III.3 DERIVATIONS.

DEFINITION 11. D is a derivation of $\mathcal{A}$ if $E \xrightarrow{D} E$ is a homomorphism (for the vector space structure) and

$$D(x_T y) = D(x)_T\ y + x_T\ D\ y$$

Note that D is linear on E and by Leibnitz's rule

$$D^n(x_T y) = \sum_{p=0}^{n} \binom{n}{p} (D^p x)_T\ (D^{n-p} y)$$

where

$$\binom{n}{p} = \frac{n!}{p!\,(n-p)!}$$

The sum of two derivations is again a derivation, but not the product since

$$D_1 D_2 (x_T y) = (D_1 D_2 x)_T\, y + (D_2\ x)_T\, (D_1 y) + (D_1 x)_T\, (D_2 y) + x_T (D_1 D_2 y).$$

However one sees immediately that $[D_1, D_2] = D_1 D_2 - D_2 D_1$ is a derivation. Hence the derivations of an algebra form a Lie algebra.

The Exponential Map.

Consider

$$e^D = \sum_{n=0}^{\infty} \frac{1}{n!} D^n$$

For an algebra $\mathcal{A}$ with a nilpotent derivation ($D^n = 0$ for some fixed N for all elements of $\mathcal{A}$) e^D is always well-defined. (If the field associated with E has finite characteristic then e^D is well-defined on $\mathcal{A}$).

Consider the following

$$e^D(x_T y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n (x_T y) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{1}{p!(n-p)!} (D^p x)_T (D^{n-p} y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (D^n x)_T \sum_{k=0}^{\infty} \frac{1}{k!} (D^k y) = (e^D x)_T\ (e^D y)$$

LEMMA 1. Given an algebra $\mathcal{A}$ with a derivation D such that e^D is well-defined on $\mathcal{A}$, then e^D is a linear map on E (vector space of $\mathcal{A}$) and an automorphism of $\mathcal{A}$.

Let Der $\mathcal{A}$ denote the set of derivations of $\mathcal{A}$. We have seen that it is a Lie algebra and it exists for any type of algebra $\mathcal{A}$. Further if we can define the exponential map e^D on $\mathcal{A}$ this provides a natural way in which to construct automorphisms of $\mathcal{A}$. Of course all automorphisms of $\mathcal{A}$ may not be obtained in this manner.

Inner Derivations.

Let a be an element of $\mathcal{A}$, then in the same manner as for groups we can define left and right translation by a:

$$x \xrightarrow{a_L} A_T x \qquad \forall x \in \mathcal{A}$$

$$x \xrightarrow{a_R} x_T a$$

Be definition of algebra these maps are linear.

For an associative algebra $\mathcal{A}$ an inner derivation is the linear map $D_a = a_R - a_L$ and

$$D_a x = - a_T x + x_T a$$

D_a is called the inner derivation associated with a.

In the case of a Lie algebra $\mathcal{A}$, left and right translation are essentially the same since

$$x \xrightarrow{a_L} a_T x = [a, x] = - x_T a$$

and $a_L = - a_R$. We then denote this mapping by

$$x \xrightarrow{a_L} a_T x = ad_a (x)$$

the adjoint mapping determined by a, or the inner derivation determined by a.

Exercise. Show directly that D_a and ad_a are derivations. For a Jordan algebra $\mathcal{A}$ let a_L the left translation corresponding to a; $x \in \mathcal{A}$, $x \xrightarrow{a_L} a_T x$. Of course $a_L \in \text{Hom}(E, E)$ i.e., a_L is an endomorphism of E the linear space structure of the algebra. We know that Hom(E, E) also often denoted $\mathcal{L}(E)$ is an associative algebra. Let $\bar{\mathcal{A}}$ the subalgebra of $\mathcal{L}(E)$ generated by all a_L, $\forall a \in \mathcal{A}$. (i.e., the smallest subalgebra of $\mathcal{L}(E)$ which contains all a_L). One shows that the inner derivation are the elements of the linear subspace of $\bar{\mathcal{A}}$ generated by all the elements $a_L b_L - b_L a_L$, $\forall a$, $\forall\ b \in \mathcal{A}$.

