Summer Study : High Energy with Polarized Beans

Argonne National Laboratory

Talk of Louis MICHEL (July 25, 1974)

"Analysis of Polarization Measurements and Test of Selection Rules and of Models"

0. Introduction

At this meeting we learned remarkable new results on p-p scattering with 2 or 3 p-polarizations measured simultaneously. Let me emphasize here that at a given s,t value ($\sim E_{cm}, \theta$) there are 5 amplitudes. Hence at least 10 observables should be measured in order to reconstruct completely these amplitudes(1). Up to now only s independent observables have been measured and the present experimental results are given in term of observables : differential cross sections and vertical polarizations of the different protons. The polarization of a spin $\frac{1}{2}$ particle is completely determined (in any frame of reference(2)) by its direction in space and its degree d_e , $0 \leq d_e \leq 1$.

1- The overall phase is not directly measurable except if one studies the Coulomb interference or uses the optical theorem (see e.g. work of Soffer and Wray). Similarly, unitarity and causality impose complicated integral relations among the amplitudes, but in principle at a given s value, the complete knowledge of some amplitudes for all t values give conditions on the others at the same s value.

2- How the polarization changes with the reference frame ? This is very simple since covariantly the polarization operator is an axial four vector orthogonal to the energy momentum, with fixed length. See my talk at the other other workshop and ref [1], [2a].

Hence all its polarization states can be described by the points of a ball (= solid sphere) in our three dimensional space (3). The center represents the unpolarized state and the point of the sphere S_2 of radius one, the completely polarized states (= pure states).

It is sad to say that most physicists do not know how this generalizes for higher spins. We follow here ref. [2] [3] [4].

1. The Polarization domain. If the polarization of several particles of spin j_i is measured simultaneously (in order to observe eventually polarization correlations) the number of observables is n^2-1 with

$$n = \prod_i (2j_i+1) \tag{1}$$

(for ν and $\bar{\nu}$, replace $2j_i+1$ by 2, 1, 1 respectively); indeed they are the real and imaginary part of the density matrix ρ, which is $n \times n$, Hermitean ($\rho^* = \rho$), trace one ($\mathrm{tr}\rho = 1$), positive ($\rho \geq 0$ i.e. its eigenvalues are ≥ 0) matrix. The set of Hermitean matrices form a n^2-dimensional Euclidean space $\mathcal{E}$ with the scalar product

$$(\rho_1, \rho_2) = \mathrm{tr}\rho_1\rho_2 \tag{2}$$

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3 - Since the photon also has only two spin states, the same ball can be used for describing light polarization. This was implicitly proposed by Stokes in 1852, ref. [7] and very explicitly by Poincaré in 1884, ref. [8]. The two circular polarizations are represented by the poles and the plane polarizations by the points of the equator. By the action of the Lorentz groups, for photons, the sphere turns only around its pole axis while for $m \neq 0$, spin $\frac{1}{2}$ it can be turned by an arbitrary rotation. For $m \neq 0$, spin ≥ 1 it is not possible to transform a pure polarization state to another arbitrary pure state by a Lorentz transformation.

The subset of positive matrices is a convex, autodual, homogeneous cone[4] $\mathcal{C}$ in $\mathcal{E}$ of vertex $\rho = 0$. The polarization Domain $\mathcal{D}$ is the intersection of $\mathcal{C}$ by the hyperplane $\mathrm{tr}\rho = 1$. The density matrices of maximal rank[5] are represented by the inside $\mathring{\mathcal{D}}$ of $\mathcal{D}$. Those of rank $\leq m < n$ correspond to a closed submanifold $\partial_m \mathcal{D}$ of the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. With

$$\dim \partial_m \mathcal{D} = 2nm - m^2 - 1 \qquad (3)$$

e.g. $\partial_1 \mathcal{D}$ is the set of pure states (= rank one projectors : $\rho^2 = \rho = \rho^*, \mathrm{tr}\ \rho = 1$).

2. Angular momentum conservation

It might be advantageous to use ρ for describing simultaneously the polarization of initial and final particles e.g. ref. [5], one then studies both polarization correlation and polarization transfer. From now on we assume that ρ describes the polarization of final particles only. Let m be the number of initial spin states (i.e. m is given by (1) for initial particles), ρ_i the $m \times m$ initial density matrix, T the $n \times m$ transition matrix; for fixed energy momenta for all particles, the differential cross section is given by

$$\sigma = \mathrm{tr}\ T\rho_i T^* \qquad (4)$$

4- Convex means : $\rho_1, \rho_2 \in \mathcal{C}$, $\forall\lambda$, $0 \leq \lambda\ 1 \Rightarrow \lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{C}$. I do not define autodual, but this is equivalent to $\rho \in \mathcal{C} \Leftrightarrow \forall\rho' \in \mathcal{C}$, $(\rho, \rho') \geq 0$. Here the interior of $\mathcal{C}$ can be identified to the homogeneous space $GL(n,\mathbb{C})/U(n)$. There is an important - and some very recent - mathematical literature on such cones.

5- The kernel of ρ is the eigenspace for the eigenvalue zero; its dimension is k , the rank of ρ is $r = n-k$.

and the final polarization is given by

$$\rho = T\rho_i T^* \sigma^{-1} \tag{5}$$

If $m < n$, angular momentum conservation requires that ρ is in the subset $\partial_m \mathcal{D}$ of $\mathcal{D}$.

