The reader is certainly familiar with the notion of neighbourhood and we advise him, if he does not know, to look at any classical book on topology in order to see how to pass from the axioms on open set to those in neighbourhoods. However these lecture notes will be self contained if one adds:

Given a subset $X \subset E$, the interior of X is the largest open set $\subset X$ (i.e., the union of all open sets in X); the closure of X is the smallest closed set $\supset X$ (i.e., the intersection of all closed sets $\supset X$).

A neighbourhood of a point $x \in E$ is a subset of E which contains X in its interior.

Of course, given two topological spaces E and F, there is a "natural" topology on the set product (product topology), on any subset (induced topology), on the quotient sets (quotient topology–this last point is more delicate). The morphisms for topological spaces are the continuous mapping. The isomorphisms of topological spaces are called homeomorphisms. $E = E'$ if these exist between them a bijective mapping f such that f and f^{-1} are continous.

II.1 DEFINITIONS.

DEFINITION 1. G is a topological group if it is both a group and a topological space such that the one variable function $G \to G$ given by $x \to x^{-1}$ and the two variable function $G \times G \to G$ given by $(x, y) \to x\,y$ are both continuous.

In other words, in a topological group, the two structures (group and topology) should be "compatible".

DEFINITION 2. The morphisms of topological groups are the continuous group homomorphisms, i.e., the continuous mappings which are homomorphisms for the group structure.

DEFINITION 3. Of course a topological subgroup of G is a subgroup which is a topological group with the induced topology from G.

Exercise 1. Show that an equivalent definition for the group G to be a topological group is that the one variable functions $x \to a\,x\,b$ are continuous for all $a, b, \in G$. Note that left (or right) translations on the group $x \xrightarrow{a} a\,x$ (or $x\,a$) are homeomorphisms so for the topological properties of G it is only necessary to consider neighbourhoods of the identity.

DEFINITION 4. Local isomorphism. Two topological groups G, G' are locally isomorphic if there exists neighbourhoods U, U' of their identities e, e', such that ∃ a homeomorphic map f: U, U' with

(a) $f(xy) = f(x)\,f(y) \qquad \forall x, y, xy \in U$

(b) $f^{-1}(x'y') = f^{-1}(x')\,f^{-1}(y') \quad \forall x', y', x'y' \in U'$.

Consider $D \lhd G$ where D is a discrete subgroup, and form the exact sequence

$$I \to D \to G \to G/D \to I.$$

Then G and G/D are locally isomorphic.

Exercise. Show that $D \lhd G$, D discrete $\to D < \mathcal{C}(G)$ if G is connected.
(Hint $d \in D$, $x \in G$, $x\,d\,x^{-1}$ is a continuous function of x on G, so it must be a constant).

EXAMPLE. Let R be the additive group of the real numbers, then $\mathbf{Z} \lhd R$ is a discrete subgroup. Further

$$R/\mathbf{Z} = U_1$$

as abstract groups. Take α irrational then $\mathbf{Z}\,\alpha \lhd R$ and $\alpha\,\mathbf{Z} \cong \mathbf{Z}$ with $\alpha\,\mathbf{Z} \cap \mathbf{Z} = \{0\}$. However the closure of

α Z + Z is the whole of R, (when X $\subset$ E and closure of X is E we say that X is dense in E).

However as abstract groups

$$Z \sim (Z + \alpha Z)/Z$$

and Z + α Z/Z is locally isomorphic to Z + α Z.

This example indicates how two groups (Z, (Z + α Z)/Z) can be isomorphic as abstract groups but quite distinct in their topological properties. Z is not dense in R, while (Z + α Z)/Z is

This shows that theorem 1 of chapter I is not true in general for topological groups. This is also the case of theorem 2, and so on. Indeed abstract groups can be considered as a special case of topological groups, if we give them the discrete topology (then every function defined on them is continuous).

Of course this notion of topological space is too general to be very useful and it would be easy to give examples of topological groups pathological to the physicist taste.

So we shall restrict ourselves, in the following, to "locally compact" group.

Definitions

If X_i are open sets of X and $\cup_{i\in I} X_i = X$, the family of X_i is called a covering of X (by the family X_i). If every family of open sets which covers X, contains a finite subfamily of X_i which covers X, then X is called "compact".

The space X is locally compact if every point has a closed compact neighbourhood.

Hence every group with the discrete topology is locally compact (and every finite group is compact). Example of a non locally compact group is that of all unitary operators acting on an infinite dimensional Hilbert-space.

For a given topological group G one can consider $Aut_c G$, the group of continuous automorphisms of G (i.e., the automorphisms of the structure of topological group). In the general case there might be several natural topologies on

$\text{Aut}_c\, G$ (which are deduced from that on G). However, for the more familiar class of groups this ambiguity does not arise. For example:

THEOREM 1. (Iwasawa) Let K be a compact group, then $\text{Out}_C\, K$ is discrete (see Iwasawa, Ann. Math. 50, 507, 1949).

Consider a non-connected topological group G. Let G_0 be the connected component which contains the identity. Since the image (by a continuous function) of a connected space is connected, for any $\alpha \in \text{Aut}_C\, G$, $\alpha(G_0) \subset G_0$ and $\alpha^{-1}(G_0) \subset G_0$ so G_0 is transformed into itself by all continuous automorphisms of G (so $G_0 \lhd G$).

Consider now a topological group G with a compact invariant subgroup K. We then deduce that K_0 (the connected component of the identity of K) is an invariant subgroup of G. Furthermore the canonical homomorphisms $G \xrightarrow{f} \text{Aut}_c\, K_0$ and $G \rightarrow \text{Aut}_c\, K$ are continuous. So $f(G_0) \subset \text{Int}\, K_0$. Hence

COROLLARY 1. Let G be a connected group and $K \lhd G$, K compact connected. Then G is a central extension of the connected G/K by K. Furthermore, if Center of K = 1, then G is the direct product K x (G/K).

Covering spaces.

Let X be a path-wise connected topological space, i.e., $\forall\, x, y \in X, \exists$ a continuous map f of the interval $[0, 1]$ into X such that $f(0) = x$, $f(1) = y$. Then X' is a covering space for X if X' is path-wise connected and $\exists$ a surjective map $\pi : X' \rightarrow X$ such that $\forall\, x \in X$, $\exists$ neighbourhood (nbhd) U(x) for which the points $x'_1, x'_2, \ldots$ of $\pi^{-1}(x)$ have neighbourhoods $U'(x'_1)$, $U'(x'_2), \ldots$ $x = \pi(x'_1) = \pi(x'_2) = \ldots$ where

(a) $\pi / U'(x'_i)$ is a homeomorphism of $U'(x'_i)$ into U(x) for each $x'_i \in \pi^{-1}(x)$.

(b) $\bigcup_i U'(x'_i) = \pi^{-1}\{U(x)\}$

(c) If $x'_i \neq x'_j$ $U'(x'_i) \cap U'(x'_j) = \phi$

π is called a covering map for x, and sometimes we say that x' lies over X.

EXAMPLE. Consider the real line R, $-\infty < x \subset \infty$, and the circle S: $|z| = 1$. The map $\pi(x) = \exp[2\pi i x]$ defines a map from $R \xrightarrow{\pi} S$ such that R is the covering space of S.

Loosely speaking if $\pi^{-1}(x)$ contains n distinct points in X' we speak of X' as an n-sheeted covering of X.

Since a topological group G is also a topological space we can speak of covering groups.

DEFINITION 6. G' is a covering group of G if G' is a group where G is homomorphic to G' as a group and as a topological space G' covers the topological space of G.

DEFINITION 7. The universal covering group $\overline{G}$ of G is a simply connected covering space of G.

It may be shown that $\overline{G}$ is uniquely defined. An example of the universal covering group is offered by SO_3 and SU_2. The latter is a simply connected covering group of the rotations and if I is the identity rotation, $\pi^{-1}(I) = \{I, \alpha\}$ where α is the rotation by 2π for the spinors. As a topological space SO_3 is double connected since if we parametrize rotations by a point on the radius vector of the unit sphere, points lying on the surface of the sphere representing a rotation of angle π, then points on the surface opposite to each other have to be identified since they lead to the same rotation. The sphere with opposite points identified is a doubly connected space. For instance there are the distinct closed paths in this space which cannot be continuously deformed into one another,

1. a closed path containing the origin but entirely within the sphere

2. a closed path containing the origin but going to the surface of the sphere and reentering at the opposite point.