One proves the following results on derivations

LEMMA 2. For any three type of algebra $\mathcal{A}$ the inner derivations form an ideal in the Lie algebra $\text{Der}\,\mathcal{A}$ of derivations.

THEOREM 1. For a semi-simple Lie algebra all derivations are inner.

THEOREM 2. For a semi-simple Jordan algebra all derivations are inner.

Remarks on the Relation Between Lie Algebras and Lie Groups.

The process by which the Lie algebra for the infinitesimal generators of a Lie group is obtained is well known to physicists, but not so the converse. Generally given a Lie algebra there are several groups having this algebraic structure. If we construct from the infinitesimal generators a simply connected topological group, we obtain the universal covering group $\overline{G}$. To each discrete group $D < \mathcal{C}(\overline{G})$ we can form $\overline{G}/D$, which has the same Lie algebra as $\overline{G}$. Stated more precisely

THEOREM 3. For each Lie algebra $\mathcal{L}$ there exists a unique simple connected group $\overline{G}$, having this Lie algebra for its generators.

THEOREM 4. All connected Lie groups G which have $\mathcal{L}$ as Lie algebra have the form $G = \overline{G}/D$, where D is a discrete invariant subgroup.

We have seen after II Definition 4 that G and $\overline{G}$ are then locally isomorphic and (following exercise) that $D \lhd \mathcal{C}(\overline{G})$. On should be careful, since it can happen that $D_1 \sim D_2$ but $G/D_1 \not\sim G/D_2$.

As simple physical examples we refer to "Istanbul" chap. I.

To each subalgebra of $\mathcal{L}$ corresponds a closed subgroup of G, while to a two-sided ideal corresponds an invariant subgroup. Characteristic subgroups of G correspond to characteristic ideals of $\mathcal{L}$, the latter being defined in the same way as for groups. Lastly to a derivation of the Lie algebra $\mathcal{L}$ corresponds an automorphism of the corresponding Lie groups G.

III.4 ALGEBRA OF OBSERVABLES.

Let us now turn our attention to the observables in a quantum mechanical theory. The basic postulates of quantum theory introduce a separable Hilbert space $\mathcal{H}$ in which the observables appear as hermitian operators, generally these operators may be bounded or unbounded. Given $\mathcal{H}$ we shall denote the set of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

DEFINITION 12. A subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a (∗) star operator algebra over $\mathcal{H}$ if

$$\forall\, S, T \in \mathcal{A} \rightarrow \alpha S + \beta T,\ ST,\ S^* \in \mathcal{A}$$

where α, β are complex numbers, S^* the hermitian conjugate of S.

In the following $\mathcal{A}$ will always mean a *-algebra unless otherwise stated.

DEFINITION 13. The *-automorphisms of $\mathcal{A}$, written *-Aut ($\mathcal{A}$), are these automorphisms ξ of $\mathcal{A}$ such that

$$\xi(S^*) = \xi(S)^*, \quad \forall S \in \mathcal{A}$$

POSTULATE 1. The set of observables for a quantum mechanical system form a *-algebra, $\mathcal{A}$.

Physicists feel how much this postulate is artificial. Indeed often only the hermitian operators of $\mathcal{A}$ are considered as observables. They form a Jordan algebra $\mathcal{J}$. Indeed if X, Y $\in$ J, (i.e., X, Y $\in$ $\mathcal{A}$ and $X^* = X$, $Y^* = Y$) then use for the Jordan multiplication of X and Y, $\frac{1}{2}(XY + YX)$ (which is a Hermitian operator of $\mathcal{A}$, hence an element of $\mathcal{J}$). This observable corresponds to the classical observable xy where x, y are the classical observable corresponding to X and Y.

As we say Jordan algebra were introduced by physicists for observable in quantum mechanics. However for the (unphysical) reason that associative algebra are simpler, physicists returned back to them.

Relativistic Invariance.

