Relativistic Invariance and Internal Symmetries

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This title is also approximately that of one of my papers (Phys. Rev. 137B, 405, 1965) which appeared at the time I accepted to give these lectures (New York meeting 1965). Then the flow of papers on SU6 was rising tremendously and it was agreed that Professor B. W. Lee would speak more specifically on SU6 and that I would have to speak about the many published theorems on "mixing" (or rather on the impossibility of mixing) Poincaré invariance and internal symmetries.

To make a review of the twenty or more papers which appeared on the subject in one year would have been tedious and not even useful; in my opinion § II.7 deals with this specific task. What I considered more important is to give the mathematical tools necessary in order to enable a physicist to evaluate by himself those papers if he wishes and to work on this subject if he likes. These mathematical tools on some aspects of structures of groups, and their extensions are not generally given in group theory books written for physicists.

So I have lectured on this subject in three summer schools (Istanbul 1962, Brandeis 1965, Cargese 1965), and the lecture notes had to be published: (same editor for the three schools).

I have tried to minimize the overlaps between these three sets of lectures. This has the bad consequence that I have here to refer too often to the two other sets (mainly for proofs not given in this one).

Another elementary mathematical object of group theory, that of homogeneous space (so useful for the theorem of the Mc Glinn type!) is studied in chapter II and I try to give to Zeeman's theorem the emphasis it deserves.

Although § 5 of chapter III deals with relations between charges and spin and § 6 of chapter IV is on polarization and isotopic or unitary spin conservation, the main theme of the title seems even more lost in these two chapters. Indeed for the content of the last part of the lectures (chap. III and IV) I have been influenced by the reaction and wishes of the audience and also the content of others' lectures (Robinson's for chap. III, Low's and Cabibbo's for chap. IV).

These notes would not exist without Mr. Challifour. Not only because he has written them, but also because he has in many places expanded them, including for instance pieces of supplementary lectures given at night on a strictly private basis and to a voluntary audience.

I did not want to suppress what he took the pain to write in detail because I trust his judgement on how to make these notes more understandable. I thank him heartily for the hard work he did. Of course, I did make some changes and I am responsible for all the crimes against the English language and all errors contained in these notes. I would have liked to work more on these notes, but the emphasis for this volume is on fast publication, so I beg the indulgence of the reader.

I have taken the opportunity of the publication of these notes to correct two mistakes that I have published:

In the book of Boulder Symposium on the Lorentz group, July 1964 (Lectures in theoretical physics, volume VIIa, University of Colorado Press, page 118), I quoted incorrectly Zeeman's theorem and there resulted a gap in the proof on my theorem that all automorphisms of the Poincaré group are continuous. This gap is filled by the slightly different form of Zeeman's theorem I prove in II.4. A more direct proof on my theorem on Aut P will be published in the 1965 Cargèse Summer School notes.

Finally § IV.6 reproduces mainly my letter to Nuov. Cimento **22**, 203 1961 "Relations between polarizations due to charge independence", in order to correct a silly mistake I made in its last equation.

I. GROUP THEORY AND GROUP EXTENSIONS.

I.0 ELEMENTARY NOTIONS.

Throughout we shall consider an abstract group G with elements $x, y, x, \ldots \in G$. An arbitrary collection of elements of G will be denoted by M when we do not wish to make reference to any algebraic structure of M. We shall write $M \subset G$. If $H \subset G$ and H is a group, then H is a subgroup of G written $H < G$.

Consider now subsets $M, N \subset G$; we denote $M \cap N$ for the elements common to both M and N and $M.N = \{ z \mid z \in G, z = xy \ \forall x \in M, \ \forall y \in N\}$.

DEFINITION 1. Let $M \subset G$ then the centralizer of M in G is

$$\mathcal{C}_G(M) = \{ x \mid x \in G, \ xy = yx \quad \forall y \in M \}$$

Clearly $\mathcal{C}_G(M)$ is the set of elements in G which commute with all the elements of M. Further $\mathcal{C}_G(M)$ is a subgroup of G under the same composition law. If G were an algebra, then $\mathcal{C}_G(M)$ would be called the <u>commutant</u> of M in G, and would be an algebra.

DEFINITION 2. The set $\mathcal{N}_G(M) = \{ x \mid x \in G, \ x \cdot M = M \cdot x \}$ is the <u>normalizer</u> of M in G.

It is an immediate consequence of these definitions that $\mathcal{C}_G(M) < \mathcal{N}_G(M) < G$. When $M = G$, $\mathcal{C}_G(G)$ is called the centre of G and is an abelian subgroup.

DEFINITION 3. K is an invariant subgroup of G if $\mathcal{N}_G(K) = G$. When K is an invariant subgroup we shall use the notation $K \lhd G$.

Exercise 1. (a) $\mathcal{C}_G(M) \lhd \mathcal{N}_G(M)$, $M \subset G$.
(b) Let M, $N < G$ then $M \cap N < G$
(c) Let M, $N \lhd G$ then $M \cap N \lhd G$

See the proof in the first pages of any text book on groups.

Consider now $M \subset G$. M generates a subgroup of G which can be found by taking products of all possible products of elements in M until we arrive at a set which is closed under this operation. Let us denote this set by $\{M\}$, the subgroup generated by M. A more precise way to define $\{M\}$ would be to consider $K_i < G$ where each K_i contains M. Then $\{M\} = \bigcap_i K_i$, $\{M\} < G$ and is the smallest subgroup of G containing M.

LEMMA 1. If H, $K < G$ then $H.K = K.H \longleftrightarrow HK < G$.

Exercise 2. (a) Prove Lemma 1. (See proof for instance in Kurosch)
(b) If $H < G$, $K \lhd G$ then $H.K = K.H < G$ (corollary of (a))
(c) If H, $K \lhd G$ then $H.K \lhd G$. (easy)

I.1 MAPPINGS AND EQUIVALENCE RELATIONS.

Given two point sets E and E' (disregard any algebraic or topological structure) a mapping f from E to E', written

$$E \xrightarrow{f} E'$$

is an association to each $x \in E$ of an element $x' = f(x) \in E'$. E is the domain of f and $\operatorname{Im} f = \{x' \mid x' \in E',\ x' = f(x), x \in E\}$ the image of f. Im f is a subset of E'. We shall require that f be single-valued i.e. to each $x \in E$ corresponds only one $f(x) \in E'$.

Various properties of f which deserve special comment are:

1. If Im f = E' f is said to be surjective (onto).
2. If $f(x) = f(y) \rightarrow x = y \quad \forall x, y \in E$, then f is injective (one to one).
3. If f is injective and surjective, f is said to be bijective.

The identity mapping I of the set E is the bijective map

$E \xrightarrow{I} E$ with $I(x) = x, \quad \forall x \in E$.

Mappings may also be composed. Consider $E \xrightarrow{f} E'$, $E' \xrightarrow{g} E''$ then the composition of g and f, $g_o f$ is a map $E \xrightarrow{g_o f} E''$ such that $g_o f(x) = g[f(x)], \forall x \in E$.

Given three maps f, g and h, an important property of their composition is its associativity $h_o(g_o f) = (h_o g)_o f$.

Exercise 3. (a) Given two maps $E \xrightarrow{f} E'$, $E' \xrightarrow{g} E$ show that $g_o f = I_E \rightarrow g$ surjective, f injective. (easy)
(b) If furthermore $f_o g = I_{E'}$, then both f and g are bijective and g is the inverse map of f and is denoted f^{-1}.

An application of the mappings of a set E is a natural definition of equivalence classes.

DEFINITION 4. $\forall x, y, z \in E$ an equivalence relation is said to hold on E if

(i) $x \sim x$
(ii) $x \sim y \rightarrow y \sim x$
(iii) $x \sim y,\ y \sim z \rightarrow x \sim z$

where $x \sim y$ means x equivalent to y. The set of elements equivalent to a given $x \in E$ is called the equivalence class with representative x. By (iii) an equivalence class is independent of which member is chosen as a representative.

In particular for a set E, an equivalence class would be the elements of a given subset. Let E' be the set of subsets of E, and f a map from E to E', f maps elements of E into the subset (equivalence class) containing them. Then $\forall x, y \in E$, $x \sim y$ if $f(x) = f(y)$ is an equivalence relation.

Exercise 4. Prove that the above equivalence satisfies Definition 4.

Let K be the set of equivalence classes in E defined by this relation. If we associate to each $x \in E$ the equivalence class to which it belongs we have a map f_1 from E to K, such that the following sequence of mappings

$$E \xrightarrow{f_1} K \xrightarrow{g} \text{Im } f_1 \xrightarrow{f} E'$$

has f_1 surjective, g bijective, and f injective. Further if the elements of K are K_i, then

$$E = \bigcup_i K_i, \quad K_i \cap K_j = \emptyset \qquad i \neq j.$$

More generally, given a map $E \xrightarrow{f} E'$, the relation between x and y : $f(x) = f(y)$ is an equivalence relation. We shall denote $f^{-1}(x')$ the equivalence class whose image by f is $x' \in E'$. This notation is useful, but note that f^{-1} is not in general a map (except if f is bijective).

Equivalence relations on E then give in a natural way the decomposition of E into mutually disjoint equivalence classes. This may now be applied to a group G.

DEFINITION 5. Let $H < G$ and $[G : H]_L$ be the set of equivalence classes in G defined by the relation

$$\forall x, y \in G \quad x \sim y \text{ if } x^{-1} y \in H.$$

That this is an equivalence relation may be verified directly

(i) $x \sim x \rightarrow$ the identity is in H. This is true since H is a subgroup.

(ii) $x \sim y \rightarrow x^{-1} y H$, but $(x^{-1}y)^{-1} = y^{-1} x$ is also in $H \rightarrow y \sim x$.

(iii) $x \sim y$, $y \sim z \rightarrow x^{-1}y$, $y^{-1}z \in H$ or $(x^{-1}y)(y^{-1}z) = x^{-1}z \in H \rightarrow x \sim z$

Let x be in a given equivalence class, then $\forall y$ in this class $x^{-1}y \in H$ or $y \in xH$. We say that y lies in the left coset of H by x. $[G:H]_L$ is just the set of left cosets of H in G, and the equivalence class decomposition of G is

$$G = \bigcup_i x_i H, \ x_i \text{ representatives in G for the left cosets of H, elements of } [G:H]_L .$$

Similar results hold for right cosets.