II.2 THE LORENTZ AND POINCARÉ GROUPS.

We recall the definitions of the various components of the group of transformations preserving the space-time metric. This group is necessarily linear.

Let T be the translation group in space-time, and L the full homogeneous Lorentz group (including space-time reflections). As usual we denote

L_+ = set of homogeneous Lorentz transformations with determinant 1.

$L^\uparrow$ = set of homogeneous Lorentz transformations preserving the sign of the time component.

then $L_+^\uparrow$ is the connected part of the homogeneous Lorentz group. More specifically if $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x})$, then $\Lambda \in L$ is for every x, y

$$x^\mu g_{\mu\nu} x^\nu = y'^\alpha g_{\alpha\beta} y'^\beta = \Lambda^\alpha{}_\mu y^\mu g_{\alpha\beta} \Lambda^\beta{}_\nu x^\nu,$$

hence

$$\Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu = g_{\mu\nu} \quad , \text{ or } \Lambda^T G \Lambda = G$$

where $G : g_{00} = - g_{11} = - g_{22} = - g_{33} = 1, \; g_{\mu\nu} = 0, \mu \neq \nu.$

This implies $(\det \Lambda)^2 = 1$

$L_+ = \{ \Lambda \mid \det \Lambda = 1 \}, \; L^\uparrow = \{ \Lambda \mid \Lambda^{00} \geq 1 \}, \; L_+^\uparrow = L^\uparrow \cap L_+.$

If T is a translation by a four vector a^μ then

$$(T x)^\mu = x^\mu + a^\mu \quad .$$

A general Lorentz transformation is an element of the Poincaré group $P = T \wedge L$ where $\forall p \in P, \; p = (a, \Lambda)$

$$(px) = (a, \Lambda)x = \Lambda x + a \qquad a \in T, \; \Lambda \in L.$$

Let Z_2 be the group of two elements $\{1, -1\}$, then

$$L = L^{\uparrow} \times Z_2 \qquad L^{\uparrow} = L_+^{\uparrow} \wedge Z_2$$

$$= L_+ \wedge Z_2 \qquad L_+ = L_+^{\uparrow} \times Z_2 \tag{1}$$

Exercise. Verify equations (1).

In a similar way we define the various components of the Poincaré group, P_+, $P^{\uparrow}$, $P_+^{\uparrow}$, etc. where $P^{(\,)} = T \wedge L^{(\,)}$.

We may define a dilatation of T by a real number $\lambda > 0$. The dilatation multiplies each translation by an amount λ, and commutes with homogeneous Lorentz transformations. The group of dilatation D, is a subgroup of $\mathrm{Aut}_C\, T$. We define

$$G = P \wedge D, \quad G^{\uparrow} = P^{\uparrow} \wedge D, \quad G_+ = P_+ \wedge D, \text{ etc.......} \tag{2}$$

G consists of triples (a, Λ, λ) where $a \in T$, $\Lambda \in L$, $\lambda \in D$ and

$$(a_1, \Lambda_1, \lambda_1)\,(a_2, \Lambda_2, \lambda_2) = (a_1 + \lambda_1 \Lambda_1 a_2, \Lambda_1\Lambda_2, \lambda_1\lambda_2)$$

One can prove

THEOREM 2. For abstract group of automorphisms

$$\mathrm{Aut}\, P = \mathrm{Aut}\, P_+ = \mathrm{Aut}\, P_+^{\uparrow} = \mathrm{Aut}\, P^{\uparrow} = G.$$

A result which states that all automorphisms of the four Poincaré group are all the same and are continuous. The proof of this theorem and also that Aut G = G are given in the Cargése Summer School lectures notes.

Covering Group of $P_+^{\uparrow}$

A standard result on the Lorentz group states that SL (2, C) is the universal covering group for $\mathcal{L}_+^{\uparrow}$. In fact

$$1 \to Z_2 \to SL(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow \to 1$$
$$\swarrow$$
$$\text{Aut } T$$

The covering group for $\mathcal{P}_+$ may then be defined as

$$\overline{\mathcal{P}}_+ = T \wedge SL(2, C) \tag{3}$$

Definitions of $\overline{\mathcal{G}}_+^\uparrow \sim \overline{\mathcal{P}}_+^\uparrow \overline{\mathcal{G}}_+^\uparrow \sim \mathcal{J}_+^\uparrow$ D follows directly.

II.3 TRANSFORMATION GROUP.

Consider a group G and a set E. The bijective mappings of E (one to one onto itself) form a group, the permutation group of E written S(E). We say that G acts upon E as a transformation group if $\exists$ a homeomorphism $G \xrightarrow{f} S(E)$. To elements x, y, $\in$ g we associate permutations f(x), f(y) of E such that f(x) f(y) = f(xy)

EXAMPLE. 1. The rotation group SO_3 acts upon euclidean three space, where rotations correspond to the class of permutations preserving length and angle.

2. In a similar way P acts upon space-time.

DEFINITION 8. Consider Ker f where $G \xrightarrow{f} S(E)$, if
(a) Ker f = 1, G acts effectively on E
(b) Ker f $\neq$ 1, G acts ineffectively on E.

The action of G on E induces equivalence classes in E. In particular $\forall \alpha, \beta \in E$ we define

$$\alpha \sim \beta \text{ if } \exists x \in G : \beta = f(x)[\alpha] \equiv x[\alpha]$$

where $x[\alpha]$ is the result of the permutation f(x) acting upon α.

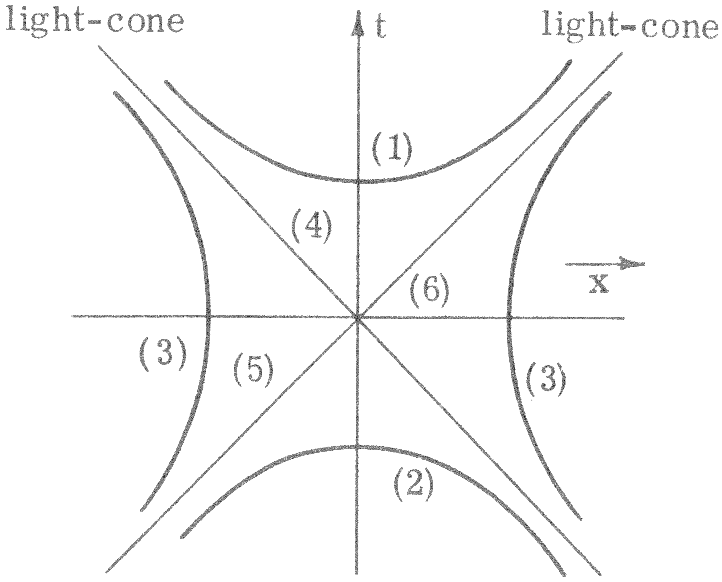
Exercise: Prove that $\alpha \sim \beta$ is indeed an equivalence relation.

DEFINITION 9. Given $\alpha \in E$, the set $O_\alpha = \{x[\alpha] \mid \forall x \in G\}$ form an orbit of G in E.

The set E is then partitioned into distinct orbits which are the equivalence classes just defined above.

Orbits of the Lorentz Group.

Let E = T. Of course Aut T $\subset$ S(E). The group L acts upon the translations. Referring to the space-time diagram below, we find



(4)

1. Orbits of $L_+^\uparrow$.

Denote the future (past) cone by $V_+ (V_-)$, its interior by $\mathring{V}_+ (\mathring{V}_-)$ and its boundary by ∂V_+ (∂V_-).

$\forall$ p $\in$ $\mathring{V}_+$, $p^2 = m^2 > 0$, $L_+^\uparrow$ transforms this vector into the hyperboloid sheet $p^2 = m^2$ $p^0 > 0$. Similarly for p $\in$ V_- . This gives the orbits (1) and (2).

Again for a space-like vector $p^2 = - k^2 < 0$, $L_+^\uparrow$ transforms p on the same hyperboloid. This gives one orbit (3).

On the light-cone we have $L_+^\uparrow$ $[\partial V_\pm] = \partial V_\pm$, orbits (4) and (5); while $\{0\}$ is an orbit itself (6).