A theory will be said to be Poincaré invariant if $P_+^{\uparrow}$ is a subgroup of the group *-Aut($\mathcal{A}$). We denote *-Int $\mathcal{A}$ the group of inner *-automorphisms of $\mathcal{A}$, i.e., the inner automorphisms $A \to U A U^{-1}$ of $\mathcal{A}$ generated by the unitary operators $U^* = U^{-1} \in \mathcal{A}$

Exercise. Show that *-Int($\mathcal{A}$) $\lhd$ *-Aut($\mathcal{A}$), and state clearly the group law.

Spatial *-automorphisms.

A special kind of automorphism plays an important role in the theory of algebra operators.

DEFINITION 14. The spatial automorphisms, Sp(Aut $\mathcal{A}$), are these elements $\xi \in$ Aut ($\mathcal{A}$):

$$\xi(S) = X S X^{-1}, \quad \forall S \in \mathcal{A} \text{ and some } S \in \mathcal{L}(\mathcal{H}).$$

If furthermore $(X S X^{-1})^* = X S^* X^{-1}$, (then $S^* X S = S X^* X$), X induces a spatial *-automorphism.

Clearly *-Int($\mathcal{A}$) $<$ Sp(*-Aut $\mathcal{A}$), and from the preceding exercise *-Int($\mathcal{A}$) $\triangleleft$ Sp(*-Aut $\mathcal{A}$).

Perhaps it is well to deal with a technical point concerning unbounded operators. For physical interest we shall have to consider unbounded operators, an unpleasant necessity. It the unbounded operators have a spectral decomposition theorem (which is always if A is self adjoint i.e., $A = A^*$ or normal i.e., $AA^* = A^* A$)

$$A = \int \lambda \, d \, E_\lambda$$

the spectral projectors E_λ are bounded. If we want to deal only with bounded operators the natural way to include the unbounded observables is to consider instead of their operators, the set of all their spectral projectors. For instance the product of two unbounded operators A, B is not defined everywhere, so is therefore their commutator AB - BA. However we shall say that AB - BA = 0 if every spectral projector of A commute with every spectral projector of B.

If all spectral projectors of A are elements of an algebra $\mathcal{A}$ we shall say that A is affiliated to $\mathcal{A}$ (A may be not an element of $\mathcal{A}$). If $\mathcal{A}$ is a *-algebra, let $\mathcal{A}'$ commutant of $\mathcal{A}$ i.e.,

$$\mathcal{A}' : \{A \text{ bounded}, \ A X = X A, \ \forall X \in \mathcal{A}\}$$

It is easy to see that $\mathcal{A}'$ is a *-algebra (of bounded operators). We denote $\mathcal{A}'' = (\mathcal{A}')'$ the double commutant, and so on ...

It is easy to see that elements of $\mathcal{A}$ are affiliated to $\mathcal{A}''$

and if $B(\mathcal{A})$ denote the (subalgebra) of spectral projectors of operators of $\mathcal{A}$, then $B(\mathcal{A}) \subset \mathcal{A}''$.

Similarly $\mathcal{A}' \subset \mathcal{A}'''$, $\mathcal{A}'' \subset \mathcal{A}^{\text{iv}}$. From $M \subset N \rightarrow \mathcal{N}' \subset \mathcal{M}'$ we deduce $\mathcal{A}''' \subset B(\mathcal{A})' = \mathcal{A}'$, $\mathcal{A}''' \subset \mathcal{A}''$, so $\mathcal{A}' = \mathcal{A}'''$ and $\mathcal{A}'' = \mathcal{A}^{\text{iv}}$ .....

DEFINITION. A von Neumann algebra is a *-algebra $\mathcal{A}$ (of bounded operators) such that $\mathcal{A} = \mathcal{A}''$.

We shall denote by $U(\mathcal{A})$ the set of unitary operators (they form a group) of the von Neumann algebra $\mathcal{A}$ and we leave to the reader to prove that $U(\mathcal{A})'' = \mathcal{A}$, i.e., a von Neumann algebra is generated by its unitary operators.

