

# Properties of the Breaking of Hadronic Internal Symmetry

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The directions of breaking of the hadronic internal symmetry by the electromagnetic, semileptonic—and nonleptonic—weak and  $CP$  violating interactions are characterized by remarkable mathematical properties. These directions correspond to idempotents or nilpotents of an algebra and they are critical, i.e., every invariant function for the symmetry group, e.g.,  $(SU(3) \times SU(3)) \times (1, P, C, PC)$  has an extremum on these directions.

## I. INTRODUCTION

In the last ten years, several internal symmetry groups have been considered for hadronic physics. We will limit here our study to  $SU(3) \times SU(3)$ , some of its extensions by discrete operations ( $C, P, CP$ ), and some of their subgroups. The symbol  $G$  will denote any of them. We will be more specially interested in the subgroups belonging to the following inclusion scheme:

$$\begin{array}{ccc}
 & SU(3) & \\
 \subset & & \subset \\
 U_y(2) & & SU(3) \times SU(3), \quad (1) \\
 \subset & & \subset \\
 & SU_y(2) \times SU_y(2) \times U_y^d(1) &
 \end{array}$$

where  $SU(3)$  is the diagonal subgroup and  $U_y(2)$  contains the isotopic spin group and the hypercharge phase transformations. When  $G$  is larger than  $U(2)$ , electromagnetic, weak, and strong interactions violate the corresponding symmetry, but these violations follow, to a good approximation, well-defined selection rules.

This can be explained by the assumption that the corresponding interaction terms in the total Hamiltonian define some special directions in the representation spaces of the symmetry group  $G$ .

The physical characterization of the directions of breaking of the  $G$ -symmetry has been discussed by several authors. In this paper, we shall instead analyze their mathematical characterization. It seems to us that the remarkable mathematical properties of these directions constitute an interesting empirical fact which probably cannot be ignored if one wants to understand the breaking of hadronic symmetry.

We give only the essential results in the text, and we confine in the appendices the definition of the mathematical concepts not familiar to the majority of high energy physicists and the explicit realization of the mathematical objects defined and used. This paper contains no proof of the essential results. Some of the proofs are already contained in previous publications [1, 2] and the others will be published subsequently.

## II. THE INTERNAL SYMMETRY OF HADRONS

Let  $\mathcal{H}$  be the Hilbert space of the physical states which is used to describe nuclear and subnuclear phenomena. This space is the tensor product

$$\mathcal{H} = \mathcal{H}_H \otimes \mathcal{H}_\Gamma \otimes \mathcal{H}_L, \quad (2)$$

where the three terms in the right side are the Hilbert spaces of hadronic, photonic, and leptonic states, respectively. The internal symmetry group  $SU(3) \times SU(3)$  acts trivially on  $\mathcal{H}_\Gamma$  and  $\mathcal{H}_L$  and acts on  $\mathcal{H}_H$  through a unitary representation. Therefore, the action of  $G$  on  $\mathcal{H}$  is also a unitary representation

$$G \ni g \rightsquigarrow U(g) = (U(g)^{-1})^* \quad (3)$$

which is a function of time. This time-dependence reflects the fact that  $SU(3) \times SU(3)$  (and its subgroups) is an approximate symmetry.

Equation (3) defines also an action of  $G$  on  $\mathcal{L}(\mathcal{H})$ , the vector space of the linear operators on  $\mathcal{H}$ ,

$$\mathcal{L}(\mathcal{H}) \ni A \rightsquigarrow U(g) A U(g)^* = U(g) A U(g)^{-1}. \quad (4)$$

This is a linear representation equivalent to  $U \otimes \bar{U} = U \otimes (U^{-1})^T$ , the tensor product of  $U(g)$  and its complex conjugate.

If we assume that the representation  $U(g)$  is infinitely differentiable, we can

derive from it a representation (up to a factor  $i$ ) of the Lie algebra  $\mathcal{G}$  of  $G$  by self-adjoint operators on  $\mathcal{H}$ :

$$\mathcal{G} \ni a \rightsquigarrow F(a) \tag{5}$$

$$[F(a), F(b)] = iF(a \wedge b) \tag{6}$$

where  $\wedge$  denotes the Lie algebra law.

Let  $\mathcal{E}$  be a vector space,  $D$  a linear differentiable representation of  $G$  on  $\mathcal{E}$  and  $L$  the corresponding representation of  $\mathcal{G}$ .

A  $G$ -equivariant<sup>1</sup> operator  $T$  between two representation spaces  $\mathcal{E}_1$  and  $\mathcal{E}_2$  is a vector space homomorphism  $\mathcal{E}_1 \rightarrow \mathcal{E}_2$  which commutes with the group action on each space:

$$\forall g \in G : TD_1(g) = D_2(g) T. \tag{7}$$

This is equivalent to

$$TL_1(a) = L_2(a) T. \tag{8}$$

If  $T_1$  and  $T_2$  are  $G$ -equivariant operators, so are  $\lambda T_1$  and  $T_1 + T_2$ . Hence the equivariant operators between  $\mathcal{E}_1$  and  $\mathcal{E}_2$  form a vector space whose dimension<sup>2</sup> will be denoted by  $\nu(\mathcal{E}_1, \mathcal{E}_2) = \nu(\mathcal{E}_2, \mathcal{E}_1)$ .

In the physics literature, an  $\mathcal{E}$ -tensor operator for  $G$  on  $\mathcal{H}$  is a  $G$ -equivariant operator  $\mathcal{E} \rightarrow \mathcal{L}(\mathcal{H})$ . Explicitly,

$$\forall g \in G, \quad \forall \mathbf{m} \in \mathcal{E} : U(g) T(\mathbf{m}) U(g)^{-1} = T(D(g) \mathbf{m}). \tag{9}$$

This implies

$$[F(a), T(\mathbf{m})] = iT(L(a) \mathbf{m}). \tag{10}$$

Equation (6) is a special case of Eq. (10) for  $\mathcal{E} = \mathcal{G}$ , the vector space of  $\mathcal{G}$ , and  $T = F$ . If  $D$  is an irreducible representation,  $T$  will be called an irreducible tensor operator. It is important to note that the values  $T(\mathbf{m})$  of  $T$  are operators on  $\mathcal{H}$ , but  $T$  itself is not. When all values of  $T$  are self-adjoint operators on  $\mathcal{H}$ , the representation space  $\mathcal{E}$  is real, and the representation  $D$  of  $G$  on  $\mathcal{E}$  is a real orthogonal representation. We denote by  $(\mathbf{m}_1, \mathbf{m}_2)$  the Euclidean scalar product that it leaves invariant.

The linearity of the dependence of  $T$  on  $\mathbf{m}$  is essential even though in the physical

<sup>1</sup> A rarer synonym of “equivariant” in the mathematical literature is “intertwining.”

<sup>2</sup> Sometimes called the intertwining number.

applications we will consider the values  $T(\mathbf{m})$  of  $T$  only for vectors  $\mathbf{m}$  normalized to one.<sup>3</sup>

We will denote (when it does not lead to ambiguities) the irreducible representations of  $SU(3)$  by their dimension: for example, (1), (3), ( $\bar{3}$ ), (8) are, respectively, the trivial, the fundamental representation, its complex conjugate, and the adjoint representation, whose space we shall also call the octet space. The irreducible representations of  $SU(3) \times SU(3)$  are the tensor product of two irreducible representations of  $SU(3)$ . We will denote them by a pair  $(m, n)$ , where  $(m)$  and  $(n)$  are the irreducible representations of the two  $SU(3)$  factors which are called the  $SU(3)$  chiral groups. The adjoint representation  $(1, 8) \oplus (8, 1)$  of  $SU(3) \times SU(3)$  is reducible. The representation  $(3, \bar{3}) \oplus (\bar{3}, 3)$  is reducible on the complex but is irreducible as a real representation.

We use latin letters  $a, b, \dots$  for the elements of the octet space. The elements of the Lie algebra space of  $SU(3) \times SU(3)$  will instead be denoted by  $\tilde{a} = a_+ \oplus a_-$  where  $a_+$  and  $a_-$  are, respectively, elements of the first and second octet spaces with opposite chirality.

A vector  $\tilde{a}$  belongs to the Lie algebra of the diagonal subgroup  $SU(3)^d$  if  $a_+ = a_- = a$ , i.e., if  $\tilde{a} = a \oplus a$ .

The usefulness of considering the action of  $SU(3) \times SU(3)$  on  $\mathcal{H}$  is due to the fact that, at each instant  $t$ , the observable operators on  $\mathcal{H}$  can be approximately considered as the values of  $SU(3) \times SU(3)$  tensor operators. Here, we only list the assumptions that are commonly made about the most important operators.