Let us define for every a $\in$ T, ϵ(a) = 0 if a = 0 or $a^2 < 0$, = 1 if $a^2 \geq 0$ and $a^0 > 0$, = - 1 if $a^2 \geq 0$ and $a^0 < 0$. Then x belongs to the orbit of a if and only if $x^2 = a^2$ and $\epsilon(x) = \epsilon(a)$.

2. Orbits of L_+.

Since time-reversal I_t takes $V_\pm \to V_\mp$ orbits (1), (2) combine under L_+. Similarly $I_t\, \partial V_\pm = \partial V_\mp$ so (4) and (5) combine. All remaining orbits are unaffected since (3) is invariant under space-time reflection $I_s\, I_t$.

For L_+ there are only four different families of orbits. This is the maximum number. Orbits are given by the value of a^2, when $a \neq 0$, since $\{0\}$ is an orbit by itself. Orbits of L and of L_+ are identical.

3. Orbits of $G_+^\uparrow$ and G_+.

Since the dilatations D multiply a by any $\lambda > 0$, they change a^2 into $\lambda^2\, a^2$. Hence G has only 6 orbits: $a^2 > 0$ and $\epsilon(a) > 0$, $a^2 > 0$ and $\epsilon(a) < 0$, $a^2 = 0$ and $\epsilon(a) > 0$, $a^2 = 0$ and $\epsilon(a) < 0$, $a^2 < 0$ and $\{0\}$ (which is characterized by $a^2 = 0$ $\epsilon(a) = 0$.).
The group G_+ and G have the same orbits; there are only four: $a^2 > 0$, $a^2 < 0$, $a^2 = 0$ and $a \neq 0$, $a = 0$.

More generally since G acts upon E it acts upon $E \times E$ in a natural way. Let $E = T$ above, then $T \times T$ is the set of pairs (a, b) of four vectors We may then ask when do the pairs (a,b), (a', b') lie on the same orbit of L. Equivalence is defined by $(a,b) \sim (a', b')$ if $\exists\, \Lambda \in L$: $(a', b') = (\Lambda a, \Lambda b)$.

Clearly $a^2 = a'^2$, $b^2 = b'^2$, $a.b = a'.b'$ are necessary conditions, but they are not sufficient as shown by the following example:

$$a' = b' = (1, 0, 1, 1) \quad a = (0, 0, 0, 1), \quad b = (1, 1, 0, 1)$$

The reason is that the four vectors lie in a space with an indefinite metric, and on an orbit we need to preserve the euclidean dimension of the vector space spanned by a and b.

LEMMA 1. An orbit of $L_+^\uparrow$ on the space $T \times T$ is uniquely characterized by the conditions

$$\epsilon(a) = \epsilon(a'),\ \epsilon(b) = \epsilon(b'),\quad a^2 = a'^2,\ b^2 = b'^2,$$

$$ab = a'b',\ \dim(a,b) = \dim(a',b')$$

where (a,b), (a',b') lie on the same orbit in $T \times T$.

For the corresponding general theorem see Artin geometric algebra (Interscience Pub. New York 1957) theorem 3/16 p. 126.

II.4 ZEEMAN'S THEOREM.

Let $E = T$ and consider $S(E)$. If $f \in S(E)$ then f^{-1} exists. Denote the future cone of x by

$$V_+(x) = \{\ y \mid (y-x)^2 \geq 0, (y-x)^0 > 0\ \}.$$

A partial ordering relation may be defined on E by

$$\forall x, y \in E \quad x < y \text{ if } y \in \mathring{V}_+(x)$$

This defines a causal ordering among the four vectors, and we wish to study the most general mappings which preserve "causality".

DEFINITION 10. $f \in S(E)$ preserves the relation $<$ if $x < y \rightarrow f(x) < f(y)$.

THEOREM 2. (Zeeman) If f and f^{-1} preserve causality $(<)$ then $f \in G^{\uparrow}$.

For the proof, see E. C. Zeeman's paper, J. Math. Phys. 5, 490 (1964). Prof. Zeeman gives another version of his theorem (see below Theorem 2').

This result is remarkable in that no continuity or linearity has been assumed for f.

If $x, y \in E$ let us denote

$$x\ T\ y \text{ if } (x-y)^2 > 0$$
$$x\ S\ y \text{ if } (x-y)^2 < 0$$
$$x\ L\ y \text{ if } (x-y)^2 = 0,\ x-y \neq 0$$

DEFINITION (Alternative to 10). f reverses $x < y$ if $x < y \rightarrow f(y) > f(x)$. For the negation of a statement $T(x\ T\ y)$ denote $\overline{T}$, similarly for S, L.

LEMMA 2. f preserves T, S, L $\longleftrightarrow f^{-1}$ does.

PROOF: If f preserves T then f^{-1} preserves $\overline{T}$ or S and L. Further f preserves S and L $\rightarrow f^{-1}$ preserves T.

DEFINITION 11. $x <\cdot\, y$ if $x\ L\ y$ and $x^0 < y^0$.

THEOREM 2'. (Other version given in Zeeman's paper of theorem 2). A permutations f of space time is $\epsilon\ G^{\uparrow}$ if and only if f and f^{-1} preserves the relation $<\circ$.

If my note on the Automorphisms of the Poincaré group, Theoretical Physics Lectures VIIa in Boulder (1964) I quoted incorrectly theorem 2'. I give here a version of the same theorem, which is needed for the Boulder paper.

THEOREM 2''. The group of permutations which preserve the relations of space like, time like or light separation between points of space time is G.

I write here the proof that Prof. Zeeman gave me once at lunch table in Bures.

Let us first remark that $a\ T\ b \longleftrightarrow a \epsilon\ \mathring{V}(b)$ where $\mathring{V}$ is the interior of the light cone with b as summit. Suppose $x\ L\ z$, $x\ L\ y$, $y\ L\ z$, then y between x and z on the same light ray means either $x <\cdot\, y <\cdot\, z$ or $z <\cdot\, y <\cdot\, x$. This is equivalent to $\mathring{V}(x) \cap \mathring{V}(z) \subset \mathring{V}(y)$. Which can also be translated by:

$$a\ T\ x,\ a\ T\ z \rightarrow a\ T\ y.$$

To summarize:

If L z, z L y, y L x, then "y between x and z" $\longleftrightarrow$ a T x, a T z $\rightarrow$ a T y.

If f preserves T, S, L and f preserves $\lessdot$ for any two points on a light ray, then f must preserve $\lessdot$ for all points on the light ray. Similarly if f reverses $\lessdot$, it does so at all points on a light ray. This is an immediate result of the above construction.

Consider two light rays with a common point y, and x a point on the first ray and z a point on the second ray.

Two cases arise

1. $y \lessdot x$ and $y \lessdot z$ or $x \lessdot y$ and $z \lessdot y \longleftrightarrow x$ L y, y L z and x S z
2. $x \lessdot y \lessdot z$ or $z \lessdot y \lessdot x \longleftrightarrow x$ L y, y L z and x T z <u>or</u> x L z.

Consequently if f preserves T, S, L; f must either preserve $\lessdot$ or reverse $\lessdot$ on both of the light rays. This result may be extended to arbitrary light rays l_1 and l_2, since there is always a third light ray l_3 which intersects both l_1 and l_2. If f preserves S, T, L so does f^{-1}; and both these maps either preserve or reverse $\lessdot$ on l_1, l_3; and hence also on l_2, l_3. Then f, f^{-1} either preserve or reverse $\lessdot$ on both l_1, and l_2 which are arbitrary light rays.

Hence, from theorem 2' we have shown theorem 2''.

II.5 HOMOGENEOUS SPACES.

DEFINITION 12. A topological space X is homogeneous if $\forall$ x, y $\in$ X $\exists$ a homeomorphism f: $X \to X$ such that $f(x) = y$.

A topological group G is a homogeneous space and $x \to y = a\,x$ with $a = y\,x^{-1}$ is the desired homeomorphism. Clearly the orbits of G acting on E are homogeneous spaces directly from their definition.

In the following we shall forget about topology. So E, homogeneous space of G is just a another name for E "orbit of G". We also say that G acts transitively on $\mathfrak{G}$ and the E is a G-homogeneous space.