If the initial particles are polarized, we might have rank $\rho_i = m' < m$; then the final states are in $\partial_{m'} \mathcal{D}$.

We will study later parity conservation.

3. Incomplete observations

Often polarization measurements are incomplete. This might be a choice of the observer; for example, the polarization of some, but not all, final particles is observed. This corresponds to observe instead of $\rho \in \mathcal{D} \subset \mathcal{E}$, the projection $\rho' \in \mathcal{D}' \subset \mathcal{E}'$ where $\dim \mathcal{E}' < \dim \mathcal{E}$; in this case one has still $\rho' \geq 0$, but the rank condition is less stringent and generally dissappears; this is certainly the case when also the energy momentum of some final particles is not observed i.e. inclusive or partially inclusive reactions[6]. In some cases, it is technically impossible to measure completely the polarization, even for one particle; e.g. the angular distribution in a parity conserving two-body decay (such as $\rho \rightarrow 2\pi$) allows only to observe the even multipole of the polarization, that we call for short the even part of the polarization[7]. In that case ρ' is still a linear projection of P and its domain $\mathcal{D}'$ is a linear projection of $\mathcal{D}$, the domain of ρ. If one replaces by zero the matrix elements of ρ' which are

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6 - Indeed, the projection $\rho \rightarrow \rho'$ can also be described by an average with a normalized Stieljes positive measure $d\mu$, on the phase space and polarization space : $\int d\mu = 1$, $\rho' = \int \rho d\mu$, so $\rho \geq 0 \Rightarrow \rho' \geq 0$; remark that if $d\mu$ is discrete : $\rho' = \Sigma \lambda_i \rho$ with $\lambda_i > 0$, $\Sigma_i \lambda_i = 1$, rank $\rho' \leq n_i$ rank ρ where n_i = number of λ_i.

7 - This is called "alignment" in nuclear physics.

not observed, one obtains a matrix ρ'_o which might not be positive; (however $\rho' \geq 0$ if the polarization of only one final particle is observed and rank $\rho'_o \leq 2$ rank ρ). More generally, inaccuracy or bias in measurements modify the representation of ρ ; for instance if one uses bins for t (or θ) values, one replaces the family of $\rho(t)$ by the barycenter $\rho_{ob} = \int \rho(t) d\mu(t)$. If angular momentum conservation required that $\rho(t) \in \partial\mathcal{D}$, ρ_{ob} will be inside $\mathcal{D}$ but near its surface[8].

4. Parity conservation.

Let ν be the dimension of the subspace of the Minkowski space spanned by the energy momenta of the reaction. When ν is smaller than its maximal value 4, there are conditions on the polarizations imposed by parity conservation. For example in a two body reaction such as $\pi + p \to \rho + \Delta$, $\nu = 3$, and when the final particle are in the forward ($\theta_{cm} = 0°$) or the backward direction ($\theta_{cm} = 180°$), $\nu = 2$. The energy momenta are left invariant by the reflexion through a hyperplane (unique for $\nu = 3$) which contains them. On the initial and final polarization space such a reflexion is represented by $m \times m$ and $n \times n$ unitary matrices B_i, B_j of square identity ($B^* = B^{-1} = B$). Parity conservation in the reaction requires

$$T = B_j T B_i^* \tag{6}$$

when the initial density matrix is also invariant by the reflection. ($\rho_i = B_i \rho_i B_i^*$) e.g. $\rho_i = \frac{1}{m} I$, the unpolarized state, then ρ is also invariant

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3 - At a given s one passes from a s to t or u channel by a "crossing rotation" which depends on t . The formation of "bins" does not commute with the change of channel dependent frame of reference. So we advize experimentalists to present their polarization data in different channel dependent frames.

$$\rho = B_j \rho B_j^{-1} \quad . \tag{7}$$

With the same hypothesis, equation (7) is also valid for reactions involving more than 2 final particles when ν', the number of observed linear independent energy momenta, is smaller than 4 (e.g. inclusive reactions).

We will denote by $\mathcal{D}_B$, $\mathcal{D}_F$ the polarization domains of ρ which satisfy (7) for $\nu = 3$, $\nu = 2$ respectively ($\mathcal{D}_F \subset \mathcal{D}_B \subset \mathcal{D}$). Of course, with partial polarization measurements, the domain of observed polarization is the projection $\mathcal{D}'_B$ or $\mathcal{D}'_F$.

5. Spin 1.

To simplify we assume that (7) is satisfied : e.g.,

$$0^- + \tfrac{1}{2} \to 1^- + X \tag{8}$$

with unpolarized target and only the even polarization of the spin 1 particle is observed. (e.g. $\rho \to 2\pi$, $K^* \to K + \pi$). Then $\mathcal{D}'_B$ is an axially symmetric cone in 3-dimensions whose diametral section is an equilateral triangle. (See Fig. 1). Every point of this cone has a physical meaning. The vertex P_1 means: one particle or one Regge trajectory exchange with natural parity (1^-, 2^+, 3^-, 4^+ etc...). The points of the basis circle represents natural parity exchange of a meson, or a Regge trajectory; in particular, P_2 represents one pseudo scalar meson exchange. If the spin 1 particle is in the forward or backward direction, its polarization must be on the segment $P_o P_2$ (which of course contains O, the unpolarized state). The height in the cone measures the ratio of natural/ unnatural exchange (it is the sum $\rho_{11} + \rho_{1-1}$ in the much used Gottfried Jackson frame, and the corresponding axis have been drawn by dotted lines in Fig. 1 - they are non-orthogonal).