Define a map $G \xrightarrow{f} [G:H]_L$ by $f(x) = xH, \forall x \in G$. Consider a representative in G for each left coset in $[G:H]_L$, this defines a map $[G:H]_L \xrightarrow{k} G$. Clearly

$$f_0 k = I \quad \text{identity in } [G:H]_L$$

but $k_0 f \neq i$. Such k is called a section of G for the base $[G:H]_L$.

I.2 HOMOMORPHISMS OF GROUPS.

Let G and G' be two abstract groups and f a map $G \xrightarrow{f} G'$.

DEFINITION 6. If $x, y \in G$, $f(x)\, f(y) = f(xy)$, f is a homomorphism from G to G'.

The essential point is that as a mapping a homomorphism preserves the group law. When G = G', then f is called an endomorphism.

DEFINITION 7. The set Ker $f = \{x \mid x \in G \text{ and } f(x) = i\}$ is called the Kernel of f.

Both Ker f and Im f are subgroups of G and G' respectively.

Exercise 5. (a) Show that Ker $f \lhd G$
(b) Under what conditions is Im $f \lhd G'$
(c) $H' \lhd G' \rightarrow f^{-1}(H') \lhd G$. (Ker f is the particular case Ker $f = f^{-1}(i_{G'})$)

Important special cases of homomorphisms deserve comment

1. f injective, Ker f = i_G and vice versa.
2. f surjective, Im f = G'
3. If Ker f = i_G and Im f = G' then f is an isomorphism and G = G'.

EXAMPLES. An endomorphism can be injective without being surjective. For instance we denote **Z** the additive abelian group of the integer and

$$\mathbf{Z} \xrightarrow{f} \mathbf{Z}$$

is a map defined by f(p) = 2p, $\forall p \in \mathbf{Z}$. f is an endomorphism and ker f = 0, but Im f = 2**Z**, the even integer which is a proper subgroup of **Z**.

A typical method in the theory of groups is to partition G into equivalence classes defined by a homomorphism f. Let K = Ker f, then $K \triangleleft G$ and xK = Kx, $\forall x \in G$. Also $[G:H]_L = [G:K]_R = [G:K]$, that is the equivalence classes of right and left cosets of K are identical. Denote

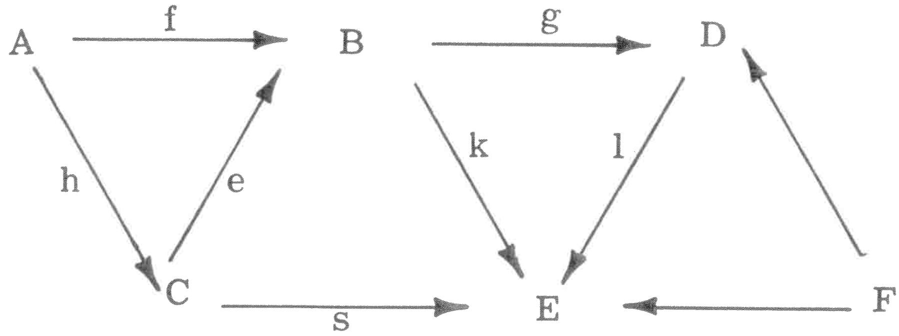
$$[G:K] = G/K$$

and define a multiplication in G/K by (xK) (yK) = (xyK). Then $G \xrightarrow{f} G/K$ is the canonical homomorphism between G and its factor group by K, Im f = G/K. Conversely given any $K \triangleleft G$ we can define a surjective homomorphism from G to G/K such that K is the kernel.

I.3 EXACT SEQUENCES AND COMMUTATIVE DIAGRAMS.

In the following the identity in a given group G will be denoted by i. When G is abelian additive notation may be used, with 0 for the unit.

Given a collection of groups A, B, C,..., G,... with homomorphisms f, g, h,... between them, it frequently happens that not all these homomorphisms are independent, and the relations between them may be represented by a diagram



A diagram is commutative if the homomorphisms given by composition of maps corresponding to a given sequence of arrows are independent of the path, e.g. commutativity of the above diagram requires

$$f = e_0 h, \quad k_0 f = s_0 h \quad k_0 e_0 h = l_0 g_0 f, \quad \text{etc. ...}$$

Frequently a given commutative diagram is not complete in the sense that the given homomorphisms imply further homomorphisms, or a subset of them is implied by a smaller subset of maps. We shall use $\rightsquigarrow$ for implied homomorphisms, and call a diagram complete if no further $\rightsquigarrow$ lines can be inserted.

It may happen that a collection of groups G_1, G_2,..., G_n,... and homomorphisms f_1, f_2,... such that

$$G_n \xrightarrow{f_n} G_{n+1}$$

have the property

$$\text{Im}\, f_{n-1} = \text{Ker}\, f_n \;, \; \forall_n \,.$$

In this case we say that f_1, f_2,..., f_n,... is an exact sequence of homomorphisms and write diagramatically

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots \longrightarrow G_{n-1} \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} G_{n+1} \xrightarrow{f_{n+1}} \cdots$$

EXAMPLES. 1. If $K \lhd G$ and $G \xrightarrow{f} G/K$ we can write

$$1 \to K \xrightarrow{i} G \xrightarrow{f} G/K \to 1$$

where $K \xrightarrow{i} G$ is the injection map, i.e. $\forall x \in K$, $i(x) = x \in G$. An exact sequence of this type is called short exact. The diagram $1 \to K \xrightarrow{f} G$ means that Ker f = 1, i.e. f is injective. $1 \to K$ is the injection of the unit into K. f is surjective is expressed by $G \xrightarrow{f} K \to 1$

2. $1 \to K \xrightarrow{f} G \to 1$ gives f injective i.e. K = G.

3. For any homomorphism f there is always the short exact sequence

$$1 \to \mathrm{Ker} f \xrightarrow{i} G \xrightarrow{f} \mathrm{Im}\, f \to 1 .$$

From now on, every diagram written shall be commutative and every sequence of homomorphisms represented by arrows on the same straight line shall be exact.

Application to Abelian Groups

Consider an abelian group A and $\forall a \in A$ write

$$\overbrace{a + a + \ldots + a}^{n} = na .$$

Define a homomorphism from A to A by $a \to na$, i.e.

$$A \xrightarrow{n} A , \quad \mathrm{Im}\, n = nA.$$

The subgroup of elements of order n in A is the kernel of N, we write Ker n = ${}_nA$. These properties may be written

$$0 \to {}_nA \to A \xrightarrow{n} A$$

For the infinite cyclic group generated by one element **Z** (additive group of integers) Im n = n**Z** ◁ **Z**. Denote **Z**/n**Z** = $\mathbf{Z}_n$. Then

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \xrightarrow{f} \mathbf{Z}n \to 0$$

Let U_1 be the one dimensional unitary group, and $U_1 \xrightarrow{n} U_1$ means

$$\forall e^{i\phi} \in U_1 \xrightarrow{n} e^{in\phi} \in U_1,$$

ϕ a real number modulo 2π. For a given n denote the n-th roots of unity by

$$_nU_1 = \{ e^{i2\pi/n} , \text{ n fixed } \}$$

Then

$$0 \rightarrow {}_nU_1 \rightarrow U_1 \xrightarrow{n} U_1 \rightarrow 0$$

There is a relation between U_1 and **Z** given by:

Exercise 6. Give definitions for the homomorphisms in

$$0 \rightarrow {}_n\mathbf{Z} \rightarrow U_1 \xrightarrow{n} U_1 \rightarrow 0$$

Returning to A, the <u>cokernel</u> of n is defined by $A_n = A/nA$ and prove:

$$0 \rightarrow {}_nA \rightarrow A \xrightarrow{n} A \xrightarrow{f} A_n \rightarrow 0.$$

This shows that exact sequences may be combined into a longer one; indeed the previous exact sequence is equivalent to the two following:

$$0 \rightarrow {}_nA \rightarrow A \xrightarrow{n} nA \rightarrow 0$$

$$0 \rightarrow nA \rightarrow A \xrightarrow{f} A_n \rightarrow 0$$

Application to Automorphisms.

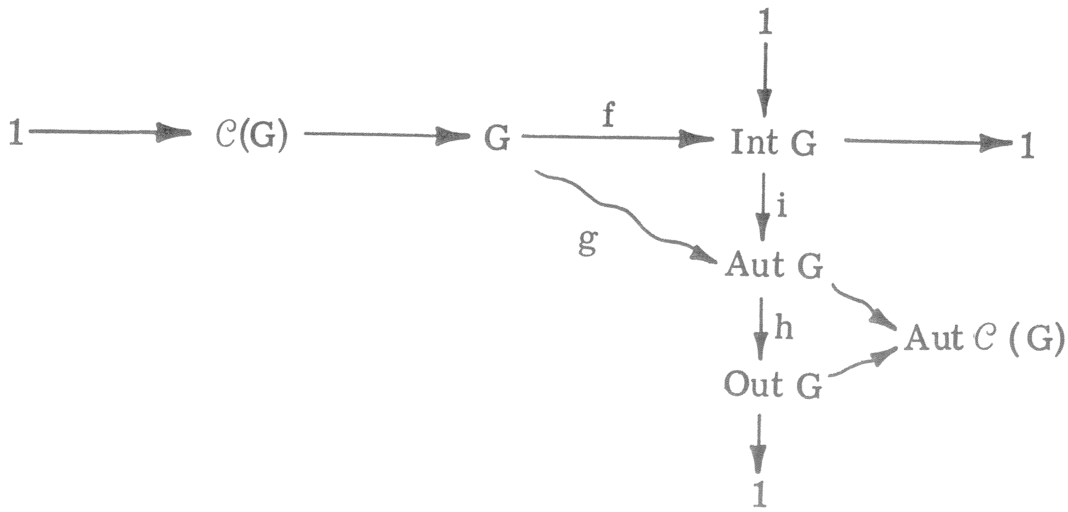
Recall that an automorphism of G is a bijective endomorphism, and among these are a special class, the inner automorphisms. Let Int G be the set of all such inner automorphisms, i.e.

$$\forall \alpha \in \text{Int } G, \ \exists a \in G : \forall x \in G \ \ x \xrightarrow{\alpha} a\,x\,a^{-1}$$

Int G becomes a group under composition of inner automorphisms. Denote the set of all automorphisms of G by Aut G. This also is a group under composition.