DEFINITION 13. $\forall \alpha \in E$ consider $G_\alpha = \{x \mid x \in G\ x[\alpha] = \alpha\}$. G_α is a group called the little group of α.

In the mathematical literature G_α is frequently termed the isotropy group or stabilizer of α since $G_\alpha[\alpha] = \alpha$.

Let β be a point in E and a a transformation such that $\beta = a[\alpha]$, then

$$a\, G_\alpha\, a^{-1}\, [\beta] = \beta$$

and the little group of β is $G_\beta = a\, G_\alpha\, a^{-1}$. So the homogenous space E is completely characterized by the little group of a point in the space and the other little groups may be obtained by conjugations of a given little group.

EXAMPLE 1. E = G i.e., let G act upon itself by left translation $G \xrightarrow{x} S(G)$ where $a \to x\, a$ for arbitrary a and given x, elements of G. We shall say that G acts transitively on itself. The set of left translations form a group isomorphic to G, and the isotropy group of any element is just the identity of G:

G can also act on itself by right translations: $a \xrightarrow{x} a\, x^{-1}$.

2. Suppose $H < G$. To say that a and $b \in G$ belong to the same left coset of H (i.e., the same element of $[G:H]_L$) is equivalent to $b^{-1}\, a \in H$. But $b^{-1}\, a = b^{-1} x^{-1} x\, a = (x\, b)^{-1} (x\, a)$ so left translations map left cosets of H into left cosets and it is easy to verify that this defines an action of G on $[G:H]_L$. The little group of $H \in [G:H]_L$ is H itself (as subgroup of G). Similarly right translations permute right cosets.

Homomorphism of Homogeneous Spaces.

Let E and E' be two homogeneous spaces of a group G acting on them. We ask for a map $E \overset{f}{\dashrightarrow} E'$ which preserves the way in which G acts upon these spaces.

This may be done with the diagram

$$\begin{array}{ccc} E & \overset{f}{\dashrightarrow} & E' \\ x \,\vdots & & \vdots\, x \\ E & \overset{f}{\dashrightarrow} & E' \end{array} \qquad x \in G. \tag{5}$$

f is the desired morphism provided the diagram is commutative. For an isomorphism f is required to be bijective.

Let us see in detail that the commutativity of (5) preserves the action of G. If α, $\beta \in E$, α', $\beta' \in E'$ such that under $x \in G$, $\beta \to x[\alpha]$, $\beta' \to x[\alpha']$. If $\alpha' = f(\alpha)$ commutativity requires that $\beta' = f(\beta)$ i.e.,

$$\forall x \in G, \quad \forall \alpha \in E, \; x[\beta'] = x[f(\alpha)] = f(x[\alpha])$$

and $\forall$ x, y $\in$ G

$$y[x[\beta']] = y[x[f(\alpha)]] = y[f(x[\alpha])] = f(y\,x\,[\alpha])$$

so the action of G is carried over to E' by f. Conversely let $x[\beta'] = x[f(\alpha)]$ and $x\,y\,[\beta'] = x[y[\beta']]$, then

$$f(x\,y\,[\alpha]) = x[\,y\,[\,f(\alpha)]]$$

Let y = 1 then

$$f(x\,[\alpha]) = x\,[\,f(\alpha)]$$

implying the commutativity of (5).

EXAMPLE. Given $H < G$, the G-homogeneous spaces $[G : H]_L$, $[G : H]_R$ are isomorphic. The map f is chosen to be $f(a\,H) = H\,a^{-1}$ $\forall$ $x\,H \in [G : H]_L$. This is a bijective map.

One shows the commutativity of (5) immediately, the action of G on these spaces being given by

$$a\,H \xrightarrow{x} x\,a\,H$$

$$H\,a \xrightarrow{x} H\,a\,x^{-1}$$

hence $x[f(a\,H)] = H\,a^{-1}x^{-1} = H(x\,a)^{-1} = f(x\,a\,H) = f(x[a\,H])$.

II.6 THE HOMOGENEOUS SPACES OF A GROUP.

We leave as an exercise to prove that if E is a G-homogeneous space with little group $H < G$ (H is defined up to a conjugation, as we have seen) it is isomorphic, as G-homogeneous space, to $[G : H]_L$. Hence the very important result.

There is a one to one correspondence between the G-homogeneous space (modulo an isomorphism) and the subgroup, up to a conjugation, of G.

Let us consider the case where G acts ineffectively and let K be the kernel of the homomorphism

$$G \xrightarrow{f} S([G : H]_L) \text{ i.e., } \forall k \in K, \; \forall a \in G, \alpha = aH \in [G : H]_L,$$

$k[\alpha] = \alpha$ so k a and a are in the same left coset of H: $a^{-1} k a \in H$. With $a = 1$ this shows $K < H$ and from $K \lhd G$, we have $K \lhd H$. Let K' be another invariant subgroup of H and $k' \in K' \lhd H$; since $\forall a \in G$, $a^{-1} k' a \in K' \lhd H$, we deduce that $K' < K$. Hence the kernel K is the largest invariant subgroup of G contained in H (such a largest group does exist; it is the product K', K'', K''', ... of all invariant subgroups of G in H).

EXAMPLES. 1. Lorentz transformations. The Poincaré group P acts transitively upon $[P : L]$, the space-time continuum. Also P acts effectively on this homogeneous space and the stabilizer of a point is indeed isomorphic to L. Of course P does not act effectively on $[P : T]_L$ which is just the Lorentz group L acting on itself by left translations.

2. Many topological spaces arise as homogeneous spaces of the type $[G : H]$, which frequently gives some insight into the action of G on these spaces. Consider for example the n-dimensional sphere S_n embedded in E^{n+1}, euclidean n + 1-space. The group SO_{n+1} acts transitively on S_n.

To find the little group of a point $\alpha \in S_n$, choose a coordinate system with one axis containing α. The little group of α is $G_\alpha = SO_n$. Let us examine the homogeneous space $[SO_{n+1} : SO_n]$.

The elements of SO_{n+1} may be represented by $(n+1) \times (n+1)$ dimensional matrices such that

$$\sum_i M_{ik} M_{ij} = \delta_{kj} \tag{5}$$

The columns $\{M_i\}_k$ $k = 1, 2, \ldots, n+1$ of M form an orthonormal set in E^{n+1} and $SO_n < SO_{n+1}$ appears as the set of matrices of the form

$$\begin{pmatrix} 1 & 0 \ldots\ldots & 0 \\ 0 & & \\ \cdot & & \\ \cdot & N & \\ \cdot & & \\ \cdot & & \\ 0 & & \end{pmatrix} \quad \det N = 1, \; N^T N = 1$$

The elements of $[SO_{n+1} : SO_n]$ are just the cosets of SO_n, which may be characterized as follows: $A, B \in SO_{n+1}$ lie in the same coset of $SO_n \Longleftrightarrow A^{-1} B \in SO_n$, ie $A^T B \in SO_n$, in particular

$$\sum_i A_{i1} \quad B_{ik} = S_{1k}$$

The vector $\{A_i\}_1$ is then orthogonal to $\{B_i\}_k$ $k = 2, 3, \ldots, n+1$. This implies $\{A_i\} = \lambda \{B_i\}_1$, from equation (5),

$$\Sigma_i \{A_i\}_1 \{A_i\}_1 = 1 = \sum_i \{B_i\}_1 \{B_i\}_1$$

we then deduce $\lambda^2 = 1$ and, from det A = det B = 1, we obtain

$$\lambda = 1 \text{ i.e. } \{A_L\}_1 = \{B_L\}_1.$$

Hence the cosets of SO_n are distinguished by the first column of the matrices in SO_{n+1}, and the matrices belong to the same coset $\longleftrightarrow$ their first columns are identical. As

$$\sum_{i=1}^{n+1} (A_{i\,1})^2 = 1$$

the elements of the first column lie on the n-sphere and conversely. Consequently

$$[SO_{n+1} : SO_n] = S_n$$

3. In a similar manner to 2.

$$[SU_{n+1} : SU_n] = S_{2\,n+1}.$$

(For the proof use the hermetian instead of the euclidean product for column vectors.) For n = 2, $[SU_3 : SU_2] = S_5$ which is homomorphic to the phase-space for three particles of finite mass and bounded total energy. (of. A. J. Dragt, Jour. Math. Phys.)