I wonder why this cone is not more widely used by physicists; it plays for polarization a role similar to that of the Dalitz plot for energy momenta[9].

6. Spin $\frac{3}{2}$.

For a spin $\frac{3}{2}$ particle the dimension of $\mathcal{D}$ is 15, that of $\mathcal{D}_B$ is 7; when only the even polarization is measured (e.g. $\Delta \to N\pi$) the corresponding domains $\mathcal{D}'$ and $\mathcal{D}'_B$ are the 5 and 3 dimensional balls bounded by the spheres S_4 , S_2 respectively. The latter case corresponds to (7), e.g., $\pi N \to \pi\Delta$ with unpolarized target. The pure ρ exchange (e.g. Stodolsky Sakurai ρ dominance [10]) or $SU(6)_W$ or the quark model predict a density matrix ρ represented by the south pole P_1; for forward or backward spin $\frac{3}{2}$ particle, the polarization must be on the diameter P_oOP_2 .

Is it too complicated to use a sphere ? Again every point of the domain has a physical meaning. Remark that, as for spin 1, the position of the points of the domain can be fixed with respect to $OP_oP_1P_2$ which are established by their physical meaning and not by reference to a special coordinate system !

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9 - We refrain here to make more than three very general remarks :

a) If a 1^+ meson instead of a 1^- is produced in reaction (8), permute natural and unnatural parity in the text.

b) If the measurement is on the lateral surface of the cone, the odd polarization is zero and hence is indirectly measured.

c) If the measured density matrix elements (see the axes of Fig. 1) yield a point outside the cone, not only this means that ρ is not ≥ 0 (i.e. the data violates angular momentum conservation) but if the polarization has been analyzed by a decay $1^- \to 0^- + 0^-$ (e.g. $\rho \to 2\pi$, $K^* \to K\pi$) , it means that the angular distribution of this decay is not positive !

7. <u>Test of models by polarization measurements</u>.

Any model, any theory which measures angular momentum and parity should predict that the polarization is a subdomain $\mathcal{B}$ of the polarization domains obtained from angular momentum and parity conservation. ($\mathcal{D}$ or $\partial_m\mathcal{D}$ or $\mathcal{D}_B$ or $\partial_m\mathcal{D}$ or $\mathcal{D}_B$, etc...). If $\mathcal{B}$ is not strictly smaller, the model has no predictive power for the polarization (e.g. the quark model in N-N scattering). For a partial measurement of polarization, one should compute $\mathcal{B}'$, the projection of $\mathcal{B}$ and compare it to $\mathcal{D}'_{ob}$ (i.e. $\mathcal{D}'$ or $(\partial_m\mathcal{D})'$ or $\mathcal{D}'_B$ etc...) for assessing the predictivity of the model.

Since there is a natural metric in $\mathcal{D}'_{ob}$, one can evaluate the shortest distance between the experimental point $\rho' \in \mathcal{D}'_{ob}$ and the subdomain $\mathcal{B}' \subset \mathcal{D}'_{ob}$. The model predicts it is zero. Of course the value of the test depends not only on this distance, but also on the accuracy of the measurement. If the size of the errors is of the same magnitude than the size of $\mathcal{D}'_{ob}$ the measurement has only a very weak meaning and therefore .... a good χ^2 test for any theoretical prediction ! We give here two examples of such a discussion, made in ref. [6], on a data published in CERN. The reaction is

$$K + N \rightarrow Y^*(3/2) + X \qquad (9)$$

Quark model assume the factorization at the vertex $N \rightarrow Y^*$ with spin flip of quark. The partial T matrix which describes the vertex $\frac{1}{2} \rightarrow \frac{3}{2}$ has a spin 1 part and a spin 2 part. The latter is excluded : it cannot describe a spin $\frac{1}{2}$ flip. The target is unpolarized, so $\rho(Y^*) = \int T(\frac{1}{2}I)T^* d\mu$ has multipoles $1 \otimes 1 = 0 \oplus 1 \oplus 2$ under SO(3). Hence the quark model (and other models) predicts that $\rho(Y^*)$ has no $L = 1$ multipole. Such a multipole cannot be observed by the angular distribution of $\Lambda \rightarrow N\pi$, but it can be observed in the sequential decays $Y^* \rightarrow \Lambda\pi$, $\Lambda \rightarrow N\pi$. When X is a π (exclusive reaction) the prediction of the quark model is stronger, it is the same that ρ dominance, $SU(6)_W$ etc. : the Stodolsky Sakurai point [10]. Fig.3 and 4 summarize the situation for two experimental results.