Exercise 7. Show that Int G $\lhd$ Aut G. (Proof in any text book).

The elements of Int G map G → G in a natural way, and furthermore if to each $a \in G$ we associate an element of Int G by taking the inner automorphism $a\,x\,a^{-1}$, $\forall\, x \in G$ thus defines a map G → Int G. f is a homomorphism under composition of elements of Int G. Ker f is the set of elements $a \in G$, such that $a\,x\,a^{-1} = x$, $\forall x \in G$. This is just the centre of G. All of these statements are contained in the diagram



from which we find Int G = G / $\mathcal{C}$(G), and Out G = Aut G/Int G. The last defines the group of class of outer automorphisms of G modulo the inner automorphisms. These are all the automorphisms which are not inner and are only determined up to inner automorphisms.

Given that $g = i \circ f$ we deduce

$$1 \to \mathcal{C}(G) \to G \xrightarrow{g} \text{Aut } G \xrightarrow{h} \text{Out } G \to 1$$

requiring Im g = Int G. If g is injective, $\mathcal{C}(G) = 1$. While if g is surjective, Int G = Aut G and G has only trivial outer automorphisms. Thus, $\mathcal{C}(G) = 1$ and Out G = 1 imply that g is an isomorphism.

DEFINITION 8. An abstract group G is complete if $\mathcal{C}(G) = 1$, Out G = 1.

We have seen that for complete groups there is a natural isomorphism between the group and its automorphism group.

EXAMPLES. 1. The rotation group in three dimensions, S O (3), is a complete group.
2. The group of permutations of n objects for $n > 2$, $n \neq 6$ is a complete group.
3. The group of automorphisms of the Poincaré group is complete. (for a proof see L. Michel's lectures in Cargese).

DEFINITION 9. A subgroup $K < G$ is called a characteristic subgroup of G if K is globally invariant under all automorphisms of G.

By this we mean the following. If $\alpha \in$ Aut G, $x \in G$, denote the action of α on x by x^α, i.e. $x \xrightarrow{\alpha} x_\alpha$. Then

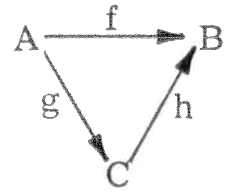
$$\forall_\alpha \in \text{Aut}\, G, \ \forall x \in K < G, \ x^\alpha \in K$$

EXAMPLES. 1. $\mathcal{C}(G)$, the center of G, is characteristic subgroup of G.
2. If $K \lhd G$ and if there is no other invariant subgroup of G isomorphic to K, then K is characteristic subgroup of G. For instance, T the translation subgroup of the Poincaré group.

Exercise 8. Show that $\mathcal{C}(G) = 1 \rightarrow \mathcal{C}(\text{Aut}\, G) = 1$. (See e.g. Kurosch).

Some Results for Commutative Diagrams.

Consider the diagram



If it is commutative then $f = h_{\circ} g$.

This is equivalent to an exact sequence obtained as follows. Clearly

$$\text{Ker } g \subset \text{Ker } f \text{ and } \text{Ker } g \xrightarrow{\text{injection}} \text{Ker } f.$$

Let

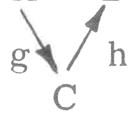
$$\emptyset = g_{|\text{Ker } f}$$

the restriction of g to the domain Ker f, then by commutativity

$$f(\text{Ker } f) = \text{Im } h_{\circ} \emptyset = 0 \;\rightarrow\; \text{Im } \emptyset \subset \text{Ker } h.$$

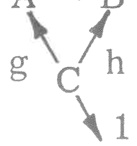
Thus $\emptyset$ defines a map from Ker f to Ker h with Ker $\emptyset$ = Ker g. The reader may verify that

Exercise 9. (a) $A \xrightarrow{f} B \rightarrow 1 \rightarrow \text{Ker } g \xrightarrow{i} \text{Ker } f \xrightarrow{\emptyset} \text{Ker } h$



is exact.

(b) $A \xrightarrow{f} B \rightarrow 1 \rightarrow \text{Ker } g \rightarrow \text{Ker } f \rightarrow \text{Ker } h \rightarrow 1$



(Proofs in Istanbul p. 157 and 172).

LEMMA 2. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \rightarrow & 1 \\ & & a\downarrow & & \downarrow b & & \downarrow c & & \\ 1 & \rightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C' & & \end{array}$$

This diagram is complete.

PROOF: We have to show how to construct the homomorphism c so that $c_{\circ} p = p'_{\circ} b$. Denote group elements by

$\alpha, \beta, \gamma \in$ A, B, C and α', β', γ' A', B', C' respectively.

1. p is surjective $\rightarrow \forall\gamma \in C, \exists\beta \in B : p(\beta) = \gamma$. Let $\beta_1, \beta_2 \in B : \gamma = p(\beta_1) = p(\beta_2)$ then $\beta_2^{-1}\beta_1 \in$ Ker p.

2. Ker p = Im i $\rightarrow \beta_2^{-1}\beta_1 = i(\alpha)$ for some $\alpha \in$ A. Since i is injective, α is unique.

3. By commutativity of the diagram $p' \circ b \circ i = p' \circ i' \circ a$ hence

$$p' \circ b \circ i(\alpha) = p' \circ b(\beta_2^{-1}\beta_1)$$

$$p' \circ b(\beta_1) = p' \circ b(\beta_2)$$

Thus the map $c \circ p = p' \circ b$ is well-defined and can be used to define c. That c is a homomorphism is easily seen, since by 1. $\forall\gamma \in C, \exists\beta \in B : \gamma = p(\beta)$

Now $\forall \gamma_1, \gamma_2 \in C$

$$c(\gamma_1\gamma_2) = c \circ p(\beta_1\beta_2) = p' \circ b(\beta_1\beta_2) = p' \circ b(\beta_1)\, p' \circ b(\beta_2) =$$
$$= c(\gamma_1)\, c(\gamma_2)$$

where commutativity of the diagram has been used.

COROLLARY 1. p', b surjective $\rightarrow$ c surjective.

COROLLARY 2. b injective, a surjective $\rightarrow$ c injective.

Exercises (a) Prove corollary 1.
(b) Prove corollary 2. This proof is equivalent to the "Five Lemma" of Cartan and Eilenberg: Hilton and Wylie - Algebraic Topology page 208.

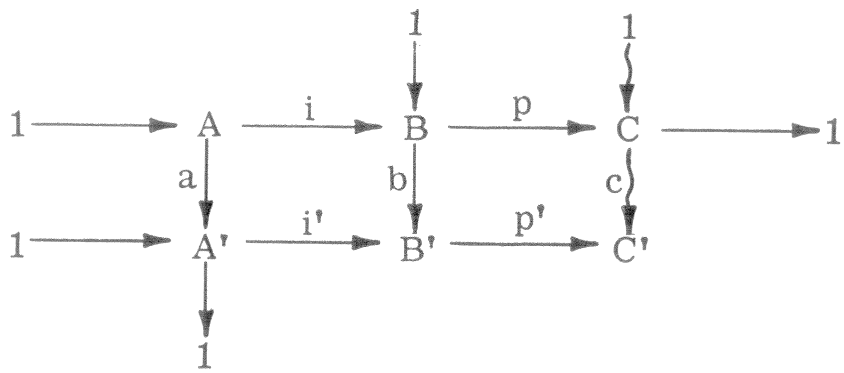
For further results on commutative diagram the reader should look at

1. L. Michel's article in the Istanbul Summer School.
2. Hilton and Wylie - Algebraic Topology § 5.3, 5.6.
3. Cartan and Eilenberg - Homological Algebra Chapter 1, Princeton University Press.

4. Mac Lane: Homology (Springer).
5. Any book on "Homological Algebra", the simplest one seems that of Northcott (Cambridge University Press).

PROOF OF COROLLARY 1. By 1. in Lemma 2 Im c = Im $(c_{\circ}p)$ = Im $(p'_{\circ}b)$ = C'.

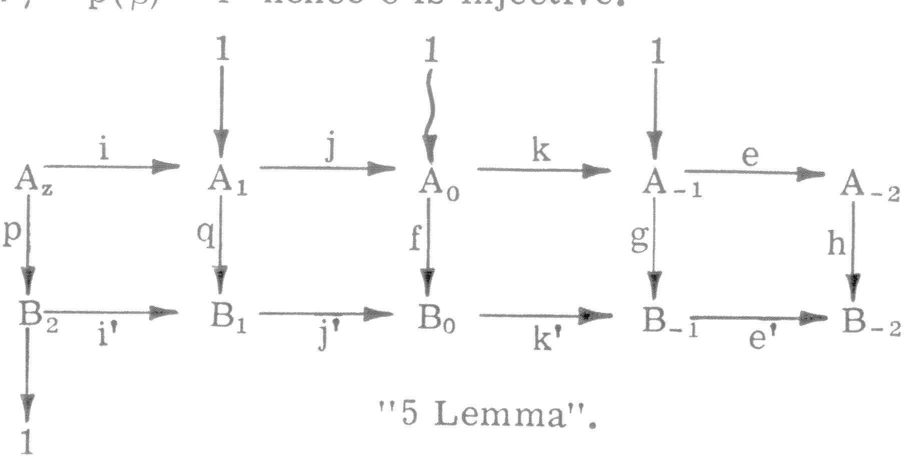
PROOF OF COROLLARY 2. We wish to prove the diagram



Since b is injective so is $b_{\circ}i$. Then by commutativity $\mathrm{Ker}(i'_{\circ}a) = 1$. But Ker $a \subset \mathrm{Ker}(i'_{\circ}a) \rightarrow$ a is injective. Thus a is bijective and A = A'. In the following we shall not distinguish A and A'.

$$\forall \gamma \in \mathrm{Ker}\ c,\ \exists \beta \in B : \gamma = p(\beta) \text{ and } \beta \in \mathrm{Ker}\,(c_{\circ}p) =$$
$$= \mathrm{Ker}\ (p'_{\circ}b)$$

Hence $b(\beta) \in$ Ker p' = Im c'. Since c', a, b, c are injective, so is $c'_{\circ}a = b_{\circ}c$. Furthermore a is bijective, so there is a unique $\alpha \in A$ such that $c'_{\circ}\, a(\alpha) = b_{\circ}\, c(\alpha)\ b(\beta)$, $c(\alpha) = \beta$, and $\gamma = p(\beta) = i$ hence c is injective.