The only invariant subroup of SU_3 is its center Z_3. Since $Z_3 \wedge SU_2 = \{1\}$ the group SU_3 acts effectively on S_5. Generally for even n only the orthogonal groups act effectively on S_n, while for odd n the unitary groups give also effective action.

4. Let $\mathbf{C}^{n+1}$ be the space of n + 1-uples of complex number $z = (z_1, \ldots\ldots, z_{n+1})$. Define an equivalence relation in $\mathbf{C}^{n+1}$ by $z_1 \sim z_2$ if $z_2 = \lambda z_1$, λ a complex number. The set of equivalence classes so defined is a topological space called complex projective n-space, written $P_n(\mathbf{C})$. In fact $P_n(\mathbf{C})$ is an analytic manifold with n complex dimensions.

The n-dimensional unitary group $U_n < SU_{n+1}$, and may be considered as the matrices of the form

$$\begin{pmatrix} (\det U_n)^{-1} & 0 \ldots\ldots 0 \\ 0 & \\ \vdots & U_n \\ 0 & \end{pmatrix}$$

U_n are $n \times n$ unitary matrix.

Then as homogeneous spaces

$$[SU_{n+1} : U_n] = P_n(\mathbf{C})$$

For n = 2, $[SU_3 : U_2] = P_2(\mathbf{C})$; but SU_3 does not act effectively on $P_2(\mathbf{C})$ since its center Z_3 is $Z_3 \lhd U_2$. The adjoint group $SU_3/\mathbf{Z}_3$ acts effectively on the homogeneous space $P_2(\mathbf{C})$ with little group isomorphic to $U_2/\mathbf{Z}_3$. (See E. Cartan Atti Congr. int. Matem. 1928 c. 4, p 252).

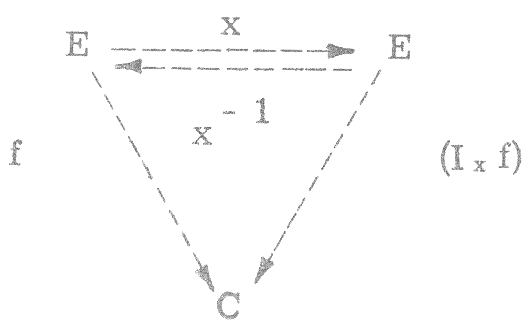
An application of this result has been suggested by Komar, Phys. Rev. Letter. 13. 220, 1964.

Application to representations.

Again let $H < G$, and let G act transitively in $E = [G:H]$. Denote by f a map from E to the complex numbers, $f \in M(E, \mathbf{C})$. For any such f an element $x \in G$ induces a map $M \xrightarrow{x} M$ given by

$$(I_x f)[\alpha] = f(x^{-1}\alpha) \quad \forall \alpha \in E \tag{6}$$

Diagramatically since x is bijective



The functionals $(I_x f)$ form a representation of G on $M(E, \mathbf{C})$ since $\forall\, x, y \in G$

$$(I_x \{I_y f\})[\alpha] = (I_y f)[x^{-1}\alpha] = f(y^{-1}x^{-1}\alpha) = (I_{xy} f)[\alpha] \tag{7}$$

$\forall\, \alpha \in E$. This construction allows a natural way to build representations of G once the functionals $M(E, \mathbf{C})$ are given.

When H is the identity subgroup (6) gives the left regular representation of G. The right regular representation is built on the functionals from the space of right cosets isomorphic to $[G : H]_R$. The two representations are clearly equivalent. We shall denote the representation (6) by U and call it the representation of G induced by the trivial representation of H.

An important generalization of (6) is due to Mackey. Let H have an irreducible representation L on a Hilbert space $\mathscr{H}$, ie to each $a \in H$ there corresponds a bounded operator L_a on $\mathscr{H}$:

$$\forall a, b \in H \qquad L_a L_b f = L_{ab} f \qquad \forall f \in \mathscr{H}.$$

$$L_e f = f$$

Define $L(E, \mathscr{H})$ to be the set of vectors defined on the space E with values in $\mathscr{H}$, is if $f \in L(E, \mathscr{H})$, $\alpha \in E$; $f(\alpha) \in \mathscr{H}$.

DEFINITION 14. The representation U of G induced by L on the Hilbert space $\mathscr{H}$ is the set of operators U_x, $\forall x \in G$ such that

$$(U_x f)[y] = L_a f(\alpha) \quad \forall y \in G, \alpha \in E = f(x^{-1}y) \; x^{-1} y =$$

$$= a^{-1}\alpha, \; a \in H.$$

The group property follows immediately by analogy with (7)

$$(U_{xy} f)[z] = (U_x U_y f)[z] \tag{8}$$

When L is the trivial representation of H on $\mathscr{H}$ then U^L reduces to U.

For the case when G is locally compact and H is compact, L may be chosen to be unitary. One may then show that U is also unitary.

Let us return to the simplest case of G compact, and D^i the unitary irreducible representations of G on some Hilbert space. If R is the regular representation of G, it may be completely reduced to the form

$R = \sum_i c_i D^i$ where c_i = number of times D^i appears in R.

is equal to the dimension of D^L.

A generalization of this result is given by the following:

THEOREM 3. Let $H < G$, and $[G : H] = E$. Let U be the regular representation of G induced by H on $M(G, \mathbf{C})$, then

$$U = \sum_i c_i \; D^i$$

where D^i is an irreducible representation of G, and c_i is the number of times the trivial representation of H appears in D^i restricted to H.

More generally:

THEOREM 4. Le $H < G$ and $E = [G : H]$. Denote the irreducible representations of G and H by D^i, d^i respectively. If D^i_H is D^i restricted to H and U^{d^i} is the induced representation of G corresponding to d^i, then in

$$U^{d^i} = \sum_m c_m D^m$$

$$D^m_H = \sum_j \gamma_j \; d^j$$

$$\gamma_i = c_m .$$

EXAMPLES. 1. Consider the reduction of the irreducible representations D^j of SO_3 over the sphere S_2. From § 2.5 $S_2 = [SO_3 : SO_2]$ so apply theorem 3 to

$$D^j{}_{SO_2} = \sum_{i=-j}^{j} d^i$$

If j is half-integral the identity does not appear in the regular representation of SO_3 corresponding to d^j and $D^j_{SO_2}$ does not contain the spinor representations. When j is an integer the corresponding trivial representation appears only once.

For the spinor representations it is necessary to consider $S_2 = [SU_2 : U_1]$ since $P_1(C)$ is just the 2-sphere.

2. Consider $S_3 = [SO_4 : SO_3]$. The irreducible representations of SO_4 may be labelled by pairs (j_1, j_2) where j_1, j_2 are integral or half-integral. If these representations are regular representations induced by SO_3 one can show $j_1 = j_2$. Physically such representations label bound states of hydrogen and they appear only once. This SO_4 symmetry of hydrogen seems to have been found by Fock Z. Phys. 98, 145 (1935). However see also the beautiful paper by Pauli Z. Phys. 26, 336 (1926).

3. Consider $P(4, R)$ i.e., the inhomogeneous $SO(4,R)$ group. $P(4, R)$ acts in the translation and there are only two types of orbits, $\{0\}$, and the spheres S_3 of radius λ, which are indeed $[SO_4 : SO_3]$. Hence there are two types of irreducible unitary representations of $P(4, R)$ those for which $(\lambda = 0, j_1, j_2)$ with $(-1)^{j_1} = (-1)^{j_2}$ where the translations are represented trivially (they are representation of SO_4) and those for which (λ, j), $\lambda > 0$, $2j$ integer ≥ 0.

By a generalization of theorem 4:

$$U(\lambda, j)\,|_{SO^4} = c_m D^{(m)}, \quad D^{(m)} = \gamma_j\, d^j, \quad c_m = \gamma_j$$

where $D^{(m)}$ and d^j are irreducible unitary representations of $SO(4,R)$ and $SO(3,R)$ respectively.

Since

$$D^{(m)}\,|_{SO_3} \equiv (j_1, j_2)\,|_{SO_3} = \sum_{j=|j_1-j_2|}^{j_1+j_2} d^j$$

we deduce

$$U(\lambda, j)\,|_{SO_4} = \Sigma_{j_1, j_2}\, (j_1, j_2)$$

when the summation is over all j_1, j_2 such that $j_1 + j_2 =$ integer and $|j_1 - j_2| \leq j \leq j_1 + j_2$.