8. Internal symmetries, e.g. isospin conservation.

To test the validity of these symmetries from polarization measurements is similar to testing the validity of a model. I published the first papers on this problem [9]. Doncel, Minnaert and I have given a complete solution of the problem ([2 d, e] and to be published). Let us just see here the solution for the most common and useful case : three reactions are going through two channels of external symmetry (e.g. $\pi^+p^+ \to \pi^+p^+$, $\pi^-p^+ \to \pi^-p^+$, $\pi^-p^+ \to \pi^0 n$ isospin $\frac{1}{2}$ and $\frac{3}{2}$; in such example [2a] gives all relation to be satisfied by the spin rotation parameters APR). Then the three transition matrices satisfy the linear relations

$$\Sigma_{\alpha=1}^{3} \gamma_q T_\alpha = 0 \tag{10}$$

We define $s_\alpha = |\gamma_\alpha|^2 \sigma_\alpha$, the three weighted cross section. To simplify here, we assume that the initial polarization are identical and that the same observation ρ_α is eventually made on the final polarization(10). Since the sum of any three weighted amplitudes vanishes : $\Sigma\, \gamma_\alpha < f|T_\alpha|i > = 0$ their moduli (which are square root of cross section) form a triangle. This is still true, after eventually spin average and partial integration over phase space, e.g. for the $\sqrt{s_\alpha}$:

$$|\cos \omega_{12}| = \left| \frac{s_3 - s_1 - s_2}{2\sqrt{s_1 s_2}} \right| \leq 1 \tag{11}$$

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10 - In general it is a projection ρ'_α , but from now we drop the prime. For isospin these assumptions are very realistic. For SU(3) symmetry, when one compar[illegible] strange and non strange particle polarizations, different observa[illegible] generally be made.

No other relation appears if only one polarization is measured. When two polarizations are measured, the new relation is[11]

$$\left| \frac{s_3 - s_1 - s_2}{2\sqrt{s_1 s_2}} \right| \leq \mathrm{tr}\sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}} \qquad (\leq 1) \tag{12}$$

This formula is true for an arbitrary number of particles for arbitrary spin values, as long as the partially observed ρ_α have to be positive[12] (see §3).

If the 3 cross-sections and the three polarization are measured, internal symmetry conservation is equivalent for the three completely observed matrices.

$$R_\alpha = s_\alpha \rho_\alpha = |\gamma_\alpha|^2 \sigma_\alpha \rho_\alpha \quad . \tag{13}$$

to the relation :

$$\forall\, \theta \;,\; 0 \leq \theta \leq 2\pi \qquad \Sigma_{\alpha=1}^{3} \lambda_\alpha R_\alpha \geq 0 \quad , \quad \text{with} \quad \lambda_\alpha = \frac{1}{3}\left[1 - 2\cos\left(\theta - \frac{2\pi}{3}\alpha\right)\right] \tag{14}$$

In the euclidean space of the Hermitean matrices, the three R_α define (in general) a 2-plane Z and form a triangle. Equation (14) require that the corresponding ellipse (see Fig. 5) is inside the intersection by Z of the cone $\mathbb{C}$ of positive matrices. The same relation holds for an arbitrary projection : $R_\alpha \to R'_\alpha$

$$\forall\, \theta \quad , \qquad \Sigma_{\alpha=1}^{3} \lambda_\alpha(\theta) R'_\alpha \in \mathbb{C}'_{ob} \quad . \tag{14'}$$

Of course for low spin values we can give a more explicit form of relations (14) or (14').

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11 - The right part of this formula is symmetric in 1,2

12 - Positive matrices have a unique positive square root. If ρ_1, or ρ_2 are not positive (12) has no meaning.