"5 Lemma".

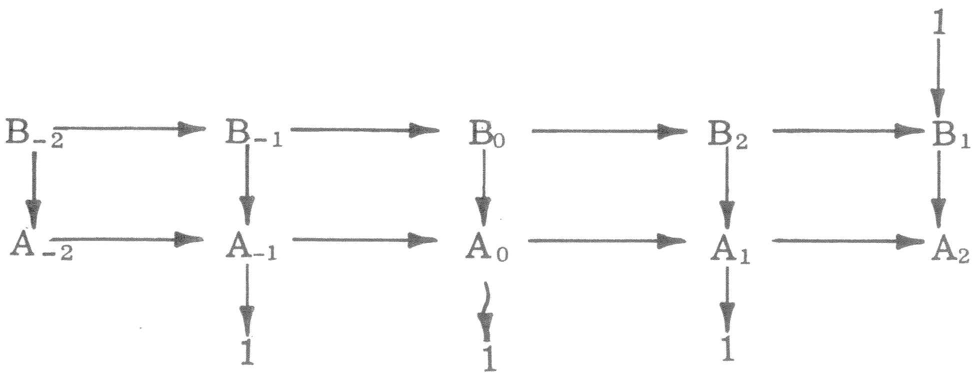
PROOF: Let $\alpha_0 \in$ Ker f then $\alpha_0 \in$ Ker $(k' \circ f)$ = Ker $(g \circ k)$ and $K(\alpha_0) \in$ Ker g = 1 $\rightarrow \alpha_0 \in$ Ker k. $\therefore$ $\alpha_0 = j(\alpha_1)$ some $\alpha_1 \in A_1$.

Now $\alpha_1 \in$ Ker $(j' \circ q)$ and $q(\alpha_1) \in$ Ker j', i.e. $\exists \beta_2 \in B_2 : q(\alpha_1) = i(\beta_2)$.

Since p is surjective $\exists \alpha_2 \in A_2 : p(\alpha_2) = \beta_2$ and $q(\alpha_1) = i' \circ p(\alpha_2) = q \circ i(\alpha_2)$.

But since q is injective $\alpha_1 = i(\alpha_2)$ and $\alpha_0 = j \circ i(\alpha_2) = 1$.

Note that the proof depends only on the diagram structure, but not on the particular group law. It is also valid for any kind of mathematical structure whose "morphisms" can form exact sequences. Note also that by "reversing the arrows" of a diagram one obtains a "dual" diagram; so each theorem has its dual. For instance, "dual 5 Lemma" is



Try to prove it directly.

I.4 THE ISOMORPHISM THEOREMS.

We shall meet often the situation of a group G with two distinct invariant subroups H and K. What can we deduce from this. Let us first consider the particular case $K < H$

THEOREM 1. (Second Isomorphism Theorem). Let G be a group and $K \lhd G$, $K < H \lhd G$, then

$$G/H = (G/K)/(H/K).$$

PROOF: Lemma 2, corollary 1 yields

$$\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & K & \xrightarrow{i} & G & \xrightarrow{p} & G/K & \longrightarrow & 1 \\
 & & \downarrow a & & \downarrow I & & \downarrow c & & \\
1 & \longrightarrow & H & \xrightarrow{i'} & G & \xrightarrow{p'} & G/H & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

and Exercise 9b tells us $1 \to \text{Ker } p \to \text{Ker } p' \circ I \to \text{Ker } c \to 1$
Hence

$$\text{Ker } c = H/K \quad \text{so} \quad \frac{G}{K} \Big/ \frac{H}{K} = \frac{G}{H}$$

In the general case, $K \lhd G$, $H \lhd G$ have a non trivial intersection $H \cap K$ which is also invariant subgroup of G. From Exercise 2c we also know $H \circ K \lhd G$.

THEOREM 1'. (Second isomorphism theorem). If

$$H \,\Delta\, G,\ K \,\Delta\, G, \ \text{then} \ \frac{H}{H \cap K} = \frac{H.K}{K}$$

By symmetry

$$\frac{K}{H \cap K} = \frac{H.K}{H}$$

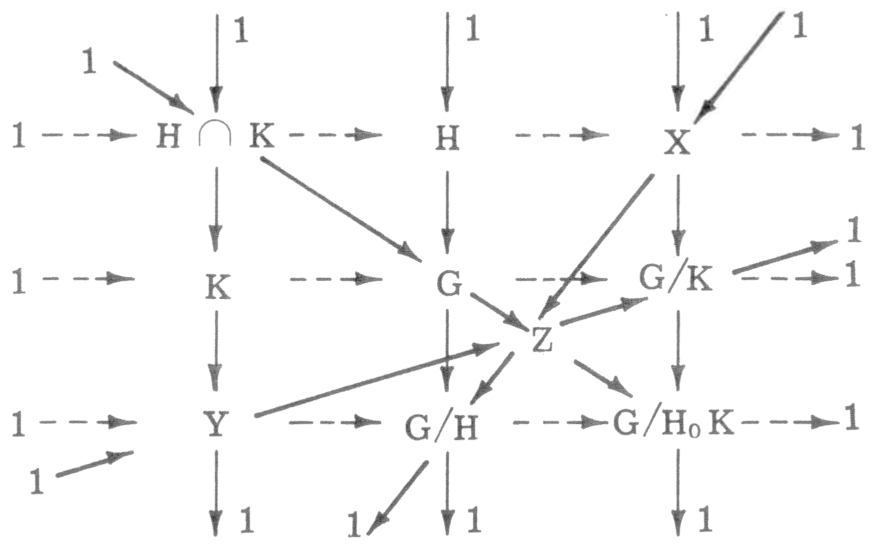
PROOF: Lemma 2 tells us

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & H \cap K & \longrightarrow & H & \xrightarrow{p} & \dfrac{H}{H \cap K} & \longrightarrow & 1 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
1 & \longrightarrow & K & \longrightarrow & H.K & \xrightarrow{p'} & \dfrac{H.K}{K} & \longrightarrow & 1
\end{array}$$

Let $\gamma' \in H.K/K$; there exist $h \in H$, $k \in K$ such that $p'(i'(k) \cdot b(h)) = \gamma'$ hence $\gamma' = p' \circ b(h) = c \circ p(h)$ so c is surjective. Furthermore $\text{Ker } c \circ p = \text{Ker } p' \circ b = (H \cap K) =$ $= \text{Ker } p$. From Exercise 7.b $1 \to \text{Ker } p \to \text{Ker } c \circ p \to \text{Ker } c \to 1$, so c is injective, hence the "second isomorphism".

The whole information concerning the general situation is condensed in the following diagram.

THEOREM 2. If $K \lhd G$, $H \lhd G$, then the diagram 1.



with $X = \frac{H}{H \cap K} = \frac{H.K}{K}$

$Y = \frac{K}{H \cap K} = \frac{H.K}{H}$

$Z = \frac{G}{H \cap K}$

diagram 2

Theorem 2 is made by the combination of all subdiagrams given by Theorems 1 and 1' or Lemma 2. We leave to the reader to check that the union of these commutative diagrams still makes a commutative diagram.

I.5 DIRECT AND SEMI-DIRECT PRODUCT.

Our next consideration is the study of two methods for forming a new group from two arbitrary groups A and B.

DEFINITION 10. The direct product of A and B is $G = A \times B = \{(a,b) \mid \forall a, a' \in A, b, b' \in B\}$ $(a,b)(a', b') = (aa', bb')$. G is a group under the indicated composition law.

The subgroups $A_1 = \{(a, 1)\} < G$, $B_1 = \{(1, b)\} < G$ are isomorphic to A and B respectively. Further $A_1, B_1 \lhd G$ since $\forall x \in G$, $x A_2 x^{-1}$ is the set of elements of the form $(a,b)(a_1, 1)(a^{-1}, b^{-1}) = (a\, a_1 a^{-1}, 1) \in A_1$; so

LEMMA 3. G is isomorphic to $A \times B \rightarrow A \lhd G$, $B \lhd G$, $A \cap B = 1$, and $G = A.B$.

Let us prove the converse lemma. If we feed its hypothesis into the diagram of Theorem 2, this diagram simplifies to :

LEMMA 3' diagram 3 $\rightarrow G \sim A \times B$.

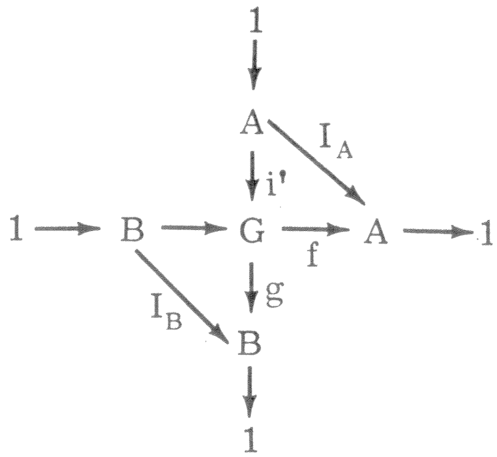


diagram 3

Indeed consider the homomorphism $G \xrightarrow{f \times g} A \times B$ which transforms $x \in G$ into $(f(x), g(x)) \in A \times B$. Any x is $\in \operatorname{Ker} f \times g$ if and only if $f(x) = 1$, $g(x) = 1$ so $x \in \operatorname{Ker} f \cap \operatorname{Ker} g$. Hence for such an x there exist $a \in A$, $B \in b$ such that $i(b) = i'(a)$. So $1 = f_o i(b) = f_o i'(a) = I(a) = a$ hence $a = 1$; similarly $b = 1$, hence $f \times g$ is injective. It is also surjective; given $a \in A$, $b \in B$, then $x = i'(a)i(b) \in G$ is such that $(f \times g)(x) = (a, b) \in A \times B$. Indeed $(f \times g)(i'(a)i(b)) = (f_o i'(a) \cdot f_o i(b),\ g_o i'(a) \cdot g_o i(b)) = (a,b)$. So $f \times g$ is an isomorphism.

DEFINITION 11. Given two groups A and B and a homomorphism $B \xrightarrow{g} \operatorname{Aut} A$, the semi-direct product $G = A \wedge B$ is the group whose set of elements if the set product of the set A by the set B, (i.e. elements of G are pairs (a, b), $a \in A$, $b \in B$) and whose group law is

$$(a_1, b_1)(a_2, b_2) = (a_1\, a_2^{\,b_1},\ b_1 b_2)$$

where $a_2^{\,b_1} = g(b_1)\,[a_2]$, the transformed of a_2 by the automorphism $g(b_1)$.