As an example

$$U(\lambda,\ 0) = \Sigma_{i=0}^{\infty} \left(\frac{1}{2},\ \frac{1}{2}\right)$$

Hence the bound states of the hydrogen atom span the space of the unitary irreducible representation $U(\lambda,\ 0)$ of the group $P(4) \equiv$ inhomogeneous $SO(4,R)$.

Dual space of a group. It is the set of all classes (up to an equivalence) of the irreducible unitary representations, let us denote it by $\hat{G}$ for the group G and not consider here the related questions of topology. Since Aut G acts on G, it also acts on $\hat{G}$. It is obvious that Int G acts trivially on $\hat{G}$.

Indeed, let $D(x)$, $x \in G$ be a representation of G and $x \xrightarrow{\alpha} x^{\alpha}$ an automorphism $\alpha \in$ Aut G. Then $D(x^{\alpha})$ is another representation of G. It is obviously equivalent to $D(x)$ if $\alpha \in$ Int G induced by the element $a \in G$. Indeed

$$D(x^{\alpha}) = D(a)\ D(x)\ D(a)^{-1}.$$

EXAMPLE. We already remarked that Aut $P_+^{\uparrow} = G$ and

$$\text{Out } P_+^{\uparrow} = G / P_+^{\uparrow} = Z_2 \times Z_2 \wedge D$$

the two Z_2's corresponding to space and time reflections.

Given a mass $m > 0$ irreducible representation of $P_+^{\uparrow}$, the space reflection leads to an equivalent representation. This is no longer true for $m = 0$ since space reflection reverses the helicity states of the massless particle.

Including the dilatation in (9) transform a given mass $m > 0$ representation into a representation of arbitrary positive mass, which is clearly inequivalent to the initial one. Thus for $m \neq 0$ it is the dilatation contribution to Out $P_+^{\uparrow}$ which generates inequivalent representations of $P_+^{\uparrow}$.

Consider now a group G which contains $P_+^{\uparrow}$ as a subgroup. Let $N = \mathcal{N}_G(P_+^{\uparrow})$ i.e., the normalizer of P_+ in G.

If N contains the dilatations D, then for any unitary representation U of G, $U|P_+^\uparrow$ contains the whole spectrum of positive m if it contains an irreducible representation of $P_+^\uparrow$ with $m > 0$.

II. 7 MC GLINN'S THEOREM.

An interesting application of the theory of homogeneous spaces relates to an invariance group G of a physical theory. G is to contain P, the Poincaré group, and an internal symmetry group S. If G has P as a quotient, then G is an extension of P. The idea behind the following is to use the representations of G to obtain mass formulae due to the combination of S and P.

The following lemma is a generalization of a result first published by Mc Glinn, (Phys. Rev. Letters, 12, 467, 1964).

LEMMA 3. (Michel) Let

(i) $G = S.\, P_+^\uparrow$, $S \cap P_+^\uparrow = \{1\}$

(ii) $P = T \wedge L$

(iii) $\exists$ a non-trivial $p \in P_+^\uparrow$, $p \notin T$

such that

$$s^{-1}\, p\, s \in P_+^\uparrow, \quad \forall s \in S,$$

then $$G = P_+^\uparrow \wedge S.$$

As a short hand, we write P' for $P_+^\uparrow$.

PROOF. Consider the homogeneous space $[G : P']$, and G act acting transitively on it. Since $G = S\,.\,P'$, $\exists$ a one-one map between S and $[G : P']$.

G does not act effectively on $[G : P]$ since $\forall s$, $s^{-1}\, p\, s \in P'$ or $p\, s \in s\, P'$; this means

$$s\,P' \xrightarrow{p} p\,s\,P' = s\,P'$$

hence $p \in \operatorname{Ker} f$ where

$$G \xrightarrow{f} S([G : P']) \quad .$$

Now, $p \in \text{Ker } f \cap P' \triangleleft P'$. But the only invariant subgroups of P' are $\{1\}$, T, and P' itself. Hence $\text{Ker } f \cap P' = P'$, and $s^{-1} ps \in P'$, $\forall p \in P$; or $P' \triangleleft G$.

G is then an extension of S by P', hence by Lemma 1.4,

$$G = P' \wedge S$$

Alternative form of Lemma 3. Let $G = S.P'$, $s \cap P' = \{1\}$, and S a simple group (no invariant subgroup) and $\exists$ a non-trivial $s \in S : p^{-1} s p \in S \;\; \forall p \in P'$. Then

$$G = S \wedge P' .$$

This hypothesis is weak and physical: at least one internal symmetry s should be Poincaré invariant. But the decomposition $G = S . P'$ and the hypothese S simple are in fact very strong.

For a more general discussion see L. Michel Phys. Rev. 137 B, 405, 1965.

II.8 IMPRIMITIVITY CLASSES.

DEFINITION 15. A subgroup $K < G$ is maximal if $K < K' < G \rightarrow K' = K$ or $K' = G$.

Exercise. Show SU_n is maximal in $SL(n, C)$.

If a subgroup H is not maximal in G we say that the corresponding homogeneous space $[G : H]$ is a G-homogeneous space which is imprimitive. The reason for this is the following. Let K be maximal in G and $H < K < G$.

Consider then the surjective morphism of G-homogeneous space $[G : H] \xrightarrow{f} [G : K]$. The surjective mapping f is such that $f(aH) = f(bH) \longleftrightarrow a^{-1} b \in K$ i.e., a and b are in the same K cosets. Since f commute with the group actions on the two G-homogeneous space (see diagram 1) the subsets $f^{-1}(\alpha)$ corresponding to every point $\alpha \in [G : K]$ are also permuted "in blocks". These subsets are called imprimitivity classes.

EXAMPLE. We have studied the orbits of $L_+^\uparrow$ on the translations T. For a time-like vector the orbit is one sheet of a 2-sheet hyperboloid the little group is SO_3 (up to a conjugation) which is maximal in $L_+^\uparrow$, so no imprimitivity classes; but the orbit of L for time-like vector has a little group $\sim 0_3$ which is a subgroup of the maximal $L^\uparrow$ subgroup of L. The homogeneous space $[L : L^\uparrow]$ has two elements corresponding to the two sheets of the hyperboloid. Similarly for space-like translation the orbits are one sheet hyperboloids, the little group for $L_+^\uparrow$ is SO(2, 1) which is maximal in $L_+^\uparrow$; we leave to the reader to study the little group for L, to show that it has four connected pieces, so that it is maximal. The more interesting case is for light vectors; then we have seen that the two halves of the light-cone (minus the summit) are two orbit for $L_+^\uparrow$, the little group is $P(2)_+$ (Poincaré group = euclidean group in two dimensions) it is not maximal in $L_+^\uparrow$. We leave to the reader to study by himself this case and to show that the imprimitivity classes are the generators of the light-cone. The set of generators is homeomorph to S_2 (two-dimensional sphere = set of points at infinity of the light-cone). If we consider SL(2, C), the universal covering of $L_+^\uparrow$, then the maximal subgroup is the group ST(2, C) of triangular matrices

$$\begin{pmatrix} z & y \\ 0 & z^{-1} \end{pmatrix}$$

of determinant 1 and $[SL(2, C): ST(2, C)] = S_2$. As another example of "imprimitivity" in Physics literature, the reader is advised to look at the remarkable Wightman's paper "On the localizability of quantum mechanical systems" Rev. Mod. Physics **34**, 845, 1962.

II.9 IWASAWA DECOMPOSITION.

We said in 2.7 that the decomposition of a group $G = S.P$ with $S \cap P = \{1\}$ was a strong condition, so many students thought this was a "rare" phenomenon. Another inquiry showed that the majority of students did not know that it was possible for instance to choose representative of cosets of SO(3) in $L_+^\uparrow$ so that they form a group. So let us learn the existence of the Iwasawa decomposition. All students know what was a Lie group. Let us just say here to be complete that every finite dimensional Lie group is locally isomorphic to a group of n by n matrices, n finite integer (Ado theorem) with the induced topology of C^{n^2} or R^{n^2}. This may not be true globally (e.g. universal covering of SL(2, R)). We speak of real Lie group if the matrices can be taken real.