(i) The hadronic electromagnetic current  $j_{em}^\mu(x)$  and the charged vector and axial vector weak current  $v_\pm^\mu(x)$  and  $a_\pm^\mu(x)$  are the values of a  $(8, 1) \oplus (1, 8)$  tensor operator  $h^\mu(x)$  for different vectors of  $\mathcal{E}_{16}$ , the space of the representation  $(8, 1) \oplus (1, 8)$ . Precisely,

$$j_{em}^\mu(x) = -\sqrt{\frac{2}{3}} h^\mu(x; \tilde{q}), \quad (11)$$

$$v_\pm^\mu(x) = \frac{1}{2} h^\mu(x; c_\pm \oplus c_\pm), \quad (12)$$

$$a_\pm^\mu(x) = \frac{1}{2} h^\mu(x; c_\pm \oplus -c_\pm), \quad (13)$$

$$v_\pm^\mu - a_\pm^\mu = h^\mu(x; \tilde{c}_\pm), \quad (14)$$

where

$$\tilde{q} = \frac{1}{\sqrt{2}} (q \oplus q), \quad (15)$$

$$\tilde{c}_\pm = \frac{1}{2} (\tilde{c}_1 \pm i\tilde{c}_2) = (0 \oplus c_1) \pm (0 \oplus ic_2), \quad (16)$$

<sup>3</sup> For example, for the rotation group,  $J(\mathbf{n}) = n_i J_i$ , also denoted  $\mathbf{J} \cdot \mathbf{n}$ , is the angular momentum operator in the direction  $\mathbf{n}$  when  $\mathbf{n}^2 = 1$  but it has no name for  $\mathbf{n}^2 = (\mathbf{n} \cdot \mathbf{n}) \neq 1$ .

and  $q, c_1, c_2$  are normalized real vectors of the octet; thus  $\tilde{q}, \tilde{c}_1, \tilde{c}_2$  are normalized real vectors of  $\mathcal{E}_{16}$ .

(ii) The representation of the  $SU(3) \times SU(3)$  Lie algebra on  $\mathcal{H}$  is given at each time  $t$  by the space integral of the time component of the tensor operator  $h^\mu(x)$ :

$$F(\tilde{a}) = \int d^3\mathbf{x} h^0(\mathbf{x}, t; \tilde{a}). \quad (17)$$

The operators  $F(a)$  would be time-independent if for all  $\tilde{a}$ 's

$$\partial_\mu h^\mu(x; \tilde{a}) = 0. \quad (18)$$

The operator

$$Q_H = -\sqrt{\frac{2}{3}} \int d^3\mathbf{x} h^0(x; \tilde{q}) = \int d^3\mathbf{x} j_{em}^0(x) = -\sqrt{\frac{2}{3}} F(\tilde{q}) \quad (19)$$

represents the hadronic electric charge (in units of the proton charge).  $Q_H$  is conserved in all but the weak semileptonic transitions.

The hypercharge operator  $Y$  is of the same form

$$Y = \sqrt{\frac{2}{3}} F(\tilde{y}), \quad (20)$$

where

$$\tilde{y} = \frac{1}{\sqrt{2}} (y \oplus y), \quad (21)$$

and  $y$  and  $q$  are two normalized octet vectors on the same  $SU(3)$  orbit.

(iii) The component  $H_{em} = \int d^3\mathbf{x} j_{em}^\mu(x) A_\mu(x)$  of the total energy  $H$  which arises from the interaction of the hadrons with the electromagnetic field is the value at  $\tilde{q}$  of a  $(8,1) \oplus (1,8)$  tensor operator.

Similarly, the component

$$H_w = \frac{G}{\sqrt{2}} \sum_{\epsilon=\pm} \int d^3\mathbf{x} h_\mu(x; \tilde{c}_\epsilon) l_{(-\epsilon)}^\mu(x), \quad (22)$$

which arises from the interaction with the charged leptonic currents is the value of the direct sum of two  $(8,1) \oplus (1,8)$ -tensor operators taken, respectively, at  $\tilde{c}_+$  and  $\tilde{c}_-$ .

Assumptions (i), (ii), and (iii) appear to be well verified by experiment. Somewhat more uncertain and still in need of precise verification are the following assumptions:

(iv) The component  $H_S$  of the total energy operator which arises from the strong interactions is the sum of two terms  $H_S = H_0 + H(\mathbf{m})$ , where the first term is an invariant operator and the second is the value at  $\mathbf{m} \in \mathcal{E}_{18}$  of a  $(3, \bar{3}) \oplus (\bar{3}, 3)$ -tensor operator.

(v) The nonleptonic Hamiltonian  $H_{NL}$  when expanded in terms of irreducible  $SU(3) \times SU(3)$  tensor operators contains predominantly a  $(8, 1) \oplus (1, 8)$  component in a direction  $\tilde{z} = 0 \oplus z$ . This assumption is the simplest generalization of the assumption that  $H_{NL}$  transforms like an octet for the diagonal  $SU(3)$  [3]. Actually this, in turn, is the simplest explanation, in the frame of  $SU(3)$  invariance, of the experimentally observed selection rules for isospin and hypercharge:  $\Delta T = 1$  or  $0$ ,  $\Delta Y = 0$  and  $\Delta T = \frac{1}{2}$ ,  $\Delta Y = 1$ .

With these assumptions, the total hadronic Hamiltonian can be written as

$$H = H_0 + H_S(\mathbf{m}) + H_{em}(\tilde{q}) + \sum_{i=1,2} H_W(\tilde{c}_i) + H_{NL}(\tilde{z}). \tag{23}$$

$H-H_0$  is thus the value of a reducible tensor operator for the vector

$$\hat{r} = \mathbf{m} \oplus \tilde{q} \oplus \tilde{c}_1 \oplus \tilde{c}_2 \oplus \tilde{z} \in \mathcal{E} = \mathcal{E}_{18} \oplus \mathcal{E}_{16} \oplus \mathcal{E}_{16} \oplus \mathcal{E}_{16} \oplus \mathcal{E}_{16} \tag{24}$$

(see Appendix 1 for the direct sum of irreducible tensor operators). If also  $H_S(\mathbf{m})$  is, like the other terms, the integral over space of a scalar (under the Lorentz group) energy density, we can write  $H$  in the form

$$H = H_0 + \int d^3x \mathfrak{H}(x; \hat{r}), \tag{25}$$

where only the second term in the right hand side contributes to the time dependence of the operators  $F(\hat{a})$ . Equation (18) then becomes

$$\partial_\mu h^\mu(x; \hat{a}) = \mathfrak{H}(L(\hat{a}) \hat{r}), \tag{26}$$

where  $\hat{r} \in \mathcal{E}$  is defined in (24) and  $L$  is the representation of the Lie algebra on this space. Equation (26) is a generalization of the equations considered by Veltman [4].

### III. G-INVARIANT ALGEBRAS

Let  $\mathcal{E}$  be the space of the finite-dimensional representation  $D$  of  $G$ . The tensor product  $D \otimes D$  is the representation of  $G$  on  $\mathcal{E} \otimes \mathcal{E}$ . An algebra  $\mathcal{A}$  on  $\mathcal{E}$  is a homomorphism

$$\mathcal{E} \otimes \mathcal{E} \xrightarrow{\alpha} \mathcal{E}. \tag{27}$$

If  $\mathcal{O}$  is a  $G$ -equivariant mapping,  $G$  is an automorphism group of the algebra  $\mathcal{O}$ . We will call such an algebra a  $G$ -invariant algebra. If  $\nu(\mathcal{E} \otimes \mathcal{E}, \mathcal{E}) = 1$ , this algebra is uniquely defined up to a dilation ( $\mathbf{x} \rightarrow \lambda \mathbf{x}$ ). The decomposition of the tensor product  $\mathcal{E} \otimes \mathcal{E}$  into symmetrical and antisymmetrical components

$$\mathcal{E} \otimes \mathcal{E} = (\mathcal{E} \overset{S}{\otimes} \mathcal{E}) + (\mathcal{E} \overset{A}{\otimes} \mathcal{E}) \tag{28}$$

is compatible with the group action.

Let  $G$  be  $SU(n)$  or  $U(n)$  or the direct product of several such groups. If  $\mathcal{E}$  is the space of the adjoint representation,  $\nu(\mathcal{E} \overset{A}{\oplus} \mathcal{E}, \mathcal{E}) = 1$  and the corresponding  $G$ -invariant algebra is the Lie algebra. For  $n > 2$ ,  $\nu(\mathcal{E} \overset{S}{\otimes} \mathcal{E}, \mathcal{E}) = 1$ . The corresponding  $G$ -invariant algebra will be called the symmetric algebra. For  $SU(3)$  we denote its law by  $\vee$ .

We use the same symbol  $\vee$  for the law of the  $SU(3) \times SU(3)$  invariant symmetric algebra on the space of the adjoint representation

$$a \vee b = (a_{+\vee} b_{+}) \oplus (a_{-\vee} b_{-}) \tag{29}$$

For the space  $\mathcal{E}_{18}$  of the representation  $(3, \bar{3}) \oplus (\bar{3}, 3)$ , we have

$$\nu(\mathcal{E}_{18} \otimes \mathcal{E}_{18}, \mathcal{E}_{18}) = \nu(\mathcal{E}_{18} \overset{S}{\otimes} \mathcal{E}_{18}, \mathcal{E}_{18}) = 1. \tag{30}$$

This defines uniquely a  $SU(3) \times SU(3)$ -invariant symmetric algebra on  $\mathcal{E}_{18}$  whose law will be denoted by  $\top$ . In Appendix 2 we give an explicit realization of these algebras.