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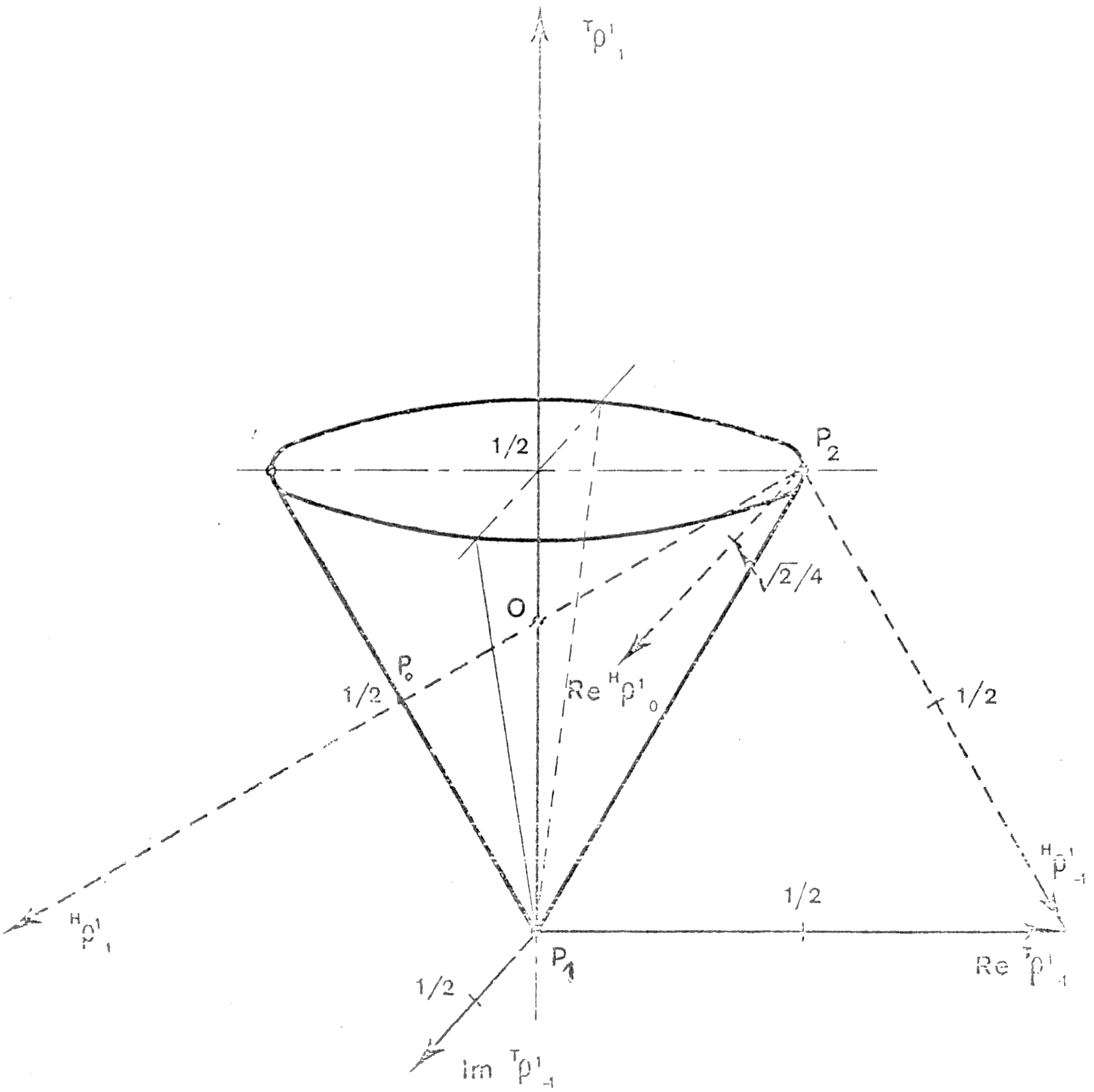

Fig. 1. This cone is the three dimensional even-polarization domain for a spin 1^ϵ particle with condition (7) for parity conservation (two body collision or one particle inclusive reaction). The matrix elements in helicity ($^H\rho$) or transversity ($^T\rho$) quantization form a basis in this polarization space. The physical meaning of each point of the cone is well defined once P_o and P_2 have been fixed : the polarization for forward or backward reaction is on the segment P_oP_2 . For spin 1^ϵ (ϵ = parity) production by a one meson exchange or a single Regge trajectory exchange, the basis circle corresponds to an exchange for $\epsilon = 1$ of natural parity, for $\epsilon = -1$ of unnatural parity (then P_2 corresponds to one π exchange). The vertex P_1 corresponds to the opposite naturality (for $\epsilon = -1$ e.g. ρ exchange, which has natural parity).