Let G be a non compact real simple Lie group (simple = here the only possible invariant subgroups are discrete and in the center). It is always possible to find a maximal compact subgroup K, where the structure of G is given by

$$G = K \; . \; A \; . \; N \cdot$$

Here A is a maximal abelian subgroup homeomorphic to that of a vector space and N is a nilpotent subgroup (isomorphic to group of triangular matrices with 1 on the diagonal and zeros below) are homeomorphic to a vector space.

This is a remarkable theorem in that this decomposition for G holds globally

$$(K \cap A = \{ 1 \} = A \cap N = K \cap N).$$

A further consequence of this decomposition is that any semi-simple Lie group has at least a compact homogeneous space, namely [G : H] for all H $<$ G and containing A . N.

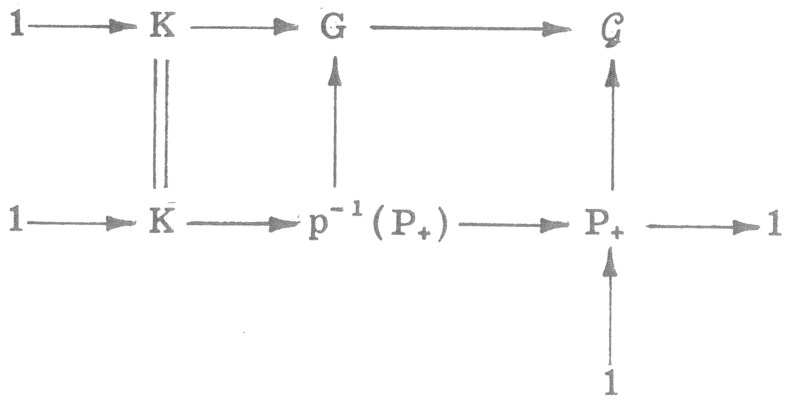
This is just the case we saw at the end of 2.8 where the compact sphere S_2 is homogeneous space for the SL(2, C). This group is a 6 real parameter Lie group with SU_2 as a maximal compact subgroup. The Iwasawa decomposition may be verified by direct computation

$$SL(2,\ C) = \begin{pmatrix} \cos\theta\ e^{i\alpha} & -\sin\theta\ e^{i\beta} \\ \sin\theta\ e^{-i\beta} & \cos\theta\ e^{-i\alpha} \end{pmatrix} \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \begin{pmatrix} 1 & \xi + i\eta \\ 0 & 1 \end{pmatrix}$$

where α, β, λ, θ, ξ, η are real.

II. 10 EXTENSIONS OF THE POINCARÉ GROUP $P_+^\uparrow$ (RESULTS ONLY).

We give here results only. For proof see L. Michel Nuclear Physics 57, 356, 1964. In chapter III we shall see the physical implication of these results. Let us just say here that if G is an invariance group for a physical relativistic theory by Zeeman's theorem (2.4) G acts upon space-time through G = Aut. Let K be the kernel. The image of p contains at last $P_+^\uparrow < G$. and $p^{-1}(P_+^\uparrow)$ is an extension of $P_+^\uparrow$



As before we denote $P_+^\uparrow$ by P' and its universal covering by $\overline{P'}$ and by $s : \overline{P'} \xrightarrow{s} P' \rightarrow 1$.

As we have seen in I.6 for searching the extensions of P' (or $\overline{P'}$) by K we must consider a homomorphism

$$g: P' \xrightarrow{g} \text{Out } K$$

$$(\text{hence } \overline{p'} \xrightarrow{g_0 s} \text{Out } K)$$

and there is a one to one correspondence between the two problems:

(K, P', g) and (C, P', $h_0 g$) where C = center of K, h is the canonical homomorphism Out K $\xrightarrow{h}$ Aut C. Correspondingly for $\overline{P'}$ the triplets are (K, $\overline{P'}$, $g_0 s$) and (C, $\overline{P'}$, $hg_0 s$).

If C is a Lie group and the extension has to be a Lie group then the only solutions are for Lie algebra the semi-direct product of the Lie algebra $\mathcal{C} \wedge \mathcal{P}'$. (See for instance L. Michel Istanbul chap. I to pass from the Lie algebra to global Lie groups, there might be then several solutions). This is an application of the last theorem of G. Hochschild and J. P. Serre, Ann. Math. 57, 591, 1953.

For abstract groups the only results established are when $h_0 g$ is trivial (central extensions of P' or $\overline{P'}$ by C). Then if C is a reduced group; i.e., if it has no divisible subgroup except $\{1\}$, the only solution is the direct product C $\times \overline{P'}$. (A abelian group is divisible if for every integer n and every $x \in A$, $\exists\, y_n \in A$ such that $x = (y_n)^n$ with the multiplicative notation. Another way to say it using the notation I.3, $\forall n$, ${}_n A = A$ i.e. $A_n = 0$).

Then the solutions of the initial problem correspond to

$$G = E_\alpha = \frac{C \times \overline{P}{}^{!}_{+}}{\mathbf{Z}_2(\alpha)} \tag{10}$$

where the notation has the following meaning. The center of $\overline{P}{}^{!}_{+}$ has two elements $1 = (0, 1)$ $(0 \in T,\ 1 \in SL(2, \mathbf{C}))$, $\omega = (0, -1)$ where $-1 = e^{i2\pi}$ is the "rotation" by 2π in SL(2, C). If $\alpha \in C = \mathcal{C}(K)$ and $\alpha^2 = 1$, then $\mathbf{Z}_2(\alpha) = \{(1, 1), (\alpha, \omega)\}$ a two element subgroup of the center of $C \times \overline{P}{}^{!}_{+}$.

If C is not a reduced group, there are still the solutions given by (10). It is not known if others exist, but if they do, they are quite pathological, as explained in the Nuclear physics 57, 356, 1964.

The meaning of (10) for the extensions of P' is that the irreducible representations of E_α are the tensor products of irreducible representations of C and of P' which map (α, ω) to the identity.

This does yield relations between quantum numbers for internal symmetries and the Poincaré group. One such relation has been found by Lurçat **and Michel:**

$$(-1)^{2j} = (-1)^{b+1}$$

where b and 1 are the Baryon and lepton numbers respectively for a system of particles and j is its total spin (see III.5 below).

III. OBSERVABLES AND ALGEBRAS.

III.1 DEFINITIONS.

With the current interest existing at the present time in symmetries of observables in quantum mechanics, it does not seem out of place to give some mathematical results on algebras in general. These will appear as the mathematical basis for discussing gauge groups and broken symmetries at an axiomatic level (cf. Robinson's lectures).

Through out we consider a linear space E over some field $\mathcal{K}$. (In the following the field cannot be arbitrary, so let us think only of the real or complex field).

DEFINITION 1. $E \otimes E = \{(x, y) \mid x, y \in E$ modulo the equivalence relation $(\lambda x, y) = (x, \lambda y),\ \lambda \in \mathcal{H}\}$

$E \otimes E$ is called the tensor product of E by itself and is the set of equivalence classes in the direct product of the abelian group of E by itself $E \times E$ modulo the above equivalence relation. Sometimes instead of (x, y) for the elements of $E \otimes E$ we write $x \otimes y$, where $\lambda x \otimes y = x \otimes \lambda y$.

DEFINITION 2. An algebra $\mathcal{A}$ is a linear space E with a homomorphism $E \otimes E \xrightarrow{h} E$.

For convenience we shall write $h(x \otimes y) = x\ _T\ y$ where $(\ \ )_T(\ \ )$ denotes the product of two elements of $\mathcal{A}$. Depending upon our choice for $(\ \ )_T(\ \ )$, we obtain different algebras. In particular we do not require $(\ \ )_T(\ \ )$ to be associative.

An algebra $\mathcal{A}$ is associative if $(x_T y)_T z = x_T (y_T z)$. It is interesting to note that an associative algebra may be defined diagramatically

$$\begin{array}{ccc} E \otimes E \otimes E & \xrightarrow{\ I \otimes h\ } & E \otimes E \\ \Big\downarrow{\scriptstyle h \otimes I} & & \Big\downarrow{\scriptstyle h} \\ E \otimes E & \xrightarrow[\ h\ ]{} & E \end{array} \qquad (1)$$

Diagram 1 is commutative. Reversing the arrows in this diagram defines a co-algebra co $\mathscr{A}$. If $\mathscr{A}$ is both an algebra and a co-algebra, then $\mathscr{A}$ is called a hyper-algebra.