The hadronic physical quantities are at each time tensor operators for  $SU(3) \times SU(3)$ . The invariance of hadronic physics under this group is expressed by the covariance of the equations satisfied by these tensors. A covariant relation between the values of tensor operators implies a geometrical equation for their arguments. If the arguments of these tensor operators belong to the same representation space  $\mathcal{E}$ , the covariant equations are expressed in terms of the invariant algebra laws on  $\mathcal{E}$ . The simplest equation for one variable on  $\mathcal{E}_{16}$  or  $\mathcal{E}_{18}$  are, respectively, of the type

$$\tilde{x} \vee \tilde{x} - \lambda \tilde{x} = 0, \tag{31}$$

$$\mathbf{x} \top \mathbf{x} - \lambda \mathbf{x} = 0, \tag{32}$$

where  $\lambda$  is an invariant which can be a function of the corresponding vector of  $x$ . Due to the unicity of these symmetric algebras all covariant quadratic equations among  $\mathcal{E}_{16}$  or  $\mathcal{E}_{18}$  tensor operators depending on the same argument, give rise to

TABLE 1<sup>a</sup>

Group	Representation space	Equation	Real or Complex	Little Group of the Orbit of Solutions	Physical Examples of Solutions
$SU(3)$	octet $\mathcal{e}_8$ (adjoint representation space)	$x_\nu x = \lambda x$	real $\lambda \neq 0$ comp $\lambda = 0$	$U_2(2)$ $W$	$y, q, z (\lambda = 1)$ $c_+, c_-$
$SU(3) \times SU(3)$	$\mathcal{e}_{16}$ (adjoint representation space)	$\tilde{x}_\nu \tilde{x} = \lambda \tilde{x}$	real $\lambda \neq 0$ real $\lambda \neq 0$ comp $\lambda = 0$ comp $\lambda = 0$	$U^d(2)$ $SU(3) \times U(2)$ $SL(3, C) \times W$ $W^d$	$\tilde{q}, \tilde{y}$ $\tilde{z}$ $\tilde{c}_+, \tilde{c}_-$
$SU(3) \times SU(3)$	$\mathcal{e}_{18}$ $(\bar{3}, \bar{3}) \oplus (\bar{3}, 3)$ representation space	$\mathbf{x}_\tau \mathbf{x} = \lambda \mathbf{x}$	real $\lambda \neq 0$ real $\lambda = 0$	$SU^d(3)$ $SU(2) \times SU(2) \times U^d(1)$	$\mathbf{n}$ $\mathbf{y}$

<sup>a</sup> The trivial solution  $x = 0$  is not indicated in the table. A solution for the real algebra is also a solution for the complex algebra; then  $\lambda$  can be complex and the little group is the complexified of a real group. The converse is not true and "complex  $\lambda$ " in the fourth column indicates solutions for the complex which do not correspond to solutions of the real algebra. The group  $W$  is a four-dimensional complex solvable Lie group that we do not explicit here.



Eq. (31) or (32). The covariance of these equations implies that when a vector is a solution, all the vectors on the same  $SU(3) \times SU(3)$  orbit<sup>4</sup> are solutions.

It is a remarkable fact that all the vectors of  $\mathcal{E}_{16}$  which we have introduced in (i), (ii), (iii), (v), are solutions of Eq. (31).

In Table I we list all the solutions of Eq. (31), of Eq. (32) and of the corresponding equation on  $\mathcal{E}_8$  for the two cases of real and complex algebras. We remark that  $\mathcal{E}_8$ ,  $\mathcal{E}_{16}$ ,  $\mathcal{E}_{18}$  are spaces of real representations, so the corresponding  $G$ -invariant algebras are real. However, these spaces and their algebras can be complexified; solutions of Eqs. (31) and (32) for the real algebra are also solutions for the complex algebra, but the converse is not necessarily true.

Equation (32) has two types of solutions (for either the real or complex algebra). We give explicitly in Appendix 2 one example for each type of solutions satisfying

$$\mathbf{n} \cdot \mathbf{n} = \sqrt{\frac{3}{2}} \mathbf{n}; \quad \mathbf{y} \cdot \mathbf{y} = 0. \tag{33}$$

These vectors would be good candidates for  $\mathbf{m}$ , the direction of  $H_S(\mathbf{m})$  (see iv), if  $SU^d(3)$  or  $SU_y(2) \times SU_y(2) \times U_y(1)^d$  were exact symmetries of the strong interaction Hamiltonian (the second case corresponds to vanishing  $\pi$ -meson mass). According to Gell-Mann, Oakes and Renner [5],  $\mathbf{m}$  is a linear combination of the two vectors (33):

$$\mathbf{m} = \mathbf{y} \left( \cos \omega - \frac{1}{\sqrt{2}} \sin \omega \right) + \sqrt{\frac{3}{2}} \mathbf{n} \sin \omega \tag{34}$$

with

$$\text{tg } \omega = \frac{\sqrt{2} - 1.25}{1 + (1.25) \sqrt{2}} = .058 \tag{35}$$

In this model, therefore,  $\mathbf{m}$  is much closer to  $\mathbf{n}$  than to  $\mathbf{y}$ .

A number of other  $G$ -invariant relations exist among the physical directions of the octet and  $\mathcal{E}_{16}$  spaces.

In the octet space, the vectors that we have defined so far are:

- $y$ , the direction of the hypercharge, Eq. (21)
- $q$ , the direction of the electric charge, Eq. (15)
- $z$ , defined in (v), that we can call as the direction of the weak hypercharge
- $c_{\pm}$ , the directions of the weak currents, Eq. (16).

<sup>4</sup> The orbit  $G(m)$  of a point  $m$  of a space is the set of all the transformed of  $m$  by the group  $G$ . The little group  $G_m$  of  $m$  is the set of transformations  $g \in G$  which leave  $m$  invariant. All points of the same orbit have conjugated little groups i.e.  $G_{gm} = gG_m g^{-1}$ . The nature of an orbit is completely characterized by the little group of one of its points.

They satisfy the following equations:

$$y_{\vee} y + y = 0; \quad q_{\vee} q + q = 0; \quad z_{\vee} z + z = 0; \quad (36)$$

$$c_{+\vee} c_+ = 0 = c_{-\vee} c_-; \quad (37)$$

and the relations

$$z = 2c_{+\vee} c_- = c_{1\vee} c_1 = c_{2\vee} c_2 = c_{3\vee} c_3, \quad (38)$$

where

$$c_{\pm} = \frac{1}{2}(c_1 \pm ic_2), \quad c_3 = c_{1\wedge} c_2, \quad (39)$$

and

$$q_{\wedge} c_{\pm} = \pm i \frac{\sqrt{3}}{2} c_{\pm}. \quad (40)$$

The Cabibbo angle  $\theta$  is defined by

$$(y, z) = 1 - \frac{3}{2} \sin^2 \theta. \quad (41)$$

It is remarkable that when  $\sin \theta \neq 0$ , the direction of the electric charge  $q$  is uniquely given by the hypercharge and the weak currents. Indeed,

$$2(1 - (y, z)) q + 2y_{\vee} z + y + z = 0. \quad (42)$$

The fact that  $y$  and  $q$  or  $q$  and  $z$  are directions of commuting charges:

$$y_{\wedge} q = 0, \quad q_{\wedge} z = 0 \quad (43)$$

is equivalent to

$$(y, q) = -\frac{1}{2} = (q, z) \quad (44)$$

or

$$y_{\vee} q = q + y; \quad q_{\vee} z = z + q \quad (45)$$

On  $\mathcal{E}_{16}$ , Eqs. (36), (37), (38), (39), (43), are still valid when one replaces  $y, q, z, c_+, c_-$  by  $\tilde{y}, \tilde{q}, \tilde{z}, \tilde{c}_+, \tilde{c}_-$ . Besides, one has

$$(\tilde{y}, \tilde{q}) = -\frac{1}{2}; \quad (\tilde{q}, \tilde{z}) = -\frac{1}{2\sqrt{2}}, \quad (\tilde{y}, \tilde{z}) = \frac{1}{\sqrt{2}} \left(1 - \frac{3}{2} \sin^2 \theta\right), \quad (46)$$

and

$$\tilde{q}_{\wedge} \tilde{c}_{\pm} = \pm \frac{i}{2} \sqrt{\frac{3}{2}} \tilde{c}_{\pm}. \quad (47)$$

IV. EXTENSION BY THE DISCRETE SYMMETRIES  $C, P, CP$ 

Strong and electromagnetic interactions are invariant under  $C$ , the charge conjugation which exchanges particle and antiparticle. Thus  $C$  should be included in the internal symmetry group  $G$ , and since  $C^2 = I$ , charge conjugation generates the two element subgroup  $Z_2(C)$  of  $G$ .