We leave to the reader to verify that this law is a group law, that $(a,b)^{-1} = ((a^{-1})^{b_1}, b^{-1}) = ((a^{b_1})^{-1}, b^{-1})$, that the elements $(a,1)$, $a \in A$, form an invariant subgroup of G, (indeed $(a,b)(a_1,1)((a^{-1})^{b_1}, b^{-1}) = (a\,a_1^{\,b}\,a^{-1}, 1)$), that the elements $(1,b)$, $b \in B$, form a subgroup of G.

In other words,

LEMMA 4. $G = A \wedge B \rightarrow A \lhd G,\ B < G,\ A \cap B = 1,\ A.B = G$
We will express the converse lemma in diagram language.

LEMMA 4'. diagram 4 $\rightarrow G = A \wedge B$

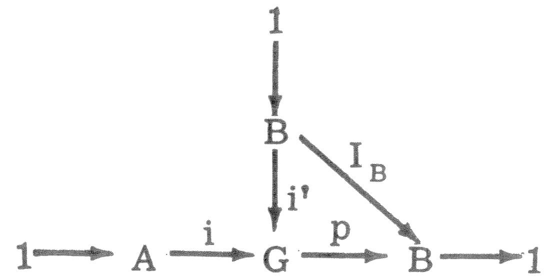


diagram 4

Indeed $i(A) \cap i'(B) = 1$ for if $i(a) = i'(b)$, $p\circ i(a) = 1 =$ $= p\circ i'(b) = b$ so $b = 1$ and since i injective, $x = 1$. So $\forall\ x \in b$, there exists a unique $a \in A$ and a unique $b \in B$ such that $x = i(a)i'(b)$. Indeed, given x, put $b =$ $= p(x)$ and $i(a) = x\cdot(i'(b))^{-1} \leftarrow$ Ker p. Since $i(A) \lhd G$, it is stable by inner automorphism of G, i.e., there is a homomorphism $G \xrightarrow{f} \mathrm{Aut}\,A$ such that $x\,i(a)\,x^{-1} = i\circ f(x)[a]$. We denote $g = f\circ i'$ so $i'(b)i(a)i'(b^{-1}) = i\circ g(b)[a] = i(a^{b})$. Then the group law of G is $x_1 x_2 = i(a_1)i'(b_1)i(a_2)i'(b_2) =$ $= i(a_1)i(a_2^{\,b_1})i'(b_1)i'(b_2)$ which is just the semi-direct product group law.

Examples of Semi-Direct Product.

Ex. 1. The Poincaré group $P = T \wedge L$ where T is the translation group (elements a, b, ...)(and L the homogeneous Lorentz group (elements Λ, M, ...)). We chose the usual additive notation for T, the group law of P is

$$(a, \Lambda)(b, M) = (a + \Lambda b, \Lambda M)$$

since physicists prefer the notation Λb to b^{Λ}.

Ex. 2. More generally in an n + m dimensional Euclidean space with pseudo metric

$$\| x \|^2 = \sum_{i=1}^{n} x_i^2 - \sum_{j=1+n}^{m+n} x_j^2 \quad x = (x_1, \ldots, x_{n+m})$$

Let $0(n, m)$ be the n + m dimensional pseudo-orthogonal group preserving this metric, then the corresponding displacement group is the semi-direct product

$$P(n, m) = T_{n+m} \wedge 0(n, m) .$$

Another example is given by Exercise 10. Note that the direct product K x Q is a particular case of the semi-direct product $K \wedge Q$ where $g : Q \xrightarrow{g} \text{Aut } K$, is the trivial homomorphism.

Exercise 10. Let $A \lhd G$ such that $B/A = \mathbf{Z}$, $\mathbf{Z}$ the infinite cyclic group = additive group of integers. Show that $G = A \wedge \mathbf{Z}$.

Hint; denote p: $G \xrightarrow{p} Z \to 1$, let $x \in G$ such that $p(x) = 1 \in \mathbf{Z}$ (with additive notation for $\mathbf{Z}$). Show that x generates a subgroup of G isomorphic to Z and that you get the diagram of Lemma 4'. The group Z is the simplest "free group". The exercise can be generalized to G/A = a "free group". (See a book for the definition of this notion)

I. 6 GROUP EXTENSIONS.

Given two groups K, Q we ask for all groups E such that $K \lhd E$ and $Q = E/K$. E is called an extension of Q by K, though this terminology is not entirely settled yet in the mathematical literature. (Some say extension of K by Q).

When E is determined, the inner automorphisms of E induce automorphisms of K in a natural way, i.e.,

$$\forall\ x \in E,\ \alpha \in K \quad \alpha \to \alpha^x = x \alpha x^{-1} = f(x)\alpha, \quad E \xrightarrow{f} \text{Aut } K$$

Stated diagramatically,

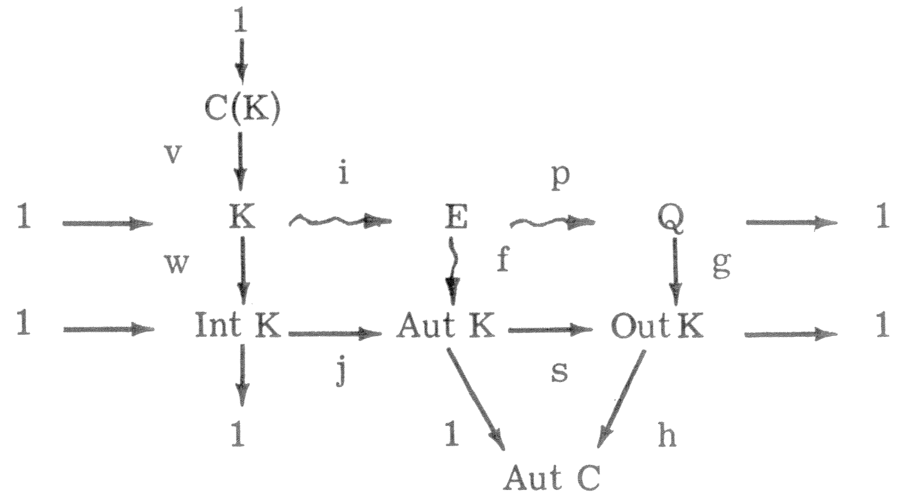


diagram 5

We may read (5) in two ways. Given the extension E, the natural map f determines g by Lemma 2, and (5) is determined.

So the problem of finding all group extensions of Q by K can be decomposed into sub-problems.

1. Find all homomorphisms g of Q into Out K. We call Hom (Q, Out K) this set. Among them, there is the trivial homomorphisms that are noted $g = 0$.

2. Given Q, K, $g \in$ Hom (Q, Out K) find all E which satisfy diagram 5.

How does one count the solutions? We have to say when two groups E are considered as the same solutions. There is a natural definition of equivalence of extensions (we will not explain here why it is natural) which exists for all kinds of mathematical structure.

DEFINITION 12. Equivalence of Extensions. Two extensions E and E' of Q by K are equivalent if there exists a homomorphism $E \xrightarrow{f} E'$ such that the diagram 6 is commutative.

Exercise 11. Show that this is an equivalence relation.

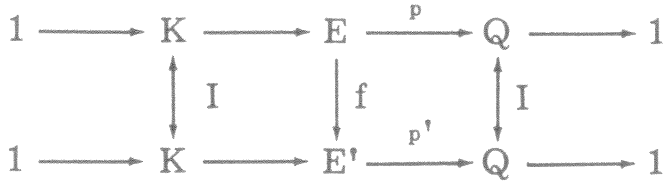


diagram 6

(The five lemma and its dual tells us that f is an isomorphism). Note that equivalent extensions are isomorphic. The converse is not generally true. See at the end of Chapter III for a counter example and the physical different meanings of isomorphic but not equivalent extensions.

The semi-direct product (and its particular case the direct product) are particular examples of extension. But in the general case of an extension E of Q by K there is no subgroup of E isomorphic to the quotient Q.

Such an example is given by SU_2 as an extension of SO_3 by Z_2:

$$1 \to Z_2 \to SU_2 \to SO_3 \to 1$$

SU_2 is the group of 2 by 2 unitary matrix of determinant 1. Its center Z_2 has two elements, the matrices 1 and - 1. Those are the only square-roots of the unit. The three dimensional rotation group SO_3 is isomorphic to SU_2/Z_2. This group has an infinity of square roots of 1: the rotations of π. (They are images in SU_2 of the zero trace matrices, which are the square roots of - 1). So SO_3 is not a subgroup of SU_2.

If L is the connected Lorentz group and SL(2, C) the group of two by two complex matrices of determinant 1, we similarly have

$$1 \to Z_2 \to SL(2,C) \to L \to 1.$$

We shall not repeat here the elementary lectures of Istanbul (1962) on group extensions and we shall try to minimize the overlap. We wish again to advise the reader to go back to the classic mathematical papers on this subject and we advise him especially:

S. Eilenberg and S. MacLane, Cohomology Theory in Abstract Groups I and II, Annals of Maths., 48, 51 and 326 (1947) and references given there, chiefly those of R. Baer.

For instance we shall not give the precise criterion for the existence of a solution to the extension problem for a given triplet K, Q, g: $Q \xrightarrow{g} \mathrm{Out}\, K$. Let us give here only a sufficient condition: the existence of a semi-direct product:

THEOREM 3. Given K, Q, $\xrightarrow{g}$ Out K, the necessary and sufficient condition for the existence of a semi-direct product as one extension K, Q, g is the existence of a homomorphism $r \in \mathrm{Hom}(Q, \mathrm{Aut}\ K)$ such that $g = r_0 s$ where s is the natural homomorphism Aut K $\xrightarrow{s}$ Out K $\to$ 1.

The condition is sufficient; indeed given Q $\xrightarrow{r}$ Aut K, we can form the corresponding semi-direct product and in diagram 5 the corresponding homomorphism g does satisfy $g = s_0 r$.