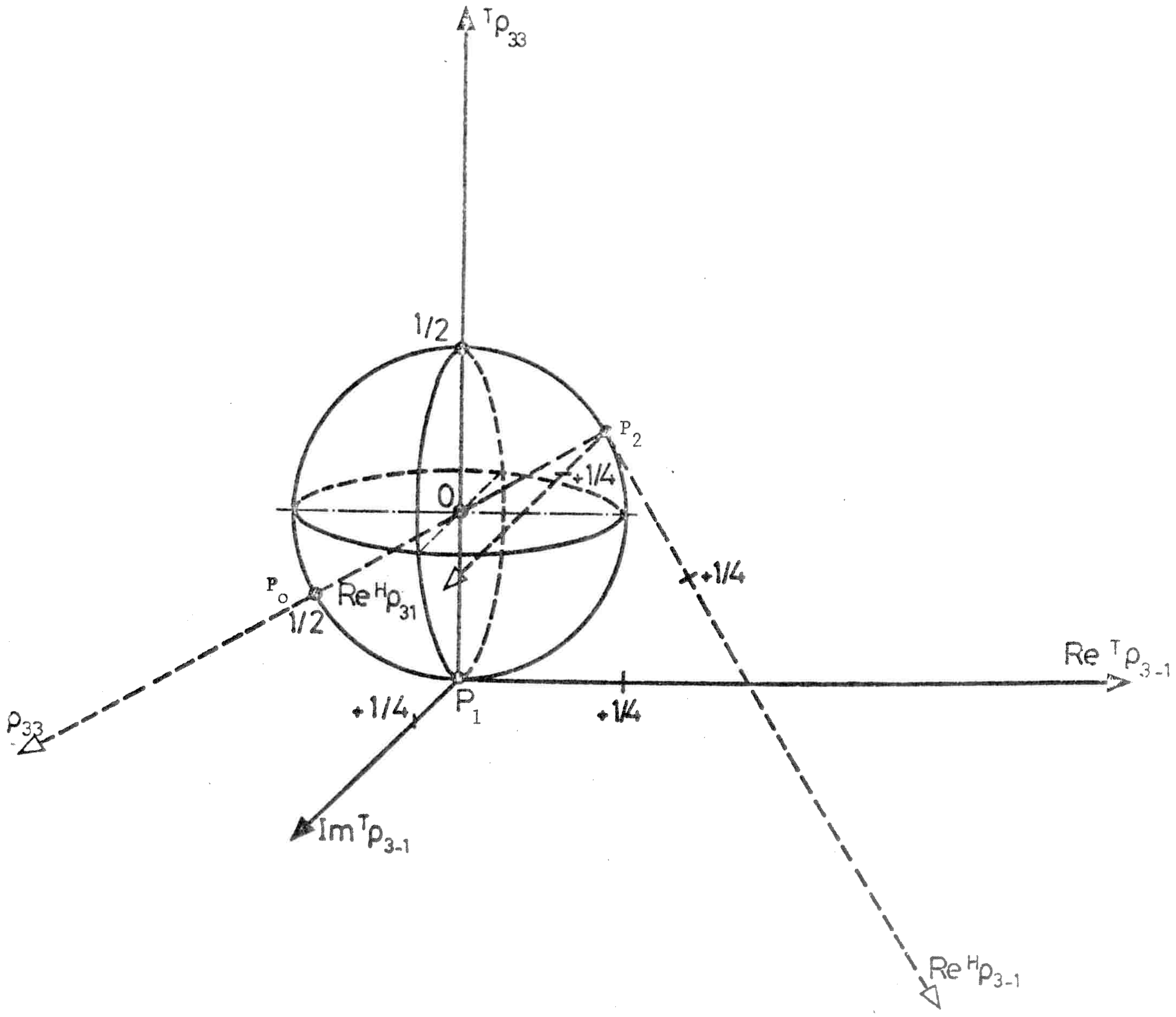

Fig. 2. This solid sphere is the three dimensional even-polarization domain for a spin $\frac{3}{2}$ particle with condition (7) for parity conservation (two body collision or one particle inclusive reaction). The matrix elements in helicity ($^H\rho$) or transversity ($^T\rho$) quantization form a basis in this polarization space. The physical meaning of each point of this ball is well defined once $P_oP_1P_2$ have been fixed. The polarization for forward or backward reaction is on the segment P_oP_2 and the south pole P_1 corresponds to vector exchange (Stodolsky Sakurai [10]).

Fig. 3. The polarization domain $\mathcal{D}$ of the Y^* (1385, spin $\frac{3}{2}$) produced in the reaction $\pi p \to KY^*$ has seven dimensions. For given s and t values ρ , the Y^* 4×4 polarization density matrix, has rank 2, so its representative point has to be in a 3 dimensional submanifold of the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. The quark model or $SU(6)_w$ yields the same prediction as Stodolsky Sakurai vector dominance : ρ_{th} . A published CERN data (37 events only!) is ρ_{exp} . The figure is in the 2-plane Z defined by the 3 polarizations ρ_o = unpolarized, ρ_{th}, ρ_{exp} . If Y^* had no odd polarization - as the model requires, the section of $\mathcal{D}'_{ob}$ by Z would be a circle : section of the sphere of Fig. 2 by the vertical Z plane). The errors are very large : in the polarization space $\mathcal{E}_7$, the ellipsoid $E(\frac{1}{2})$ (whose points have a level of confidence $\geq \frac{1}{2}$) has a volume 7 times larger than that of $\mathcal{D}$ and only .003 of the volume of $E(\frac{1}{2})$ is in $\mathcal{D}$! Comment : this measurement based on 37 events yields very little information (and does not confirm the quark model as it was claimed).