An interesting feature of chiral internal symmetry, i.e.,  $SU(3) \times SU(3)$  or  $SU(2) \times SU(2)$ , is its connection with relativistic invariance through the space symmetry  $P$  which is both an outer automorphism of the connected Poincaré group and of  $SU(3) \times SU(3)$ . Its action on the latter is to exchange the two  $SU(3)$  factors.

By definition,  $C$  commutes with every Poincaré transformation including  $P$ :

$$CP = PC \quad (48)$$

and

$$C^2 = P^2 = I = (CP)^2. \quad (49)$$

Thus  $C, P$  generate the group  $Z_2 \times Z_2$  whose elements induce, on  $SU(3) \times SU(3)$ , the following outer automorphisms:

$$(u_+, u_-) \xrightarrow{C} (\bar{u}_-, \bar{u}_+), \quad (50)$$

$$(u_+, u_-) \xrightarrow{P} (u_-, u_+), \quad (51)$$

$$(u_+, u_-) \xrightarrow{CP} (\bar{u}_+, \bar{u}_-), \quad (52)$$

where  $(u_+, u_-)$  is an element of  $SU(3) \times SU(3)$ .

From the theory of group extensions [6], it follows that there is a unique extension of  $SU(3) \times SU(3)$  which satisfies (48) and (49), namely, the semidirect product

$$G = (SU(3) \times SU(3)) \square (Z_2(P) \times Z_2(C)). \quad (53)$$

Similarly, it can be proved that the only extension of  $SU(3) \times SU(3)$  by  $Z_2(X)$ , where  $X = C, P, CP$ , is the semidirect product

$$G^{(X)} = (SU(3) \times SU(3)) \square Z_2(X). \quad (54)$$

Let  $g \rightsquigarrow D(g)$  be the representation of a group  $G_0$ , and  $X$  an automorphism of  $G_0$ ; then  $g \rightsquigarrow D(X(g))$  is a representation of  $G_0$  that we denote as  $D^X$ . When  $G_0 = SU(3) \times SU(3)$ , and  $X$  is  $C, P, CP$ , we find

$$\begin{aligned} (m, n)^C &= (\bar{n}, \bar{m}), \\ (m, n)^P &= (n, m), \\ (m, n)^{CP} &= (\bar{m}, \bar{n}). \end{aligned} \quad (55)$$

Thus, the representations  $(1, 8) \oplus (8, 1)$  and  $(3 \oplus \bar{3}) \oplus (\bar{3}, 3)$  are invariant (up to an equivalence) by  $C, P, CP$ .

The method to build the representations of an extended group  $G$  from those of  $G_0$  is well-known [7]. When we apply it to  $G_0$  equal to  $SU(3)$  or  $SU(3) \times SU(3)$ , we find, using (55) for the latter group,

(i) The octet space can carry four inequivalent representations of the group  $SU(3) \square (Z_2(P) \times Z_2(C))$  which we denote by

$${}^c(8)^p \quad \text{with} \quad c^2 = p^2 = 1. \tag{56}$$

(ii) Each space  $\mathcal{E}_{16}$  and  $\mathcal{E}_{18}$  carries two representations of the full group  $G$  [Eq. (53)]. We denote them by

$$((1, 8) \oplus (8, 1))^\epsilon, \quad \epsilon^2 = 1, \quad ((3, \bar{3}) \oplus (\bar{3}, 3))^\eta, \quad \eta^2 = 1. \tag{57}$$

Under  $C$ , the vector currents are odd and the axial vector currents are even. The time component of a vector is invariant under  $P$  while that of an axial vector changes sign. From this we deduce that the  $F(\vec{a})$ 's of Eq. (17) are odd under  $CP$  and that  $F(a \oplus a)$  is the value of a  $-8^+$  tensor operator for  $SU(3) \square (Z_2(P) \times Z_2(C))$ ,  $F(a \oplus -a)$  is the value of a  $+8^-$  tensor operator for  $SU(3) \square (Z_2(P) \times Z_2(C))$ , and the  $F(\vec{a})$ 's are the values of a  $((1, 8) \oplus (8, 1))^-$  tensor operator for  $(SU(3) \times SU(3)) \square (Z_2(P) \times Z_2(C))$ . Since strong interactions preserve  $P, C, PC$ , with Eq. (7) we find, for the direction  $\mathbf{m}$  of the strong Hamiltonian  $H_S(\mathbf{m})$ ,

$$D(P) \mathbf{m} = \mathbf{m}, \quad D(C) \mathbf{m} = \mathbf{m}. \tag{58}$$

This requires that  $D$  be the  $((3, \bar{3}) \oplus (\bar{3}, 3))^+$  representation.

We remark that neither the group nor the irreducible representations (up to an equivalence) are changed if we conjugate the automorphism  $P$ , Eq. (51), of  $G_0$  by an inner automorphism. Physically, this means the possibility to “redefine” parity in the internal symmetry space (see Ref. [1b]).

### V. THE ACTION OF THE INTERNAL SYMMETRY GROUP ON THE UNIT SPHERES OF THEIR REPRESENTATION SPACES

In Section III, we have seen that the physical directions are the solutions of simple algebraic equations covariant under the internal symmetry group  $SU(3) \times SU(3)$ . We will show in this section that for the physical models which blend a variational principle with the invariance under a compact group  $G$ , these directions are the positions of extrema which depend only on the geometry of

the group action and are independent of the function which is varied, i.e., they are independent of the details of the dynamical model.<sup>5</sup>

We begin by recalling some properties of the differentiable action of a compact Lie group  $G$  on an infinitely differentiable manifold  $M$ . This action is given by a smooth (=infinitely differentiable) mapping

$$G \times M \xrightarrow{\Phi} M$$

satisfying

$$\Phi(g_1, \Phi(g_2, m)) = \Phi(g_1 g_2, m). \quad (59)$$

We will often denote  $\Phi(g, m)$  simply by  $g.m$ . We call  $M$ , with the action  $\Phi$  of  $G$ , a  $G$ -manifold and we denote it by  $(G, M, \Phi)$ .

An equivariant smooth map  $(G, M, \Phi) \xrightarrow{\varphi} (G, M', \Phi')$  is a smooth map  $M \xrightarrow{\varphi} M'$  commuting with the action of  $G$ :

$$\forall g \in G, \quad \forall m \in M, \quad \varphi(\Phi(g, m)) = \Phi'(g, \varphi(m)). \quad (60)$$

If  $\varphi$  is bijective (i.e., one to one, onto) and if  $\varphi$  and  $\varphi^{-1}$  are smooth, then the two  $G$ -manifolds  $M$  and  $M'$  are said to be equivalent.

The orbits of  $G$  on  $M$  are closed smooth submanifolds. Two orbits are isomorphic  $G$ -manifolds iff (=if and only if) the little groups of their points are all conjugated.

We call a stratum the set of all points of  $M$  with conjugated little groups and we denote by  $S(m)$  the stratum of  $m$ . The stratum  $S(m)$  is the union of all orbits isomorphic to  $G(m)$ ; it is a submanifold of  $M$ .

Consider a real smooth function  $f$  defined on  $M$  and invariant by  $G : f(g \cdot m) = f(m)$ , i.e., constant on the orbits of  $G$ . Let  $df$  be its differential. We have conjectured the following theorem (which has been proved by one of us [8]).

**THEOREM 1.** *Let  $G$  be a compact Lie group acting smoothly on the real manifold  $M$  and let  $m \in M$ . The two properties (a), (b) are equivalent.*

(a) *The orbit  $G(m)$  is critical, i.e., the differential  $df_{m'}$  of every smooth real  $G$ -invariant function  $f$  on  $M$  vanishes for  $m' \in G(m)$ .*

(b) *The orbit  $G(m)$  is isolated in its stratum, i.e., there exists a neighborhood  $V_m$  of  $m$  such that if  $p \notin G(m)$ ,  $p \in V_m$ , then  $G_p$  is not conjugated to  $G_m$ .<sup>6</sup>*

Other useful mathematical properties of  $G$ -manifolds are given in Appendix 3.

<sup>5</sup> References for such models are given in our previous papers, ref. 1.

<sup>6</sup> This theorem is a generalization of the remark that for every smooth even function on the real:  $f(x) = f(-x)$ ,  $df/dx(0) = 0$ .

Let us call  $\{G_m\}$  the set of little groups, up to a conjugation, which appear in  $(G, M, \Phi)$ . It is known that if  $G_m$  is maximal in  $\{G_m\}$ , then  $S(m)$  is closed. We also proved [8]

**THEOREM 2.** *If  $S(m)$  is closed, either  $S(m)$  has one orbit (which is then isolated in  $S(m)$ ) or, for every invariant smooth real function  $f$  on  $M$ ,  $df$  vanishes on at least two orbits of  $S(m)$ .*

Thus there are always extrema for “maximal” little groups,<sup>7</sup> but the positions of these extrema may depend on the function.