The condition is necessary. If there is as solution of diagram 5 a semi-direct product, then we can imbed in it diagram 4 (characteristic of a semi-direct product). We then obtain diagram 7 and then homomorphism for Q $\to$ Aut K is $r = f_0 c$ which does satisfy $g = s_0 r$

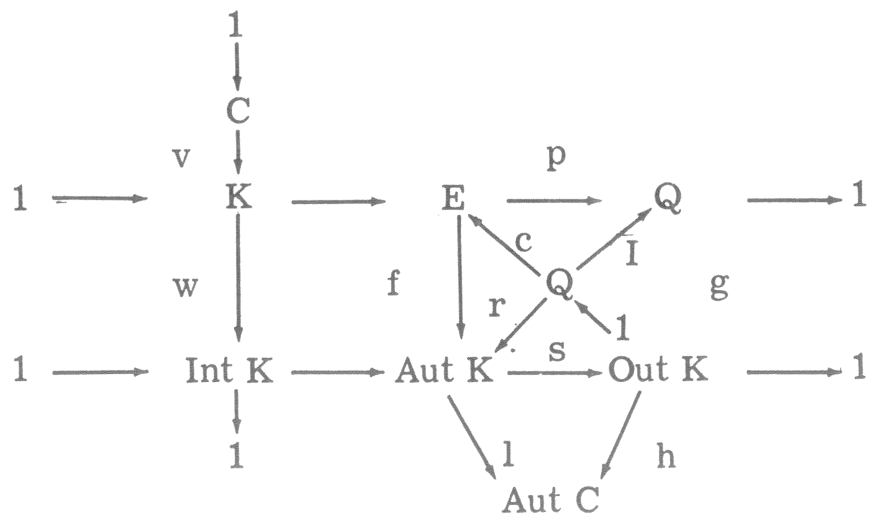


diagram 7

(We draw more diagrams than strictly necessary, but we feel this makes things clearer for the reader and helps him to memorize the situation).

Note that the condition of Lemma 5 is always satisfied if

1. Aut K = Out K, i.e., K is abelian.
2. Aut K = Int K $\wedge$ Out K. Which is the case of most of the groups met by physicists. For instance, compact Lie groups, semi-simple Lie groups. Then there is a

homomorphism k such that $s_0 k = I$ (of Out K), see diagram 8 and the corresponding homomorphism $r : Q \to \text{Aut } K$ is $r = k_0 g$. It satisfies $s_0 r = s_0 k_0 g = g$.

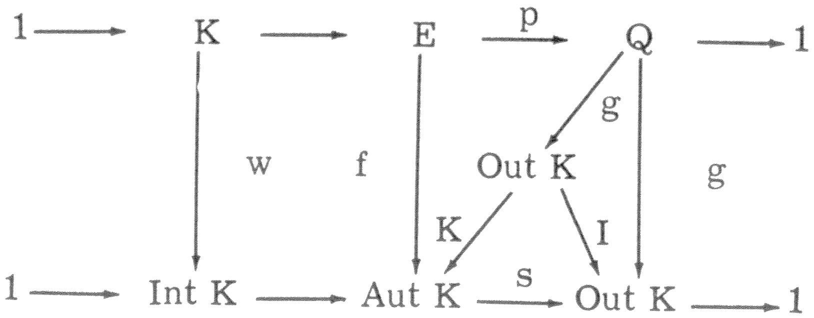


diagram 8

3. Note that it is enough that $s^{-1}(\text{Im } g)$ be a semi-direct product

$$s^{-1}(\text{Im } g) = \text{Int } K \wedge \text{Im } g.$$

Otherwise, if there are solutions for a triplet K, Q, g which do not satisfy the condition of Theorem 3, then none of them is a semi-direct product.

I.7 CENTRAL EXTENSIONS.

The only extension problem we shall study is that to find the central extension of Q by K.

DEFINITION 13. An extension E of Q by K is called central if the corresponding homomorphism $Q \xrightarrow{g} \text{Out } K$ is trivial.

In this case E is called a central extension, since if K were abelian then $K < \mathcal{C}(E)$. For central extensions the direct product K x Q is always a solution. This need not be the only one.

In (5) g trivial implies $\text{Im } f < \text{Ker } s = \text{Int } K$, thus $\forall\ x \in E$, $\forall \alpha \in K$ since $x \alpha x^{-1}$ is an automorphism of K,

$$\exists\ \xi \in K : x \alpha x^{-1} = \xi \alpha \xi^{-1} \quad \text{or} \quad \xi^{-1} x \in \mathcal{C}_E(K),$$

the centralizer of K in E. But $\xi^{-1} x \in$ Ker f hence Ker f= = H = $\mathcal{C}_E$ (K). Diagram 5 then reduces to diagram 9

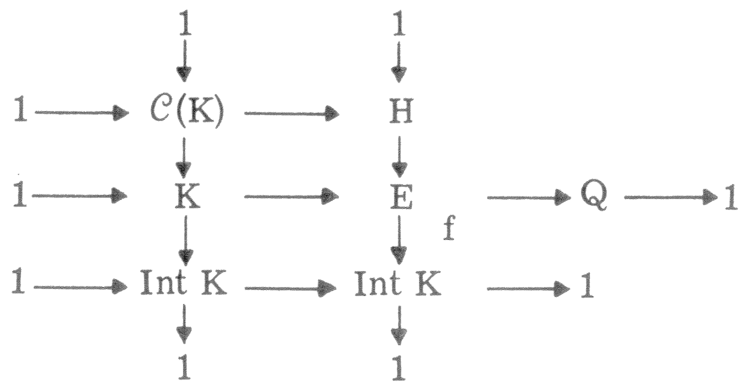


diagram 9

which gives Int K = E/H, and E is also an extension of Int K by H.

Furthermore, by definition of the center, K$\cap$C (K) = = K$\cap$H = $\mathcal{C}$(K), and K, H $\lhd$ E. By Theorem 2 we can extend (9) to

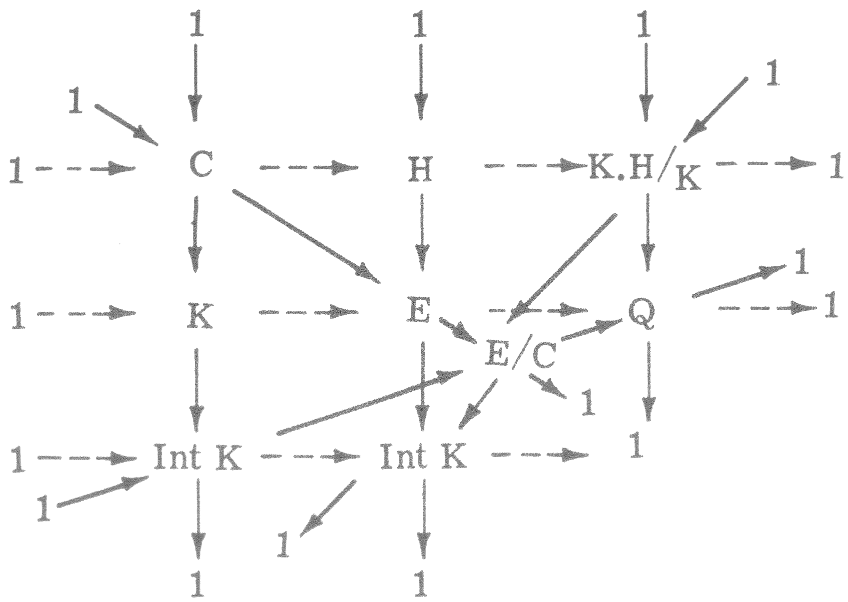


diagram 10

From (10) $E/_{H.K} = 1$ so E = H.K. Also Q = K.H/K = H. So this diagram also shows that H is a central extension of Q by $\mathcal{C}$(K), which has an extensive literature since $\mathcal{C}$(K) is abelian. A last result of (10) is, from diagram 3,

$$E/_C = \text{Int } K \times Q$$

Let us give an example of central extension which is familiar to physicists. We denote U_n the group of n by n unitary matrices. The correspondence U → det U is a homomorphism $U_n \xrightarrow{\det} U_1 \to 1$ whose kernel is by definition SU_n (unitary matrices of determinat 1). The centralizer of SU_n in U_n is the set of matrices of U_n which are a multiple of the unit : $e^{i\phi}$ 1_n. They form a group U_1 and the intersection $SU_n \cap U_1 = Z_n$ the cyclic group of n elements, represented by the matrices $e^{i\varphi n}$ 1_n with $e^{in\varphi n} = 1$. Hence the corresponding diagram 10' as an application of Theorem 2.

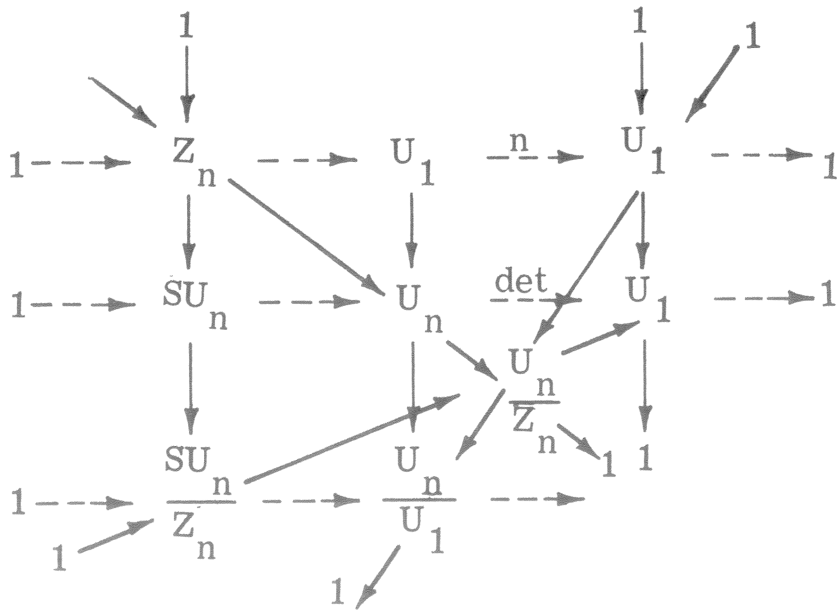


diagram 10'

The homomorphism $U_1 \xrightarrow{n} U_1$ is defined by $n(e^{i\phi}1_n) = e^{in\phi}$. Note that

$$\frac{U_n}{U_1} = \frac{SU_n}{Z_n} \text{ and } \frac{U_n}{Z_n} = \frac{SU_n}{Z_n} \times U_1 .$$

Suppose E and E' are equivalent central extensions with corresponding extensions H, H' of Q by C = $\mathcal{C}$(K) respectively. As we have seen, $H = \mathcal{C}_E(i(K))$, $H' = \mathcal{C}_{E'}(i'(K))$. Put E and E' in diagram 6

$$f[\mathcal{C}\ \epsilon\ (i(K))] = \mathcal{C}_{E'}(f_0 i(K)) = \mathcal{C}_{E'}(i'(K))$$

by commutativity. This is $H' = f(H)$.