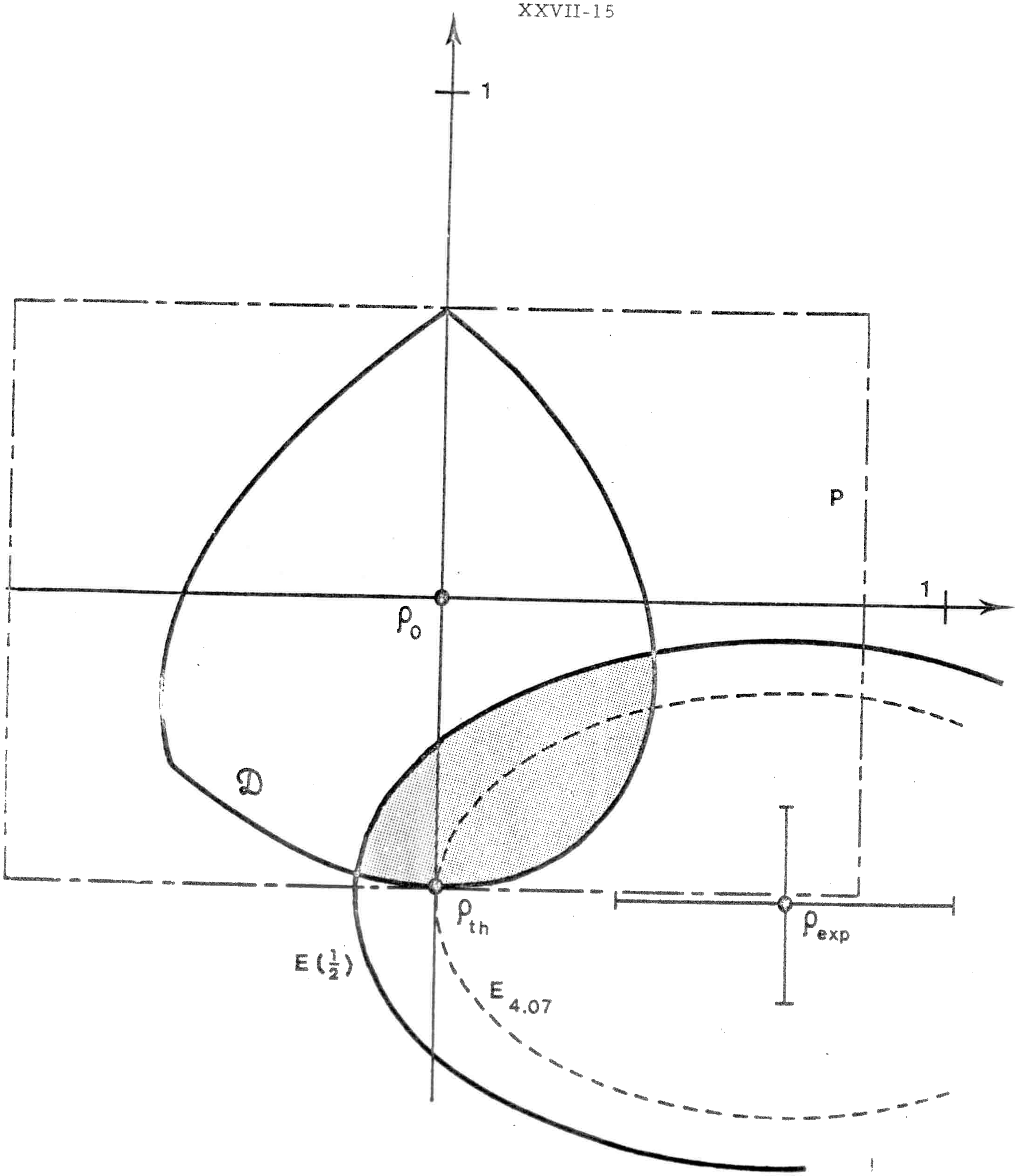

Fig. 3

Fig. 4. The polarization domain $\mathcal{D}$ of the Y^* (1385, spin $\frac{3}{2}$) produced in an inclusive reaction $K+p \to Y^* + X$ has seven dimensions. The quark model predicts that the polarization of this Y^* has no $L = 3$ multipole component i.e. it must be in the 4 dimensional subspace Q spanned by the multipole $L = 2$ and $L = 1$ components (which satisfies equation (7)). A published CERN data based on 282 events is represented by ρ_{exp}. The point ρ' is the foot of the perpendicular from ρ_{exp} to Q. The distance $\rho'\rho_{exp}$ measures the violation of the quark model. This data is good enough. The ellipsoid $E(\frac{1}{2})$ whose points have a level of confidence $\geq \frac{1}{2}$, has a volume 50 times smaller than that of $\mathcal{D}$ and .15 of the volume of $E(\frac{1}{2})$ is in $\mathcal{D}$. This data is not in favour of the quark model. It is strange that it favours the boundary of $\mathcal{D}$; indeed, in an inclusive reaction one measures the barycenter of all polarizations for the different X's and their energy momenta; the only possibility to stay very near the boundary $\partial\mathcal{D}$ is that all polarizations are practically the same (independent of the X's and their energy momenta).

In this CERN paper, only a third point (not discussed here) was actually in favour of the quark model.