In the physical applications that we study here,  $G$  is either  $SU(3)$  or  $SU(3) \times SU(3)$  or one of their extensions by  $C, P, CP$ ;  $\Phi$  is the linear action on  $\mathcal{E}_8$  or  $\mathcal{E}_{16}$  or  $\mathcal{E}_{18}$ , and  $M$  is the sphere of the unit vectors of these real spaces. Let  $x, y$  be points of one of these spaces and  $x \vee y$  be the product for the  $G$ -invariant symmetric real algebra introduced in Section III. Let  $f$  be a smooth function on the real, and  $f'$  be its derivative; then  $f(x, x \vee x)$  is a  $G$ -invariant function on the unit sphere,  $(x, x) - 1 = 0$  of  $\mathcal{E}$ . Using the Lagrange multiplier  $\lambda$ , we find that the extrema of  $f$  on the sphere are given by

$$\text{grad}(f + \lambda(1 - (x, x))) = 3x \vee x f'(x) - 2\lambda x = 0. \tag{61}$$

i.e.,  $x$  is an idempotent or nilpotent of the symmetric algebra (if  $\lambda$  or  $f'(x) \neq 0$ ).

It is convenient to introduce the orbit space  $M/G$  which is the set of orbits of  $(G, M, \Phi)$  with the quotient topology. For instance, one orbit  $G(m)$  isolated in its stratum is in  $M/G$ , a point which forms a connected component of  $\varphi(S(m))$ , the image of the strata of  $m$ . Also  $S(m)$  is closed iff  $\varphi(S(m))$  is closed.

In all the examples that we shall study,  $M$  is compact, and therefore the number of stratum is finite. Moreover, there is a stratum which is open and dense which we will call the generic stratum.

**EXAMPLE 1.**  $G = SU(3)$ ,  $M = S_7$ , the unit sphere of the octet space  $\mathcal{E}_8$ . The orbit space is the set of values of the invariant  $(x \vee x, x)$ . It is the closed interval  $A'A$  of Fig. 1:

$$-1 \leq (x \vee x, x) \leq 1. \tag{62}$$

There are two strata. The generic one is represented by the interior  $\mathcal{I}(A'A)$  of  $A'A$ . The other stratum contains two orbits  $A$  and  $A'$  which are therefore critical.

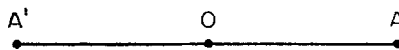


FIG. 1. Orbit space of  $S_7/SU(3)$ .

<sup>7</sup> A trivial example, when it exists, is  $G_m = G$ , the little group of the (closed) stratum of fixed points.

**EXAMPLE 2.**  $G = (SU(3) \square Z_2(C)) \times Z_2(P)$ ;  $M = S_7$  the unit sphere of the  ${}^{\circ}(8)^p$  representation with  $c + p + cp = -1$  (i.e.,  $+(8)^+$  is excluded).

This example applies to both octets of vector and axial vector currents.  $P$  changes  $x$  into  $px$ ; so it changes  $(x_{\vee} x, x)$  into  $p(x_{\vee} x, x)$ , and  $C$  changes  $(x_{\vee} x, x)$  into  $c(x_{\vee} x, x)$ . So  $M/G$  is the interval  $0A$  (see Fig. 1):

$$0 \leq (x_{\vee} x, x) \leq 1, \tag{63}$$

and there are three strata. The generic one is  $\mathcal{S}(0A)$ , and the two others have each one orbit which therefore is critical. These two orbits are:

- $A$ , whose little group is  $U(2) \times Z_2(P)$ ,
- $0$ , with little group  $(U_1 \times U_1 \times Z_2(P)) \square Z_2(C)$ .

The physical vectors  $y, q, z$  and  $c_1, c_2$  belong, respectively, to the orbits  $A$  and  $0$ . This example is summarized in Table II.

**EXAMPLE 3.**  $G = SU(3) \times SU(3)$ ;  $M$  is  $S_{15}$ , the unit sphere of the  $(1, 8) \oplus (8, 1)$  (= adjoint) representation space  $\mathcal{E}_{16}$ . The points of  $S_{15}$  are  $\tilde{x} = x_+ \oplus x_-$  such that  $(\tilde{x}, \tilde{x}) = (x_+, x_+) + (x_-, x_-) = 1$ . A complete set of linearly independent invariants is:

$$\alpha = (x_+, x_+), \quad \theta_+ = (x_{+\vee} x_+, x_+), \quad \theta_- = (x_{-\vee} x_-, x_-). \tag{64}$$

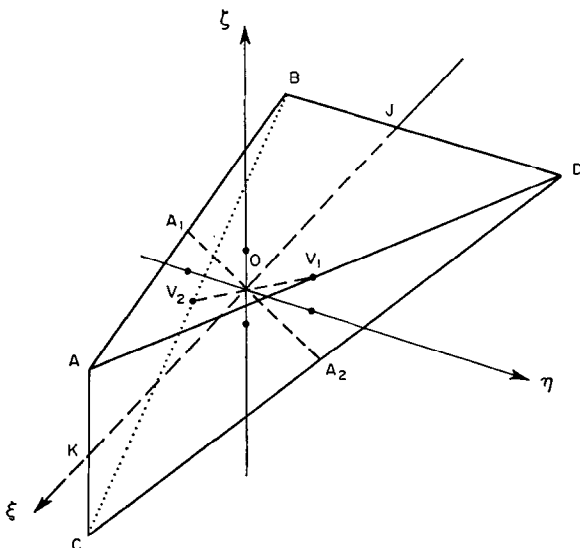


FIG. 2. Orbit space of  $S_{15}/(SU(3) \times SU(3))$ .

Strata on

Group	Manifold	Stratum $S$	$NCC_\varphi S$	$NS$	
$SU(3) \square (Z_2(C) \times Z_2(P))$	$S_7$ in the representation space of $-(8)^+$	$\mathcal{I}(0A)$	1	$g$	
		0	1	$M$	
		$A$	1	$M$	
			FIG. 1		
			Fig. 2 Interior axes/4	1	$g$
			Interior of faces/4	1	
$(SU_3 \times SU(3)) \square (Z_2(P) \times Z_2(C))$	$S_{15}$ in the representation space of $((1, 8) \oplus (8, 1))^-$	$(\mathcal{I}(AV_1) \cup \mathcal{I}(V_1D) + 3 \text{ symmetric})/4$	2		
		$(\mathcal{I}(AK) + 3 \text{ symmetric})/4$	1		
		$\mathcal{I}(0V_1) \cup \mathcal{I}(0V_2)/2$	1		
		$\mathcal{I}(0A_1) \cup \mathcal{I}(0A_2)/2$	1		
		$\mathcal{I}(0J) \cup \mathcal{I}(0K)/2$	1		
		$A+B+C+D/4$	1	$M$	
			$V_1+V_2/2$	1	$M$
			$A_1+A_2/2$	1	$M$
			$K+J/2$	1	$M$
			0	1	$M$
			FIG. 2		
	$(SU(3) \times SU(3)) \square (Z_2(P) \times Z_2(C))$	$S_{17}$ in the representation space of $((3, \bar{3}) \oplus (\bar{3}, 3))^+$	Fig. 3 Interior/2	1	$g$
$(\text{Surface-}ABB')/2$			1		
$\mathcal{I}(ABB')$			1		
		$\mathcal{I}(AB) \cup \mathcal{I}(BB') \cup \mathcal{I}(AB')$	1		
		$\Gamma-B-B'/2$	1		
		$B+B'$	2	$M$	
		$A$	1	$M$	
		FIG. 3			

This table resumes the information obtained on the examples which correspond to the full physical of the image  $\varphi(S)$  of the stratum in the orbit space;  $NS$  = nature of stratum;  $g$  = generic;  $M$  = closed  $\dim \varphi(S)$  = dimension of the image of the stratum in the orbit space;  $\dim \theta$  = dimension of orbit =

In a rectangular coordinate system

$$\xi = 1 - 2\alpha, \quad \eta = \theta_+/\sqrt{\alpha}, \quad \zeta = \theta_-/\sqrt{1-\alpha}, \quad (65)$$

the orbit space is the tetrahedron  $ABCD$  (of Fig. 2) whose vertex coordinates are

$$A(1, 0, 1), \quad C(1, 0, -1), \quad B(-1, -1, 0), \quad D(-1, 1, 0). \quad (66)$$

There are 8 strata: the interior  $\mathcal{I}(ABCD)$  is the generic stratum; two other strata are the interior of the pair of opposite faces; the interior of the edges form three strata,

$$\mathcal{I}(AB) \cup \mathcal{I}(AD) \cup \mathcal{I}(CB) \cup \mathcal{I}(CD), \mathcal{I}(AC), \mathcal{I}(BD); \quad (67)$$