Exercise 12. If $\mathcal{C}(K) = H \cap K$, $\mathcal{C}'(K) = H' \cap K$ show $f(\mathcal{C}(K)) = f'(K)$.

This shows that given an extension E (central), $H = \mathcal{C}_E(K)$ is a central extension of Q by $\mathcal{C}(K)$ and that $E \sim E' \rightarrow H \sim H'$ (where $\sim$ is the equivalence relation of extensions). Consider the case when H is given, then we ask how to construct E in accordance with (10).

Let $\alpha\ \epsilon\ K$, $\xi\ \epsilon\ H$ and consider $(\alpha, \xi)\ \epsilon\ K \times H$. Since $\mathcal{C}(K) \lhd K, H$, $C \times C < \dot{\mathcal{C}}(K \times H) \lhd K \times H$. The set $C' = \{(\gamma, \gamma^{-1}) \mid \gamma\ \epsilon\ \mathcal{C}(K)\}$ is a group isomorphic to $\mathcal{C}(K)$ but embedded in $K \times H$. It is called the antidiagonal of $C \times C$.

There is a natural map from $K \times H$ onto E given by $(\alpha, \xi) \xrightarrow{q} \alpha\xi$. This map q is a homomorphism since

$$q(\alpha\beta, \xi\eta) = \alpha\beta\xi\eta = \alpha\xi\beta\eta = [q(\alpha, \xi)][q(\beta, \eta)]$$

since $H = \mathcal{C}_E(K)$. Also $\text{Ker } q = C'$, and $\text{Im } q = E$ since by (10) $E = H.K$. We can then write the short exact sequence

$$1 \rightarrow C' \rightarrow K \times H \xrightarrow{q} E \rightarrow 1 \qquad \text{diagram 11}$$

and $E = K \times H/C'$. The reader may readily verify that H is isomorphic to $\mathcal{C}_E(K_1)$ where $K_1 = \{(\alpha, 1), \alpha\ \epsilon\ K\}$.

The equivalence of extensions is preserved under this construction. For let H and H' be equivalent extensions of Q by $\mathcal{C}(K)$, and E, E' the corresponding extensions by K. In the diagram for H, H' comparable to (6) denote the map $H \rightarrow H'$ by f, then $1 \times f$ is an isomorphism from $K \times H$ to $K \times H'$ such that diagram 11

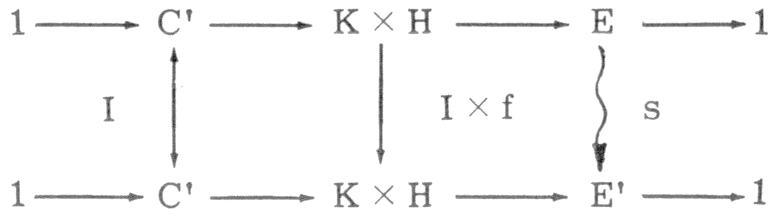


diagram 11

the map s is constructed by Lemma 2. An easy computation shows $K_1 \triangleleft K \times H/_{C'}$, so that we may write diagram 11'

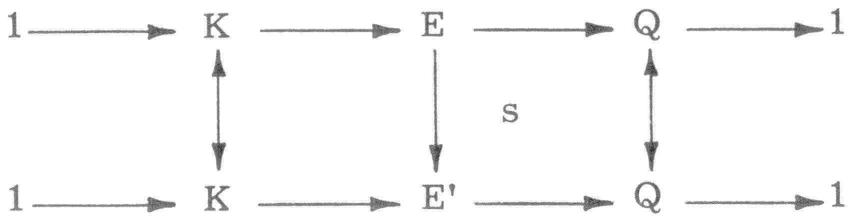


diagram 11'

We may then conclude that there is a one-one correspondence between the equivalence classes of central extensions of Q by K and of Q by C = $\mathcal{C}$(K).

EXAMPLES. 1. Suppose Out K = 1, then all extensions of kernel K are central.

2. If K is complete, i.e., C(K) = 1, Out K = 1. Any extension E of kernel K satisfies diagram 12 which is a simplified diagram 10.

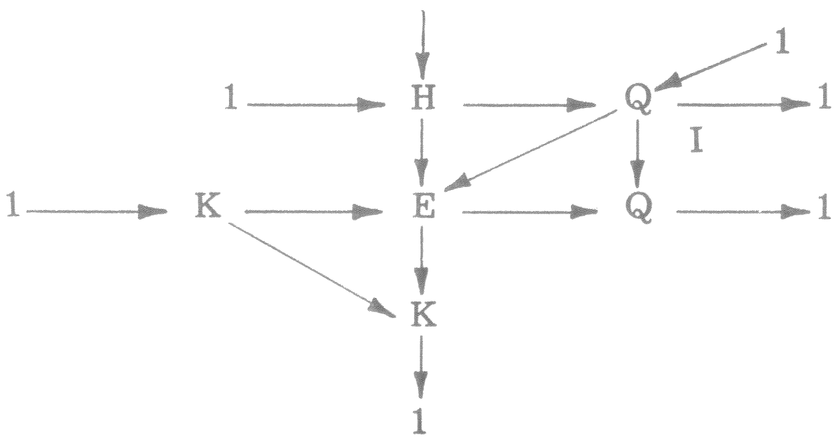


diagram 12

With diagram 3 this shows $E = K \times Q = K \times (E/K)$. The direct product is the only extension of an arbitrary group Q by a complete group K.

Exercise 13. Prove this result directly. (Proof given for instance in A. Speiser, Theorem 110, Gruppen von endlichen Ordnung).

Charge Conjugation and Isotopic Parity.

Consider the isotopic spin space of a system of particles, invariant under isotopic rotations SO_3. Adjoin to this symmetry group charge conjugation C, giving a larger symmetry group $G = \{ SO_3, C \}$. Since $C^2 = 1$, SO_3 has just two cosets in G, SO_3 and $C.SO_3$. Thus $SO_3 \triangleleft G$ and we may write

$$1 \to SO_3 \to G \to \mathbf{Z}_2 \to 1$$

where $\mathbf{Z}_2$ is a group of two elements. However SO_3 is complete, and its extension is $G = SO_3 \times \mathbf{Z}_2$. Note that G is <u>not</u> a direct product of SO_3 by $\{1, C\}$ since physically charge conjugation and isotopic spin rotations do not commute.

Denote reflections through the origin in isospace by η, then G is isomorphic to $O_3 = SO_3 \times Z_2$, the rotation group with η added. The quantum numbers of the system of particles having charge independence and charge conjugation as symmetries are just isotopic spin I and isotopic parity η.

For the π mesons we know that $C\,\pi^0 = \pi^0$ from $\pi^0 \to 2\gamma$, $\to C\gamma = -\gamma$. Choose a coordinate system in which the π^0 is along the z-axis, then C is a reflection through a plane containing the z-axis. Which plane depends upon the phases between $\pi^{\pm}$. Under isotopic parity $\pi^0 \to -\pi^0$, or $\eta(\pi^0) = -1$, since η is the product of C and a rotation of π, around an axis $\perp$ to the reflection plane of C, which reverses π^0.

More generally, for an isotensor t, if the charge conjugation quantum number of the neutral component is C, the isoparity of the isotensor is

$$\eta = C(-1)^t$$

This is equation (45) of the paper which introduced isoparity, L. Michel, N. Cim. 10, 31((1953) (called there isotopic parity)*.

In this very simple example we see that physicists consider groups with names given to elements. However, it is also useful to consider the structure of the abstract group. A problem which may then arise is the following: Given K, Q, $Q \xrightarrow{r} \mathrm{Int}\,K$, one can form the semi-direct product $K \wedge Q$ which is also a central extension. When is this semi-direct product equivalent to a direct product ?

THEOREM 3. Given K, Q, $Q \xrightarrow{r} \mathrm{Int}\,K$, this defines a semi-direct product which is a central extension $E = K \wedge Q$. The necessary and sufficient condition that it is equivalent to the direct product $E = K \times Q$ is the existence of $t \in \mathrm{Hom}(Q,K)$ such that $w_0 t = r$ where w is the natural homomorphism $K \xrightarrow{w} \mathrm{Int}\,K$.

PROOF: The condition is necessary. Let us write in diagram 13 that

$$1 \to K \to E \xrightarrow{p} Q \to 1$$

with

$$E \xrightarrow{f} \mathrm{Int}\,K$$

both a semi-direct product (diagram 3, i, p, k) and a direct direct product (diagram 4, i, p, i', p'). Then $t = p'_0 k$ and indeed $w_0 t = r$

*When T. D. Lee and C. N. Yang learned about this paper they liked it so much that they decided to propagandize for it, reproducing its main table p. 333 for annihilation of nucleon antinucleon pair, in N. Cim. 13.749 (1956); see also their footnote 3. I am very grateful to them for the great advertisement they gave to this new quantum number, although they never explained why they dislike the expression " isotopic parity" and why they choose G-parity.