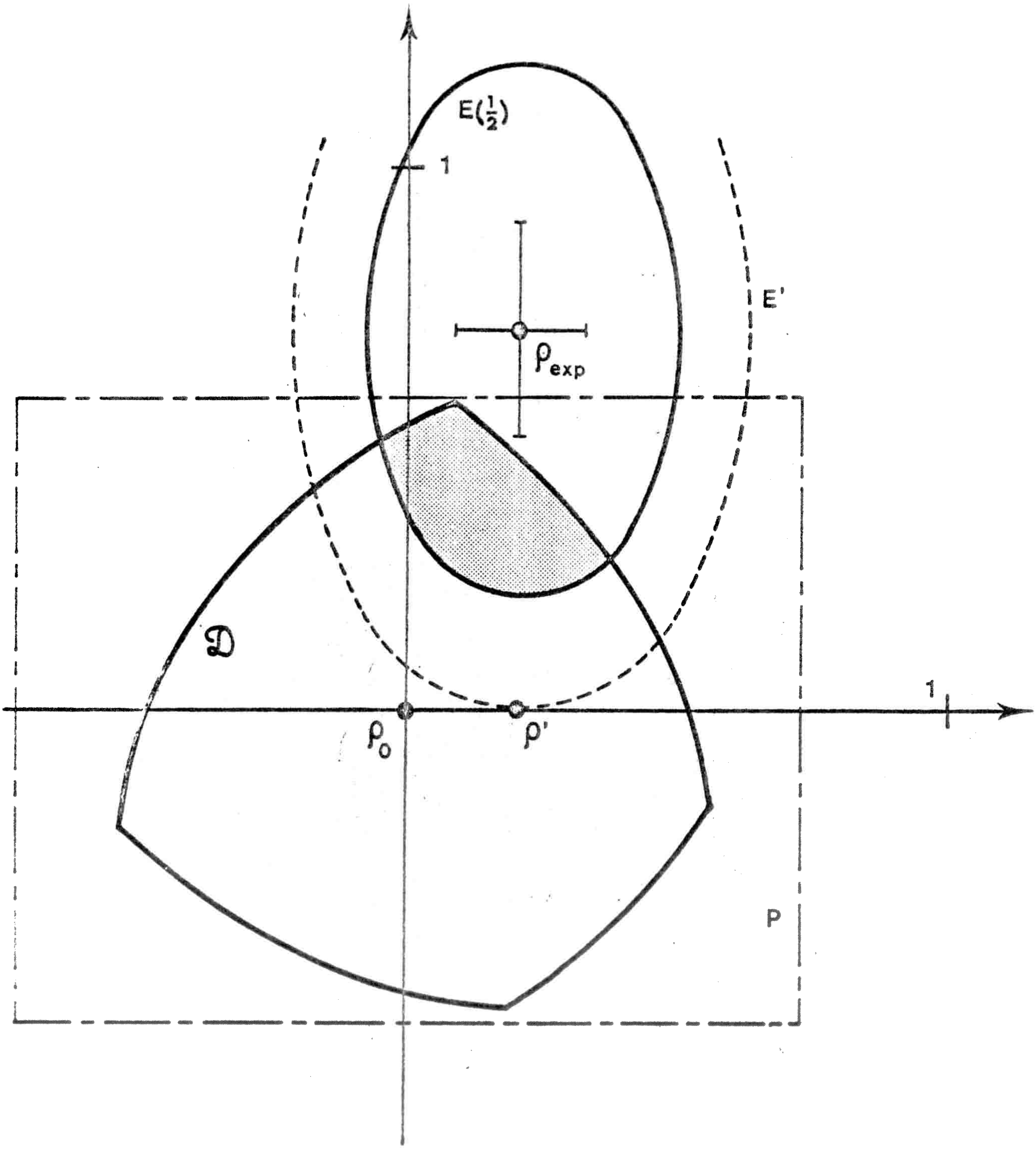

Fig. 4

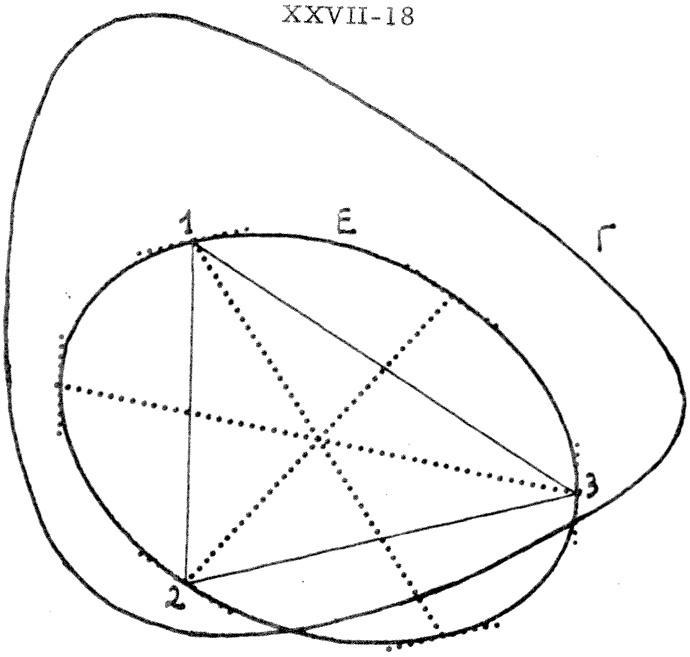

Fig. 5. For 3 reactions going through 2 isospin (or internal symmetry group) channels, the three transition matrices are linearly dependent $\Sigma\, \gamma_\alpha T_\alpha = 0$. Let σ_α be three cross sections and ρ_α three corresponding measured polarizations. If the ρ_α should be positive, (although the polarization might be only partially observed) the 3 experimental matrices $R_\alpha = |\gamma_\alpha|^2 \sigma_\alpha \rho_\alpha$ must be positive. They are represented by 3 points, in the Euclidean space of Hermitean matrices, which define a two dimensional plane Z . The points must be in $\mathbb{C}$, the convex cone of positive matrices. The figure is drawn in the plane Z . It shows the boundary $\partial\Gamma$ of the intersection $\Gamma = Z \cap \mathbb{C}$. The three points 1,2,3 defines a triangle and also an ellipse

$$E = \{R(\theta)\} \ , \ R(\theta) = \frac{1}{3} \Sigma_{\alpha=1}^{3} R_\alpha (1+2 \cos (\theta + \frac{2\pi}{3} \alpha))$$

Isospin conservation requires that $R(\theta) \subset \Gamma$; (in the figure this is not true, although each $R_\alpha \in \Gamma$). A sufficient condition for this check is to verify that one of the R_α is positive and then $\forall\, \theta$, $0 \leq \theta \leq 2\pi$, $\det R(\theta) > 0$. If the partially measured polarization density matrices have not to be positive the 3 observed R_α are projections. The ellipse E is similarly constructed from the 3 points and Γ is the intersection of 2 with the projection of $\mathbb{C}$ corresponding to this partial polarization measurement. Isospin conservation is again equivalent to $E \subset \Gamma$.