TABLE II

$S_7, S_{15}, S_{17}$

Little Group	dim $S$	dim $\varphi(S)$	dim $\emptyset$	$c$	Physical Example
$U(1) \times U(1) \times Z_2(P)$	7	1	6		
$((U(1) \times U(1) \times Z_2(P)) \square Z_2(C))$	6	0	6	$c$	$c_1, c_2$
$U(2) \times Z_2(P)$	4	0	4	$c$	$q, y, z$
$(U(1) \times U(1) \times U(1) \times U(1) = A)$	15	3	12		
$U(1) \times U(1) \times U(2)$	12	2	10		
$U(2) \times U(2)$	9	1	8		
$SU(3) \times U(1) \times U(1)$	5	1	4		
$A \square Z_2(P)$	13	1	12		
$A \square Z_2(C)$	13	1	12		
$A \square Z_2(CP)$	13	1	12		
$SU(3) \times U(2)$	4	0	4	$c$	$\tilde{x}$
$(U(2) \times U(2)) \square Z_2(P)$	8	0	8	$c$	$\tilde{q}, \tilde{y}$
$(U(2) \times U(2)) \square Z_2(C)$	8	0	8	$c$	?
$(SU(3) \times U(1) \times U(1)) \square Z_2(CP)$	6	0	6	$c$	$\tilde{c}_1, \tilde{c}_2$
$A \square (Z_2(P) \times Z_2(C))$	12	0	12	$c$	?
$(U(1) \times U(1))^d \square Z_2(CP)$	17	3	14		
$U(2)^d \square Z_2(CP)$	14	2	12		
$(U(1) \times U(1))^d \square (Z_2(P) \times Z_2(C))$	16	2	14		
$U(2)^d \square (Z_2(P) \times Z_2(C))$	13	1	12		
$SU(3)^d \square Z_2(CP)$	9	1	8		
$SU(3)^d \square (Z_2(P) \times Z_2(C))$	8	0	8	$c$	$\pm \mathbf{m}(\omega = \tan^{-1} \sqrt{2}) = \mathbf{n}$
$(SU(2) \times SU(2) \times U(1))^d \square (Z_2(P) \times Z_2(C))$	9	0	9	$c$	$\mathbf{m}(\omega = 0) = \mathbf{y}$

situation (examples 2, 4, 6). Abbreviations for the columns:  $NCC\varphi S$  = number of connected components stratum because the little group is maximal in the set of little groups;  $\dim S$  = dimension of stratum; dimension of the group minus that of the little group =  $\dim S - \dim \varphi(S)$ ;  $c$  = critical orbit.

the vertices form two strata  $\{A, C\}$  and  $\{B, D\}$  each containing 2 orbits which are the only critical ones.

EXAMPLE 4.  $G = (SU(3) \times SU(3)) \square (Z_2(P) \times Z_2(C))$ ,  $M = S_{15}$  the unit sphere of  $\mathcal{E}_{16}$ , the space of the representation  $((1, 8) \oplus (8, 1))^-$  which is that of the hadronic currents. Figure 2 still represents the orbit space when the points representing the same orbits of the complete group are identified.

Indeed,  $P$  acts on the orbit space  $ABCD$  of  $SU(3) \times SU(3)$  as a rotation by  $\pi$  around the axis  $V_1V_2$  ( $V_1$  is the middle point of  $AD$ ,  $V_2$  the middle point of  $BC$ );  $C$  acts as a rotation of  $\pi$  around the axis  $A_1, A_2$  ( $A_1$  is the middle point of  $AB$ ,  $A_2$  the middle point of  $CD$ ) and  $CP$  acts as a rotation by  $\pi$  around the axis  $JK$

( $K$  is the middle point of  $AC$ ,  $J$  the middle point of  $BD$ ). Therefore, for the orbit space we are studying, we have to identify the points of Fig. 2 which are symmetric with respect to the axes  $A_1A_2$ ,  $V_1V_2$ ,  $JK$ . In the general case, four orbits of  $SU(3) \times SU(3)$  form one orbit of  $G$ , the extended group, and the little group is unchanged. Except for the point 0, the representatives of orbits of  $SU(3) \times SU(3)$  on the symmetry axes  $V_1V_2$  or  $A_1A_2$  or  $JK$  have only one distinct symmetric point with respect to these 3 orthogonal axes and this point is the symmetric with respect to 0. These two symmetrical points form one orbit of the extended group  $G$  and the little group is doubled (i.e., extended by a  $Z_2$ ).

Table III gives the action of  $P$ ,  $C$ ,  $PC$  on the points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $V_1$ ,  $V_2$ ,  $A_1$ ,  $A_2$ ,  $I$ ,  $J$ . From it one deduces immediately the action on the edges and faces.

TABLE III  
Action of the Group  $\{I, P, C, PC\}$  on the Points of Fig. 2

Group element	Points of the tetrahedron $ABCD$										
$I$	$A$	$B$	$C$	$D$	$V_1$	$V_2$	$A_1$	$A_2$	$J$	$K$	
$P$	$D$	$C$	$B$	$A$	$V_1$	$V_2$	$A_2$	$A_1$	$K$	$J$	
$C$	$B$	$A$	$D$	$C$	$V_2$	$V_1$	$A_1$	$A_2$	$K$	$J$	
$PC$	$C$	$D$	$A$	$B$	$V_2$	$V_1$	$A_2$	$A_1$	$J$	$K$	

There are twelve strata: The generic stratum is the interior of  $ABCD$  minus the axis  $V_1V_2$ ,  $A_1A_2$ ,  $JK$ , and each orbit is represented by 4 points; the interior of the faces form one stratum (4 points, one on each face, represent one orbit); the interior of the edges form two strata; the interior of each axis forms a different stratum (indeed  $Z_2(P)$ ,  $Z_2(C)$ ,  $Z_2(CP)$  are not conjugated in  $G$ ); finally, there are five strata of one orbit each (which is therefore critical):

$$A + B + C + D, V_1 + V_2, A_1 + A_2, J + K, 0$$

Table II lists the strata of this example.

EXAMPLE 5.  $G$  is  $SU(3) \times SU(3)$ ,  $M$  is  $S_{17}$ , the unit sphere of the real representation  $(3, \bar{3}) \oplus (\bar{3}, 3)$ .

As we have already said in Section II, this representation is reducible on the complex but irreducible on the real; thus there is not only an invariant scalar product  $(\mathbf{x}, \mathbf{y})$  but also an antisymmetrical invariant bilinear form which is represented by the real matrix  $A = -A^T$  which we normalize to  $A^2 = -I$ . With the help of the symmetric algebra (see Section III and Appendix 2), we can write a

complete list of linearly independent invariants for the unit vectors  $((\mathbf{x}, \mathbf{x}) = 1)$  of  $S_{17}$  :

$$\rho = (\mathbf{x}_T \mathbf{x}, \mathbf{x}_T \mathbf{x}); \quad \sigma_1 = (\mathbf{x}_T \mathbf{x}, \mathbf{x}); \quad \sigma_2 = (\mathbf{x}_T \mathbf{x}, A\mathbf{x}). \quad (68)$$

We choose a system of orthogonal coordinates  $\xi, \eta, \zeta$ , defined by

$$\lambda' = \sigma_1^2 + \sigma_2^2 = \frac{9}{16} \lambda(4 - 4\lambda + \lambda^2 - \zeta^2), \quad (69)$$

$$\lambda = \xi^2 + \eta^2, \quad \zeta \geq 0, \quad (70)$$

$$\xi/\lambda = \sigma_1/\lambda', \quad \eta/\lambda = \sigma_2/\lambda', \quad (71)$$

$$\rho = \frac{1}{8}(4 + 4\lambda - 3\lambda^2 - \zeta^2). \quad (72)$$

The orbit space is defined by

$$0 \leq \zeta \leq 2, \quad (73)$$

and

$$\left(\frac{\zeta}{2} - 1\right)^2 - \frac{9}{4}(\xi^2 + \eta^2) \geq 0. \quad (74)$$

The orbit space is a part of a circular cone whose vertex is  $A(0, 0, 2)$  and whose basis is the circle  $\Gamma : \zeta = 0, \xi^2 + \eta^2 = 4/9$  (see Fig. 3).

There are four strata: The inside is the generic one (little group  $(U_1 \times U_1)^d$ ); the surface of the cone is another stratum (little group  $U(2)^d$ ); the circle  $\Gamma$  is a closed stratum (little group  $SU(3)$ ); the vertex  $A$  is a stratum made of one (critical) orbit with little group  $SU(2) \times SU(2) \times U(1)^d$ .

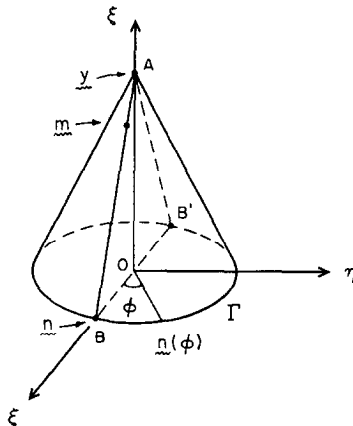


FIG. 3. Orbit space of  $S_{17}/(SU(3) \times SU(3))$ .