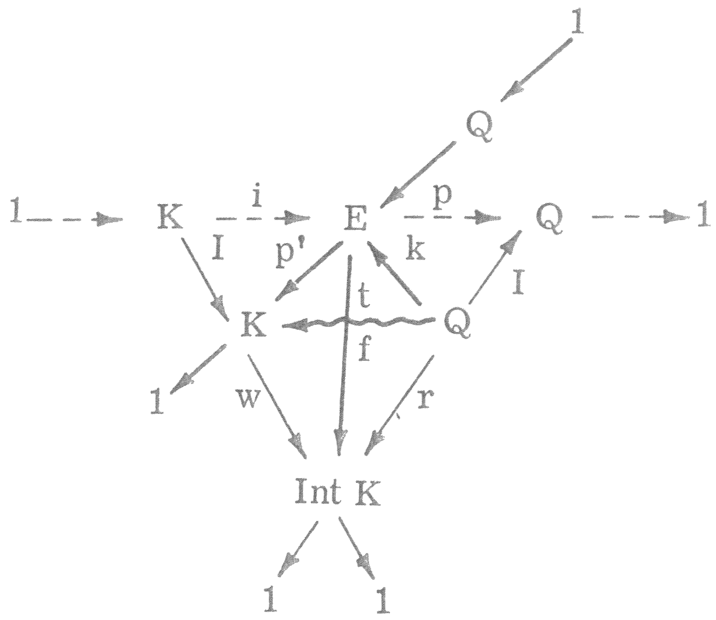


diagram 13

The condition is sufficient; $Q \xrightarrow{t} K$ is given, so is the composition law for the semi-direct product E

$$(\alpha, a)\ (\beta,\ b) = (\alpha\beta^{a},\ a\ b)$$

corresponding to $Q \xrightarrow{r} \mathrm{Int}\ K$ with $\beta = r(a)[\beta]$. The group K is isomorphic to $K' = \{\alpha' = (\alpha,\ 1)\}$, $K \lhd E$ and Q to $\{a' = (t(a^{-1}),\ a)\} = Q'$. Note that $K' \cap Q' = 1$ and $K'.Q' = E$.

Furthermore any $\alpha' \in K$ commutes with an $a' \in Q'$. Indeed

$$\alpha' a' = (\alpha\ t(a^{-1}), a) = (t(a^{-1})t(a)\alpha t(a)^{-1}, a) =$$

$$(t(a^{-1}).\ w_{o}t(a)\ [\alpha'], a) = (t(a^{-1})r(a)[\alpha], a) = (t(a^{-1})\ \alpha^{a}, a) =$$

$$= (t(a^{-1}), a)(\alpha, 1) = a'\ \alpha'$$

Hence $Q' \lhd E$ and by Lemma 3', $E = K \times Q$.

Exercise 14. Show that Hom (Q, K) is just the number of ways in which the group Q can be embedded in the direct product $K \times Q$. And Hom(K, Q) gives the number of map f for embedding K into $K \times Q$. (For the generalization to semi-direct product, see Cargèse Lectures).

Let us give an example of a semi-direct product which is a central extension but not a direct product. We have seen (in I.5) that L = SL(2, C)/Z_2 and L = P/T. Since L has no center L = Int SL(2, C). So the semi-direct product SL(2, C) ∧ P defined by P → L = Int SL(2, C) is not a direct product since Hom (P, SL(2, C)) reduces to the trivial homomorphism.

I.8 SUMMARY ON GROUP EXTENSIONS.

We will give here only results. We refer to the quoted references for the proof.

Returning to §I.6 we give several sufficient conditions for the existence of solutions to the extension problem. The direct product K × Q is always one solution. By introducing the map $Q \xrightarrow{g} \text{Out } K$ we obtain a subproblem, given K, Q, g ∈ Hom (Q, Out K) determine all E which satisfy (5). For a general map g there may be no extensions. Below are a set of sufficient conditions.

Our first claim is that if E exists then E is a solution for the extension problem $\mathcal{C}$(K), Aut K × Q, 1 × (h∘g) ∈ Hom(Aut K × Q, Aut C). This may be seen by extending (4) with (9), to

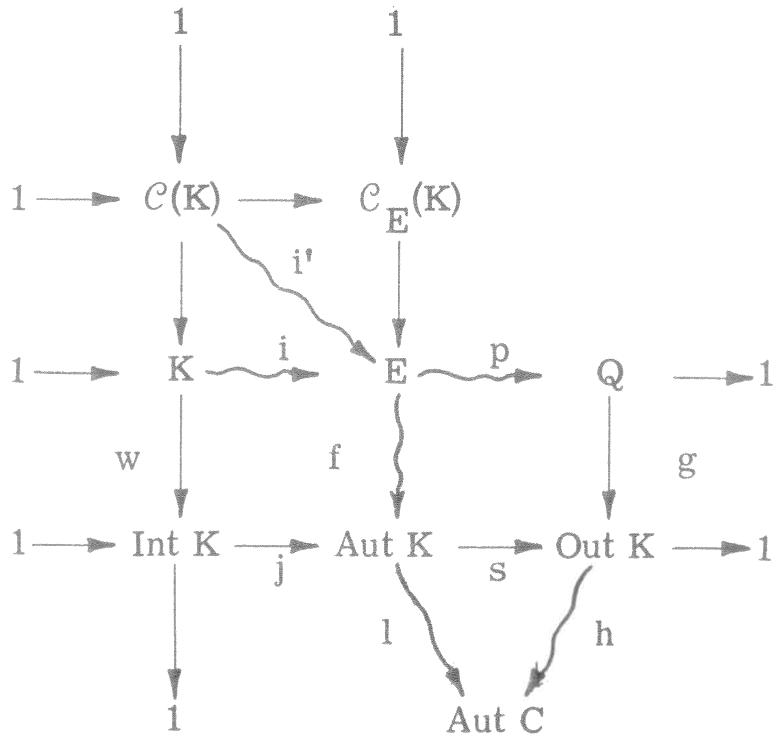


diagram 14

From (14) $f \times p$ is a map from $E \rightarrow \text{Aut } K \times Q$, where $\text{Ker } (f \times p) = \text{Ker } f \cap \text{Ker } p = \mathcal{C}_E(K) \cap i(K) = i'(\mathcal{C}(K))$. So any group extension problem can always be transformed into a problem with abelian kernel, the center of the kernel K.

DEFINITION 14. A short exact sequence $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ splits if any of the following hold:

(a) $A = 1$
(b) $C = 1$
(c) $B = A \wedge C$. (which is the only non trivial splitting!)

Sufficient Conditions for Existence of E.

This list is not complete, for further results the reader should consult the quoted literature, (Istanbul or, better, Eilenberg and MacLane).

1. $1 \rightarrow \mathcal{C}(K) \rightarrow K \rightarrow \text{Int } K \rightarrow 1$ splits.

There are three possibilities

(a) $\text{Int } K = 1 \rightarrow K$ abelian.
(b) $\mathcal{C}(K) = 1 \rightarrow \exists$ one solution for each $g \in \text{Hom}(Q, \text{Aut } K)$
(c) $K = \mathcal{C}(K) \wedge \text{Int } K$ which implies $K = \mathcal{C}(K) \times \text{Int } K$

2. $1 \rightarrow \text{Int } K \rightarrow \text{Aut } K \xrightarrow{s} \text{Out } K \rightarrow 1$ splits at Out K.

Then

(a) $\text{Int } K = 1$, this is case 1. (a).
(b) $\text{Out } K = 1$, E is a central extension.
(c) $\text{Aut } K = \text{Int } K \wedge \text{Out } K$.

As we have already seen, for case (c) it is actually enough to have the exact sequence

$$1 \rightarrow \text{Int } K \rightarrow s^{-1} (\text{Im } g) \rightarrow \text{Im } g \rightarrow 1$$

which always splits in the central case.

3. $\exists\, r \in$ Hom (Q, Aut K) : $s \circ r = g$. There exists among the solutions the semi-direct product. (See also Theorem 3) and we have seen 2. implies 3.

THEOREM 5. If there exist extensions E there is a one-to-one correspondance between equivalence classes of extensions for the two problems:

$$K,\ Q,\quad g \in \mathrm{Hom}(Q,\ \mathrm{Aut}\ K)$$

$$\mathcal{C}(K),\ Q,\quad h \circ g \in \mathrm{Hom}\ (Q,\ \mathrm{Aut}\ C)$$

NOTE: If $\mathcal{C}(K) = 1$, for any g, there is a trivial solution of the second problem. It is a semi-direct product for K, Q, g only if 3. holds.

In II.9 we give known results for the central extensions of the Poincaré group by an arbitrary group K.

II. HOMOGENEOUS SPACE AND ZEEMAN THEOREM

Introduction

In the first chapter we dealt only with groups (sets with a group law). It would be very useful to study a richer structure (i.e., with more axioms) that of topological groups (topological space with a compatible group law). We shall define it but not study it because this would increase too much the scope of these lectures. However we mention topological groups in order to give a warning to the reader: do not consider that topological groups are a special family of groups so that everything we proved in chapter I for the general case of groups is true for the particular case of topological groups. This point of view is mathematically wrong. As we shall see, groups are a particular case of topological groups. Some results of I are not true for the general case of topological groups.

We cannot here begin this chapter by a general course in general topology. We suppose that the reader knows what is an open set, a closed set, a continuous mapping (or function), a neighbourhood, even if he does not know the more general mathematical meaning of these words. For what follows let us give sufficient definitions (which are not very intuitive if seen for the first time).

We say that E is a topological space, if one has chosen a family of sets of elements of E called the open sets of E, such that E and the empty set ϕ belong to the family and the family is closed for union and finite intersection, i.e., the union of any number of open sets is an open set, the intersection of a finite number of open sets is an open set.

The closed sets are the complements of the open sets. (So ϕ and E are also closed). Let us give one definition using open and closed sets. The space E is connected if there is no proper subset of E (i.e., not E, not ϕ) that is both open and closed. Such a definition was intriguing for the majority of participants of the summer school, (but it was familiar, and therefore "intuitive" to a substantial minority). Let us just give for the mathematically untrained but curious reader the highlight of the content of a supplementary and well attended lecture.

Given two topological spaces E and E' a mapping $E \xrightarrow{f} E'$ is continuous if for any open set $X' \subset E'$, $f^{-1}(X')$ is an open set of E. Hence one can have different topologies for E, such that for some of them f is continuous and for some others f is not continuous. One may compare topology. Given two topologies on E, T_1 is finer than T_2 (and T_2 is coarser than T_1) if every open set for T_2 is an open set for T_1. Hence, if T_1 is strictly finer than T_2 (i.e., there are open sets of T_1 which are not open sets of T_2 then the identical mapping $E(T_1) \xrightarrow{I} E(T_2)$ is continuous but $E(T_2) \xrightarrow{I} E(T_1)$ is not. The finest of all topologies on E is the <u>discrete</u> topology; for it every subset of E is open (and therefore closed), in particular this is true for every point (= element) of E. Hence whatever the topology on E', any mapping $E \to E'$ is continuous for the discrete topology in E.

Of course, one can also define the coarsest topology on E such that a given mapping $E \xrightarrow{f} E'$ is continuous, and so on.