Let $\xi \in$ Sp(*-Aut $\mathcal{A}$), represented by $X \in \mathcal{B}(\mathcal{H})$ (bounded operators on $\mathcal{H}$). We have seen that X^{-1} exists and $X^* X \in \mathcal{A}$. Since $X^* X$ is a positive hermitian operator let C be its positive square root.

Since C has same spectral projections as X^*X, $C \in \mathcal{A}'$. Hence $Y = X C^{-1}$ induces the same spatial automorphisms as X: $Y S Y^{-1} = X C^{-1} S C X^{-1} = X S X^{-1}$, $\forall S \in \mathcal{A}$, but Y is also a unitary operator on $\mathcal{H}$.

We see that a spatial *-automorphism can be represented by a unitary operator on $\mathcal{H}$. Let X', X'' be two unitary operators corresponding to the same element of Sp(*-Aut $\mathcal{A}$)

$$\xi(S) = X'' S X''^{-1} = X' S X'^{-1}$$

then $X'^{-1} X'' = Y$ is a unitary operator in $\mathcal{A}'$. The converse is also true. Let $U(\mathcal{H})$ be the set of all unitary operators on $\mathcal{H}$, this set is also a group for the product of operators. We denote by $U(\mathcal{A})$ the group of unitary operators of $\mathcal{A}$. The normalizer $\mathcal{N}_{U(\mathcal{H})}(U(\mathcal{A}))$ will be denoted simply $U\mathcal{N}(\mathcal{A})$. To summarize, we have established the following results, (Diagram 1) (with $\mathcal{C}(\mathcal{A})$ = center of $\mathcal{A}$).

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U\mathcal{C}(\mathcal{A}) & \longrightarrow & U(\mathcal{A}) & \longrightarrow & *\text{-}\,\mathrm{Int}\,\mathcal{A} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U(\mathcal{A}') & \longrightarrow & U\mathcal{M}(\mathcal{A}) & \longrightarrow & \mathrm{Sp} * \mathrm{Aut}\,\mathcal{A} & \longrightarrow & 1 \end{array}$$

diagram 1

and, of course, this diagram 1 should be completed according to diagram I.2. The more elegant manner to do it is to consider $\mathcal{A}$ as the von Neumann algebra generated by all observables (which are then affiliated to $\mathcal{A}$). Then $(\mathcal{A}')' = \mathcal{A}'' = \mathcal{A}$. So $U(\mathcal{A})$ and $U(\mathcal{A}')$ a respectively centralizer of each other in the group $U\mathcal{M}(\mathcal{A})$. We leave to the reader to write in this case the full diagram I.2, which is completely symmetrical with respect to the main diagonal.

Some further results arise if we modify Postulate 1 slightly. Given an operator $S \in \mathcal{B}(\mathcal{H})$ we define the norm of S, written $\|S\|$, by

$$\|S\| = \sup_{\|f\| \leq 1} \|Sf\|$$

where f is any element of $\mathcal{H}$ lying in the unit ball. The operators norm allows us to define a topology on $\mathcal{L}(\mathcal{H})$. An open neighbourhood of $S_0 \in \mathcal{B}(\mathcal{H})$ in the norm topology is

$$N_\epsilon(S_0) = \{ S : S \in \mathcal{B}(\mathcal{H}),\ \|S - S_0\| < \epsilon \}$$

The usual notions of limit, continuity, and closure can be readily defined.

DEFINITION 15. A *-algebra $\mathcal{A}$ of operators on a Hilbert space which is closed in the norm topology is called a C*-algebra.

Let D be a derivation on an algebra $\mathcal{A}$, then D is called a spatial derivation if $\forall S \in \mathcal{A}$, $\exists \delta \in \mathcal{B}(\mathcal{H})$:

$$D(S) = \delta S - S \delta.$$

As before if $D \in \mathcal{B}(\mathcal{H})$, $e^D \in \text{Aut}\, \mathcal{A}$ and $e^{\alpha D}$ form a one parameter group of automorphisms; for $*\text{-Aut}(\mathcal{A})$, D is anti-self adjoint. An important result in the development of C*-algebras is that all its derivations are spatial.