EXAMPLE 6.  $G$  is  $(SU(3) \times SU(3))_{\square}(Z_2(P) \times Z_2(C))$  and  $M$  is  $S_{17}$ , the unit sphere of  $\mathcal{E}_{18}$ , the space of the  $((3, \bar{3}) \oplus (\bar{3}, 3))^+$  representation.

$P$  acts on the orbit space of  $SU(3) \times SU(3)$  on  $S_{17}$  as a symmetry with respect to a plane which contains the cone axis; with a suitable definition of  $P$  (which exchanges the two  $SU(3)$  factors) this is the plane  $\eta = 0$ .  $C$  has the same action as  $P$ , so  $CP$  acts trivially on the cone.

In the general case, orbits of the enlarged groups are pairs of orbits (symmetric with respect to the plane  $ABB'$ ) of  $SU(3) \times SU(3)$  and their little group is extended by  $Z_2(CP)$ . Instead the orbits of  $SU(3) \times SU(3)$  in the plane  $ABB'$  are also orbits of the enlarged group, and the little group is extended by  $Z_2 \times Z_2$ . To the four strata of the previous example, one must add two others:  $\mathcal{I}(AB) \cup \mathcal{I}(BB') \cup \mathcal{I}(B'A)$  and  $\{B, B'\}$  a stratum which contains two (critical) orbits.

This example is also tabulated in Table II.

## VI. FINAL REMARKS

The directions of the Hermitian weak currents and of the electric charge, hypercharge and weak hypercharge  $c_1, c_2, q, y, z$  and  $\tilde{c}_1, \tilde{c}_2, \tilde{q}, \tilde{y}, \tilde{z}$  are on critical orbits. In  $\mathcal{E}_{18}$ ,  $\mathbf{n}$  (with  $SU(3)$  symmetry) and  $\mathbf{y}$  ( $SU_y(2) \times SU_y(2) \times U_y(1)^a$  symmetry) belong to critical orbits; this is not the case for the vector  $\mathbf{m}$ , the direction of the strong hamiltonian  $H_S(\mathbf{m})$ , which is invariant only under  $U(2)^a \square (Z_2(P) \times Z_2(C))$ . The vector  $\mathbf{m}$  belongs to the fourth listed stratum of Example 6 whose orbits are parametrized by  $\varphi$  (defined modulo  $2\pi$ ) and  $\omega$ , with  $0 < \omega < \pi/2$ ,  $\omega \neq \tan^{-1} \sqrt{2}$ . In Eq. (34) we gave the value of  $\omega = .058$  radians in the model of Ref. [5]. The value of  $\varphi$  is related to the arbitrariness in the definition of the  $P$  operator on the internal symmetry space. We have chosen  $\varphi = 0$ .

One cannot expect to deduce the actual value of  $\omega$  from pure geometrical considerations. This value probably depends, as that of  $\theta$ , the Cabibbo angle, upon the details of the dynamics. It is tempting to assume that  $\omega$  and  $\theta$  satisfy some relation which, for instance, could explain why both are small. This relation should be built in terms of the invariants of  $SU(3) \times SU(3)$  if the dynamics correspond to a spontaneous breaking of this symmetry.

There is no invariant linear in both vectors  $\tilde{\mathbf{z}}$  and  $\mathbf{m}$ . Those linear in  $\tilde{\mathbf{z}}$ , quadratic in  $\mathbf{m}$  are (use Eq. (89))

$$(\mathbf{m}, D(\tilde{\mathbf{z}}) \mathbf{m}) = 0,$$

$$f = (\mathbf{m}, AD(\tilde{\mathbf{z}}) \mathbf{m}) = -\frac{1}{\sqrt{3}} \left(1 - \frac{3}{2} \sin^2 \theta\right) \left(1 - \frac{3}{2} \sin^2 \omega\right).$$

Many higher degree invariants are function of those of lower degree.

For example,

$$(D(\tilde{z}) \mathbf{m}, D(\tilde{z}) \mathbf{m}) = \frac{1}{2\sqrt{3}} (\sqrt{3} - f).$$

We note that in  $\mathcal{E}_{16}$  all but two critical orbits contain physical vectors. We have suggested earlier [9] that the  $CP$  violating term in the total Hamiltonian might be along a direction belonging to an unused critical orbit.

To conclude, it seems to us that the remarkable mathematical properties of the direction of symmetry breaking constitute an interesting empirical fact which probably cannot be ignored if one wants to understand the breaking of hadronic symmetry. In particular, we have established that the physical directions are

- (i) idempotents or nilpotents of a real symmetric algebra  $\mathcal{A}$ , which has the compact symmetry group  $G$  as automorphism group;
- (ii) idempotents or nilpotents of the complex algebra  $\mathcal{A}'$ , the complexified of  $\mathcal{A}$ . Then  $G'$ , the complexified of  $G$ , is a group of automorphisms of  $\mathcal{A}'$ .
- (iii) on critical orbits of  $G$  acting on the unit sphere of the vector space of  $\mathcal{A}$ .

We have shown that (i) can be deduced from (iii) but we do not yet know the connection between (ii) and (iii).

### APPENDIX 1: TENSOR OPERATORS

In this Appendix, we consider a group  $G$  and different tensor operators valued on a given Hilbert space  $\mathcal{H}$ . In the physics literature, one generally uses an orthonormal basis  $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$  on the  $k$ -dimensional representation space  $\mathcal{E}$ ; the value  $T(\mathbf{e}_i)$  of the tensor operator  $T$  at  $\mathbf{e}_i$  is often denoted by  $T_i^{(k)}$  and is called the  $i$ -th component of  $T$ . If  $\mathbf{x} = \xi_i \mathbf{e}_i$  (summation over repeated indices is always implied), we can thus write

$$T(\mathbf{x}) = \xi_i T_i^{(k)}. \tag{75}$$

Let  $T_1, T_2 \dots$  be  $\mathcal{E}_1, \mathcal{E}_2 \dots$  tensor operators; then  $T_1(\mathbf{a}_1) + T_2(\mathbf{a}_2)$  is the value at  $\mathbf{a}_1 \oplus \mathbf{a}_2$  of a  $\mathcal{E}_1 \oplus \mathcal{E}_2$  tensor operator that we will denote by  $T_1 \oplus T_2$ . The product of operators  $T_1(\mathbf{a}_1) T_2(\mathbf{a}_2)$  is the value at  $\mathbf{a}_1 \otimes \mathbf{a}_2$  of a  $\mathcal{E}_1 \otimes \mathcal{E}_2$  tensor operator that we denote by  $T_1 \otimes T_2$ . This is true when  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are either different or equivalent representation spaces of  $G$ . In the latter case,

$$T_1(\mathbf{a}) + T_2(\mathbf{a}) \quad (\text{the same } \mathbf{a} \in \mathcal{E}) \tag{76}$$

is the value for  $\mathbf{a}$  of a  $\mathcal{E}$  tensor operator that we denote by  $T_1 + T_2$ .

APPENDIX 2: REALIZATION OF THE OCTET SPACE, THE  $(1, 8) \oplus (8, 1)$  AND  $(3, \bar{3}) \oplus (\bar{3}, 3)$  REPRESENTATION SPACES; THEIR SYMMETRIC ALGEBRAS

The octet space is realized as the 8-dimensional vector space of  $3 \times 3$  hermitian ( $x^* = x$ ), traceless ( $\text{tr } x = 0$ ) matrices. The action of  $u \in SU(3)$  is defined by

$$x \rightsquigarrow u x u^{-1} = u x u^*. \quad (77)$$

It leaves invariant the Euclidean scalar product

$$(x, y) = \frac{1}{2} \text{tr } xy. \quad (78)$$

The Lie algebra law is

$$x_{\wedge} y = -\frac{i}{2} [x, y]. \quad (79)$$

The symmetric algebra law is

$$x_{\vee} y = \frac{\sqrt{3}}{2} \{x, y\} - \frac{2}{\sqrt{3}} (x, y) I. \quad (80)$$

In the physics literature, one uses for the octet space a traditional basis  $\lambda_i$ ,  $(\lambda_i, \lambda_j) = \delta_{ij}$  defined by Gell-Mann, who denotes by  $f_{ijk}$  and  $\sqrt{3} d_{ijk}$  the corresponding structure constant for the  $\wedge$  and  $\vee$  algebra:

$$(\lambda_i, \lambda_{j \wedge} \lambda_k) = f_{ijk}, \quad (81)$$

$$(\lambda_i, \lambda_{j \vee} \lambda_k) = \sqrt{3} d_{ijk}. \quad (82)$$

Every vector of the octet satisfies

$$x_{\vee} x_{\vee} x = x. \quad (83)$$

In the Gell-Mann basis,

$$y = \lambda_8, \quad q = -\frac{1}{2}(\lambda_8 + \sqrt{3} \lambda_3), \quad (84)$$

$$c_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \cos \theta + \frac{1}{2}(\lambda_4 \pm i\lambda_5) \sin \theta. \quad (85)$$

We denote by the direct sum of two vectors of the octet,  $\tilde{a} = a_+ \oplus a_-$ , the elements of the representation space  $\mathcal{E}_{16}$  of the adjoint representation  $(1, 8) \oplus (8, 1)$  of  $SU(3) \times SU(3)$ . Then

$$(\tilde{a}, \tilde{b}) = (a_+, b_+) + (a_-, b_-) \quad (86)$$

and

$$(\tilde{a} \wedge \tilde{b}) = (a_+ \wedge b_+) \oplus (a_- \wedge b_-). \quad (87)$$

The symmetric product  $\tilde{a} \vee \tilde{b}$  is defined in Eq. (29).