Given the tensor product space $E \otimes E$ we can make a decomposition into symmetric and anti-symmetric tensors

$$x \otimes y = \frac{1}{2}\left(x \otimes y + y \otimes x\right) + \frac{1}{2}\left(x \otimes y - y \otimes x\right)$$

or

$$E \otimes E = E \overset{\text{symm}}{\otimes} E + E \overset{\text{anti-symm}}{\otimes} E = E \overset{s}{\otimes} E + E \overset{a}{\otimes} E$$

Then

$$1 \to E \overset{s}{\otimes} E \to E \otimes E \overset{f}{\to} E \overset{a}{\otimes} E \to 1$$

and

$$E \overset{a}{\otimes} E = E \otimes E \,/\, E \overset{s}{\otimes} E.$$

Similarly

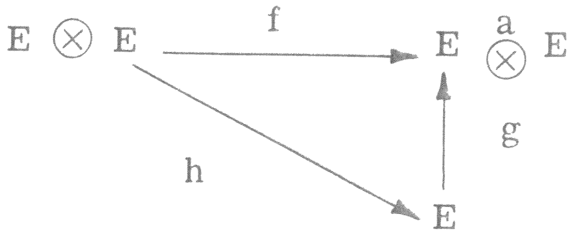
$$E \overset{s}{\otimes} E = E \otimes E \,/\, E \overset{a}{\otimes} E.$$

DEFINITION 3. An algebra $\mathscr{A}$ is called a Lie algebra if
1° h factorizes over

$$E \overset{a}{\otimes} E.$$

2° $\forall\, x, y, z,\ (x_T\,(y_T\, y)) + (y_T\,(z_T\, x)) + (s_T\,(x_T\, y)) = 0$
(Jacobi's identity)

In terms of diagrams, given the homomorphism h there exists f and g such that $h = g \circ f$ i.e. the diagram is commutative.



In a Lie algebra $x_T x = 0$ since $x_T x = h(x \otimes x) = g_0 f(x \otimes x) = 0$; and $x_T y = -y_T x$.

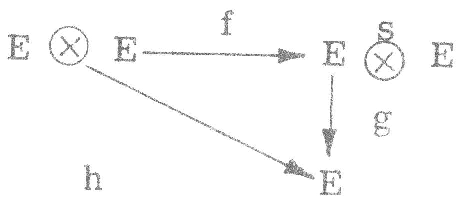
Note that Jacobi's identity can then be written

$$h_0 [I \otimes h - h \otimes I] (x \otimes y \otimes z) = h_0 (h \otimes I) (z \otimes x \otimes y).$$

This corresponds to the "lack" of associativity of h.

Exercise. If $E \otimes E \overset{h}{---} E$, $h(x \otimes y) = x_T y$ is an associative algebra and $\tilde{h}(x \otimes y) = h(y \otimes x)$ defines the "Opposed" algebra, this is associative. Furthermore show that $(h-\tilde{h})(x \otimes y) = x_T y - y_T x$ defines a Lie algebra (indeed Jacobi's identity is verified). Note that every Lie algebra is not obtained by this method from an associative algebra.

DEFINITION 4. An algebra $\mathcal{A}$ is a Jordan algebra 1° if the homomorphism h factorizes through $E \otimes E$, i.e., $\exists$ f and g such that the diagram



is commutative. Then of course $x_T y = y_T x$.

2° If $\forall$ x, y,

$$[(x_T x)_T y]_T x = (x_T x)_T (y_T x)$$

Exercise. Same notation as previous one, h and $\tilde{h}$ define opposed associative algebras. Then $(h + \tilde{h})(x \otimes y) = x_T y + y_T x$, define a Jordan algebra. The same remark is true. There are Jordan algebra which are not of this type.

The theory of Jordan algebras has its foundations in quantum mechanics where if x and y were observables then generally x y was not. However the symmetrized product was $(xy + yx)/2$ since this is hermetian if x and y are. Indeed these algebra were introduced by Jordan for quantum mechanics in 1933 and studied by P. Jordan, J. von Neumann and E. Wigner "On an algebraic generalization of the quantum mechanical formalism" Ann. Math. 35, 29, 1934.

3.2 Classification of Algebras

Given two subalgebras B_1 and $B_2 \subset \mathcal{A}$, we may form the product $B_1 {}_T B_2 = \{ b_1 {}_T b_2,\ b_1 \in B_1,\ b_2 \in B_2 \}$. In general $B_1 {}_T B_2$ is not an algebra since the vector space axioms are not satisfied. The closure $B_1 {}_T B_2$ would be a subalgebra of $\mathcal{A}$. The intersection $B_1 \cap B_2$ is always a subalgebra. (For the case of groups recall § 1-0).

DEFINITION 5. Let $\mathcal{A}$ be an algebra and $\mathcal{C}$ a linear subspace of such that $\mathcal{A}\ \mathcal{C} \subset \mathcal{C}$. $\mathcal{C}$ is called a left-ideal.

Similarly we define a right-sided ideal, and then a two-sided ideal as an ideal which is both left and right sided. For Lie algebra and Jordan algebra every ideal is two-sided.

Two-sided ideals $\mathcal{C}$ play the same role for algebras as invariant subgroups, for groups. Given a homomorphism between two algebra $\mathcal{A} \xrightarrow{f} \mathcal{A}'$ the kernel of f i.e., $f^{-1}(0)$ is a two-sided ideal of $\mathcal{A}$ and we can form the factor algebra by the kernel, i.e.,

$$1 \rightarrow g \rightarrow \mathcal{A} \xrightarrow{f} \mathcal{A}' = \mathcal{A} / g \rightarrow 1 \qquad \text{Ker } f = g.$$

Exercise. Prove that the intersection and linear sum of two ideals are also ideals.

DEFINITION 6. Let A be an algebra. There the <u>derived algebra</u> $\mathcal{A}'$ of $\mathcal{A}$ is

$$A' = A_T A$$

We can form higher derived algebras by

$\mathcal{A}'' = \mathcal{A}'_T \mathcal{A}'$, $\mathcal{A}''' = \mathcal{A}''_T \mathcal{A}''$, ... $\mathcal{A}^{(n)} = \mathcal{A}^{(n-1)}{}_T\mathcal{A}^{(n-1)}$.

Let us denote $D^0 \mathcal{A} = \mathcal{A}$, $D^n \mathcal{A} = \mathcal{A}^{(n)}$, then

$$D^0 \mathcal{A},\ D^1 A,\ D^2 \mathcal{A},\ \ldots\ldots,\ D^{(n)} \mathcal{A}, \ldots$$

is called the derived series for the algebra $\mathcal{A}$.

Exercise. (a) Show that $D^n \mathcal{A}$ is a two-sided ideal in $\mathcal{A}$, and $D^n \mathcal{A} \supset D^{n+1} \mathcal{A}$.

(b) Consider an abstract group G, and $\forall x, y \in G$ the set of products of the form $x^{-1}y^{-1}xy$ generates $G' < G$, the commutator or derived group of G. Give the derived series for G and generalize exercise (a) to groups.

(c) Show that G / G' is abelian.

DEFINITION 7. An algebra $\mathcal{A}$ is solvable if $\exists$ integer k:

$$D^n \mathcal{A} = 0 \quad \forall n > k.$$

Exercise. Generalize definition 7 to groups.

DEFINITION 8. Given an algebra $\mathcal{A}$, $\exists$ a largest solvable ideal called the radical of $\mathcal{A}$.

DEFINITION 9. $\mathcal{A}$ is semi-simple if its radical is trivial.

DEFINITION 10. $\mathcal{A}$ is simple if $\exists$ no non-trivial two-sided ideal in $\mathcal{A}$.

EXAMPLES. 1. In classical mechanics the observables of a physical system form an infinite dimensional Lie algebra, where if p, q are observables $p_T q = \{p, q\}$ the Poisson bracket. In quantum mechanics the observables formed an associative algebra and, with the commutator of any two operators, a Lie algebra. Dirac tells us in his book the general correspondence between the two Lie algebras (classical and quantal) for the same system. A detailed study for concrete problems is worthwhile.