Presently there seems to be considerable interest among physicists to develop the theory of observables as a C*-algebra $\mathcal{A}$ with the Lie algebra of the Poincaré group as a subalgebra of the algebra of derivations ot $\mathcal{A}$. This last condition implies $P_+^! < *\text{-Aut}(\mathcal{A})$. It is hoped that this section will appeal to same readers who might start their serious thinking about these ideas by sketching some consequences of the assumption $P_+^! < \text{Sp}(*\text{-Aut}\, \mathcal{A})$.

See preprints by Kastler, Borchers etc....

III.5 OBSERVABLES AND VON NEUMANN ALGEBRAS.

Given a *-algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ we may also introduce a weak neighbourhood of $S_0 \in \mathcal{A}$ by

$$N_\epsilon (S_0) = \{S: |<(S - S_0) f_i, g_i>| < \epsilon \quad i = 1, \ldots, n \ \forall f_i, g_i \in \mathcal{H}\}$$

Clearly every uniform neighbourhood is also a weak neighbourhood but not vice versa.

DEFINITION 16. A *-algebra $\mathcal{A}$ which is closed in the weak topology is called a W*-algebra. (We assume $\mathcal{A}$ contains the identity.)

With this definition a W*-algebra is also closed in the weak, strong, ultra-strong, and ultra-weak topologies (Dixmier p. 44, Les Algèbres d'opérateurs de l' Espace Hilbertian, Gauttier-Villars Paris). Hence every W*-algebra is a C*-algebra. The converse is not true.

THEOREM 6 (VON NEUMANN). $\mathcal{A}$ is a W*-algebra $\longleftrightarrow \mathcal{A} = \mathcal{A}''$. i.e., Von Neumann algebra and W*-algebras are the same objects. A Theorem recently proven by Kadison and others state that all derivations of a von Neumann algebra are inner.

To end this paragraph I present an already old work by Lurçat and I (N. Cim. 21, 574, 1961).

Our assumptions on the observables for a quantum theory may be written:

(1) The algebra $\mathcal{A}$ generated by the observables is a W*-algebra. To see the difference between C* and W*-algebra generated by the observables, we advise reading the beautiful paper of R. Haag and D. Kastler J. Math. Phys. 5, 840 (1964), its use of ϵ-equivalence, its explanation of superselection rules.

(2) $\exists$ a complete commuting set of observables $\mathcal{B}$. Mathematically we state this as $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}' = \mathcal{B}$. We say in this case $\mathcal{B}$ is maximal abelian. We refer to J. M. Jauch, Helv. Phys. Acta 33, 711 (1960) for the physical motivation of this mathematical hypothesis.

Since $\mathcal{B} \subset \mathcal{A}$, $\mathcal{A}' \subset \mathcal{B}' = \mathcal{B} \subset \mathcal{A}$, or $\mathcal{A}' \subset \mathcal{A}$. Hence $\mathcal{A}'$ = Center of $\mathcal{A}$. Such a von Neumann algebra is called a discrete algebra.

(3) The absolutely conserved charges in the theory are contained in $\mathcal{A}' = \mathcal{C}(\mathcal{A})$ and they generate it. By the charges we mean the operators generating baryon, electric, leptonic charges.

(4) $P_+^! \subset$ *- Int $\mathcal{A}$

As regards assumption (4) we note that $\mathcal{C}(\mathcal{A})$ is left invariant by $P_+^!$. Conversely if $\mathcal{C}(\mathcal{A})$ is left invariant by $P_+^!$ then a theorem of Dixmier (p. 120) has $P_+^! \subset$ *-Int$\mathcal{X}$. So 4 is implied by 1, 2, 3.