The real representation space  $\mathcal{E}_{18}$  of the  $(3, \bar{3}) \oplus (\bar{3}, 3)$  representation can be realized as the complex 9-dimensional vector space of all  $3 \times 3$  matrices  $\mathbf{m}$  with complex coefficients. The action of  $(u_+, u_-) \in SU(3) \times SU(3)$  on  $\mathbf{m}$  is defined by

$$\mathbf{m} \xrightarrow{(u_+, u_-)} u_+ \mathbf{m} u_-^* = u_+ \mathbf{m} u_-^{-1}. \quad (88)$$

The corresponding representation of the Lie algebra is

$$L(\tilde{a}) \mathbf{m} = L(a_+ \oplus a_-) \mathbf{m} = -\frac{i}{2} (a_+ \mathbf{m} - \mathbf{m} a_-). \quad (89)$$

The restriction of the  $(3, \bar{3}) \oplus (\bar{3}, 3)$  representation to the  $SU(3)^d$  subgroup yields two scalars  $\mu$  and  $\mu'$  and two octets  $m$  and  $m'$ , so we can denote  $\mathbf{m}$  by

$$\mathbf{m} = (\mu \mid m \mid \mid \mu' \mid m') \quad (90)$$

with

$$\mathbf{m} = \sqrt{\frac{2}{3}} (\mu + i\mu') I + m + im'. \quad (91)$$

In the notations of Ref. [5],

$$\mu = u_0, \quad \mu' = v_0, \quad m = u_i \lambda_i, \quad m' = v_j \lambda_j. \quad (92)$$

The  $SU(3) \times SU(3)$  invariant Euclidean scalar product is

$$(\mathbf{m}_1, \mathbf{m}_2) = \frac{1}{4} \text{tr}(\mathbf{m}_1^* \mathbf{m}_2 + \mathbf{m}_2^* \mathbf{m}_1) \quad (93)$$

$$= \mu_1 \mu_2 + \mu_1' \mu_2' + (m_1, m_2) + (m_1', m_2'). \quad (94)$$

Since the  $(3, \bar{3}) \oplus (\bar{3}, 3)$  is a real representation, irreducible on the real but reducible on the complex, this representation  $g \rightsquigarrow D(g)$  or  $\tilde{a} \rightsquigarrow L(\tilde{a})$  leaves invariant an antisymmetric bilinear form which is represented by the matrix

$$A = -A^T, \quad A^2 = -1, \quad (95)$$

$$(\mathbf{m}_1, A \mathbf{m}_2) = -(\mathbf{m}_2, A \mathbf{m}_1) = \frac{1}{4i} \text{tr}(\mathbf{m}_1^* \mathbf{m}_2 - \mathbf{m}_2^* \mathbf{m}_1) \quad (96)$$

$$= \mu_1 \mu_2' - \mu_1' \mu_2 + (m_1, m_2') - (m_1', m_2). \quad (97)$$

The  $SU(3) \times SU(3)$  invariant symmetric algebra  $\tau$  is defined by

$$\mathbf{r}_\tau \mathbf{s} = \frac{1}{2} I (\text{tr } \mathbf{r} \mathbf{r}^* \text{tr } \mathbf{s} \mathbf{s}^* - \text{tr } \mathbf{r} \mathbf{s}^* \mathbf{s}^* \mathbf{r}^*) - \frac{1}{2} \mathbf{r} \mathbf{r}^* \text{tr } \mathbf{s} \mathbf{s}^* - \frac{1}{2} \mathbf{s} \mathbf{s}^* \text{tr } \mathbf{r} \mathbf{r}^* + \frac{1}{2} \{\mathbf{r}^*, \mathbf{s}^*\}. \quad (98)$$

Useful identities are

$$(\mathbf{x}_\tau \mathbf{x})^* \mathbf{x} = I \det \mathbf{x}, \quad (\mathbf{x}_\tau \mathbf{x})_\tau (\mathbf{x}_\tau \mathbf{x}) = \mathbf{x} \det \mathbf{x}. \quad (99)$$

We define

$$\mathbf{n}(\varphi) = (\cos \varphi \mid 0 \mid \mid \sin \varphi \mid 0) \quad (100)$$

and

$$\mathbf{y}(\varphi) = (\sqrt{\frac{1}{3}} \cos \varphi \mid -y \sqrt{\frac{2}{3}} \cos \varphi \mid \mid \sqrt{\frac{1}{3}} \sin \varphi \mid -y \sqrt{\frac{2}{3}} \sin \varphi). \quad (101)$$

Then

$$\mathbf{n}(\varphi)_\tau \mathbf{n}(\varphi') = \sqrt{\frac{2}{3}} \mathbf{n}(-\varphi - \varphi'). \quad (102)$$

So

$$\mathbf{n}(0) \quad \text{and} \quad \mathbf{n}(\pi) = -\mathbf{n}(0) \quad (103)$$

are idempotents; also

$$\mathbf{y}(\varphi)_\tau \mathbf{y}(\varphi') = 0. \quad (104)$$

In the text we denote  $\mathbf{n}(0)$  and  $\mathbf{y}(0)$  simply by  $\mathbf{n}$ ,  $\mathbf{y}$ . For more details, see Ref. [1(b)].

### APPENDIX 3: SMOOTH ACTION OF A COMPACT LIE GROUP $G$ ON A MANIFOLD $M$

One can define on  $M$  a Riemann metric  $\gamma(x, y)$ . Its average, by the Haar measure of  $G$ ,

$$\hat{\gamma}(x, y) = \int_G \gamma(gx, gy) d\mu(g) \quad \text{with} \quad \int_G d\mu(g) = 1 \quad (105)$$

defines on  $M$  a  $G$ -invariant Riemann metric; so  $G$  acts on  $M$  by isometries. The choice of a geodesic coordinate system in a neighborhood  $V_m$  of  $m$  makes the action of the little group  $G_m$  linear on  $V_m$ . The operators  $D(g)$  of this linear representation of  $G_m$  are

$$D(g) = d\Phi(g)_m, \quad (106)$$



where  $\Phi(g)$  is the diffeomorphism  $m \rightsquigarrow \Phi(g, m)$  of  $M$  for a fixed  $g$ , and  $d\Phi(g)_m$  is its differential at  $m$ . When  $g \in G_m$ ,  $d\Phi(g)_m$  is also a linear operator on  $T_m(M)$ , the plane tangent to  $M$  at  $m$ . Let  $T_m(G(m))$  and  $T_m(S(m))$  be the tangent plane at  $m$  to the orbit  $G(m)$  and to the stratum  $S(m)$ . Of course,

$$T_m(G(m)) \subset T_m(S(m)) \subset T_m(M). \quad (107)$$

The representation  $D$  [Eq. (106)] of  $G_m$  on  $T_m(M)$  is real, orthogonal, and leaves invariant the subspaces  $T_m(G(m))$  and  $T_m(S(m))$ .

We split  $T_m(M)$  as the direct sum of three orthogonal subspaces;

$$T_m(M) = T_m(G(m)) \oplus N_m; \quad N_m = Q_m^{(1)} \oplus Q_m^{(2)} \quad (108)$$

such that

$$T_m(S(m)) = T_m(G(m)) \oplus Q_m^{(1)}. \quad (109)$$

One proves that the representation of  $G_m$  on  $Q_m^{(1)}$  is trivial and the representation on  $Q_m^{(2)}$  does not contain the trivial representation. One also proves that if  $f$  is a smooth  $G$ -invariant function on  $M$ , then

$$\text{grad } f \in Q_m^{(1)}. \quad (110)$$

So if  $Q_m^{(1)} = 0$ ,  $T_m(G(m)) = T_m S(m)$ , i.e., the orbit  $G(m)$  is isolated in its stratum and, as Eq. (110) shows,  $\text{grad } f = 0$ . One also proves that  $Q_m^{(1)} = 0$  is equivalent to (a) or (b) of Theorem 1.

We also point out to the attention of physicists the Theorem of Mostov: If the number of strata of  $(G, M, \varphi)$  is finite (this is the case when  $M$  and  $G$  are compact), there exists an equivariant map which embeds  $M$  into a finite dimensional Euclidean space on which  $G$  acts linearly.

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