x$ (λ may be zero).

Of course the elements of a critical orbit might be idempotent even if the orbit is not single and $\mathcal{N}_G(H)$ is strictly larger than H.

Conversely let $\mu = (x_T x, x)$. Any function of μ has extrema on the unit sphere where x is idempotent. Indeed, with the Lagrage multiplier λ, such extrema are obtained for

$$\text{grad}\ (f - \lambda((x, x) - 1)) = 0 \qquad (5.27)$$

i.e.

$$3\ \frac{df}{d\mu}\ x_T x - 2\ \lambda x = 0 \qquad (5.28)$$

This does not imply that the idempotents are on a critical orbit because the functions of μ may not be the most general G-invariant function on the unit sphere (they are so for the SU(3) invariant functions on S_7).

To conclude, the relations between idempotents and critical orbits are not very direct, but both seem to play an important role in symmetry breaking.

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