The unitary operators of A which represent the inner automorphisms are defined up to elements of $\mathcal{U}\,\mathcal{C}(\mathcal{A})$ so the group ϵ which realizes all relativity transformations is a central extension of $P_+^!$ by $\mathcal{U}\mathcal{C}(\mathcal{A})$: see diagram 2. $\mathcal{E} = p^{-1}(P_+^!)$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U\mathcal{C}(\mathcal{A}) & \longrightarrow & U(\mathcal{A}) & \longrightarrow & * - \mathrm{Int}\,\bar{\mathcal{A}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & U\mathcal{C}(A) & \longrightarrow & \mathcal{E} & \longrightarrow & P_+^{\dagger} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

diagram 2

We have already remarked in § 2. 9 that the physically interesting central extensions of $P_+^{\dagger}$ by $U = \mathcal{U}\mathcal{C}(\mathcal{A})$ are:

$$\mathcal{E}_\alpha = \frac{U \times \overline{P_+^{\dagger}}}{Z_2(\alpha)}$$

here α is a square-root of U which is generated by the charges; that is if we denote the charges in the set by q, b, l, ..., the unitary operators U are of the form

$$e^{i\pi f(q, b, l, \dots)}$$

where f is a real-valued function of the charges. The square-roots α of U correspond to integer valued functions f. In an irreducible representation of $\overline{P_+^{\dagger}}$, ω is represented by $(-1)^{2j}$. The condition on ϵ_α is that α and ω are both represented by 1 or - 1, i.e., $\alpha = (-1)^{2j}$. This gives

$$e^{i\pi f} = (-1)^f = (-1)^{2j}$$

f can be obtained as the weak limit of polynomials in q, b, l, ..., but it has to be integer-valued and f = 2j, mod. 2. One may deduce that f is linear and if $\epsilon = 0$ or $+1$

$$f = \epsilon_0 + \epsilon_q\, q + \epsilon_b\, b + \epsilon_l\, l$$

In nature it is found that $\epsilon_0 = \epsilon_q = 0$, $\epsilon_b = \epsilon_l = +1$, thus

$$(-1)^{b + l_1 + l_2} = (-1)^{2j}$$

where l_1 is the electron lepton number, and l_2 the muon lepton number.

IV. POLARIZATION.

IV.1 COVARIANT DESCRIPTION OF POLARIZATION.

Our purpose is to give a covariant description of spin for relativistic particles. A free particle is described by an irreducible unitary representation of the Poincaré group P^{μ}, $M^{\mu\nu}$ obey the following commutation relations,

$$[i P^{\mu}, iP^{\lambda}] = 0$$

$$[i P^{\lambda}, iM^{\mu\nu}] = i P^{\mu} g^{\lambda\mu} - i P^{\nu} g^{\lambda\mu} \tag{1}$$

$$[i M^{\mu\nu}, i M^{\rho\sigma}] = i M^{\mu\rho} g^{\nu\sigma} + i M^{\nu\sigma} g^{\mu\rho} - i M^{\mu\sigma} g^{\nu\rho} - i M^{\nu\rho} g^{\mu\sigma}$$

which form the Lie algebra of the Poincaré group. For a unitary representation, the P's and M's are hermetian (= observables). The infinitesimal operators of the Lie algebra are $i P^{\lambda}$ and $i M^{\mu\nu}$; it is clear that this Lie algebra is on the real field.

DEFINITION 1. Universal enveloping algebra. Physicists want to consider products of P's and M's with the only condition that they identify the commutator AB - BA of two elements with the bracket of Lie algebra [A, B]. This has a name in mathematics: it is called the universal enveloping algebra $\mathcal{E}$ of the Lie algebra $\mathcal{L}$. It has many important properties; for instance any homomorphism of $\mathcal{L}$ into an associative algebra $\mathcal{A}$ e.g., any linear representation of $\mathcal{L}$) can be extended to $\mathcal{E}$. (See a book on Lie algebra for more detail.) When physicists have to consider a mathematical structure, the associated "universal" structure must have also a physical meaning. We can say that $\mathcal{E}$ is the algebra of observables generated by the P's and M's. According to § 3.5 (2) to find a complete set of commuting observables we try to find a maximal abelian subalgebra of $\mathcal{E}$. It contains the elements of the centre the operators $P^\lambda P_\lambda = P^2$, $W^\lambda W_\lambda = W^2$ where

$$W_\lambda = -\frac{1}{2} \epsilon_{\lambda\mu\nu\rho} P^\mu M^{\nu\rho} = -\frac{1}{2} \epsilon_{\lambda\mu\nu\rho} M^{\mu\nu} P^\rho \qquad (2)$$

($\epsilon^{\gamma\mu\nu\rho}$ the anti-symmetric tensor in four dimensions).

Who introduced first this operator before the war ? It seems it was Pauli (see e.g., a reference by Lubanski, Physica in 1942). I do not know. At that time great physicists did not rush to print any thought they had (and they did have a lot of good ones) and did not fight for priority. It is known by physicists but does not seem to be proven by mathematicians that P^2 and W^2 generates the center of $\mathcal{E}$. In an irreducible representation of (1), P^2 and W^2 are multiples of the identity, hence their spectrum can be used to classify these representations.

This is not completely true, because for $P^2 = 0$ $W^2 = 0$ there is an infinity of representation. So for zero mass representation $P^2 = 0$ and "finite spin" one needs the helicity λ defined by $W = \lambda P$ to distinguish the different irreducible representations.

The commutation relations of P and W are given by

$$[W^\lambda, P^\mu] = 0, \; [W^\lambda, W^\mu] = i\, \epsilon^{\lambda\mu\nu\rho} P_\nu W_\rho \qquad (3)$$

$$[W^\lambda, M^{\mu\nu}] = i\, g^{\lambda\mu} W^\nu - i\, g^{\lambda\nu} W^\mu, \qquad (4)$$

Note that $P^\lambda W_\lambda \equiv P \cdot W = 0$ (4')

The W^λ has been built in order to commute with the P's. So if we characterize the state of a particle by the observables P (P^0 energy, **P** momentum) to complete the description of the state we shall have to use some function of the W the polarization operator. (Polarization is what is needed to form with energy and momentum a complete kinematical description of a particle static).

Let us consider an irreducible unitary representation of $\mathcal{L}$ on the Hilbert space $\mathcal{H}$. This representation extends to an irreducible representation of $\mathcal{E}$. The operators P^2

and W^2 are multiples of the identity. Let us diagonalize the P's. Since they have a continuous spectrum, this introduces a continuous basis on $\mathcal{H}$ whose vectors are not in $\mathcal{H}$. (But physicists are not embarrassed by this although mathematicians have been slow to justify them.)

$$P^{\mu} \mid p > = p^{\mu} \mid p > \tag{5}$$

The operators in the enveloping algebra which commute with P^{μ} are functions only of P^{μ}, W^{μ}. Restricted to the vectors given by (5) further specification of the states is given by a function F(W).

$$P^{\mu} F(W) \mid p> = p^{\mu} F(W) \mid p > \tag{6}$$

To each p^{μ} there corresponds a Hilbert space of polarization states which is reduced by F(W). For a representation [M, j], M > 0, the degeneracy of $\mid p>$ is of degree 2j + 1. We shall denote the restriction of W^{μ} to this 2j + 1 dimensional subspace $\mathcal{H}_p \not\subset \mathcal{H}$ by W_p^{μ}. From now on, we consider only the case M > 0.

For each point p^{μ} in the spectrum of P^{μ} choose a tetrad, i.e., orthonomal basis of four vectors $n^{(\alpha)}(p)$, α = 1, 2, 3, 4 where $n_{\mu}^{(0)} = p\mu/M$.

$$n_{\mu}^{(\alpha)} \; n^{(\beta)\mu} = g^{\alpha\beta} \tag{7}$$

$$\epsilon^{\lambda\mu\nu\rho} \; n_{\lambda}^{(\alpha)} \; n_{\mu}^{(\beta)} \; n_{\nu}^{(\gamma)} \; n_{\rho}^{(\delta)} = -\epsilon^{\alpha\beta\gamma\delta} \tag{7'}$$

(This last relation chooses a right handed orientation.)

The completeness of the basis is expressed by

$$n_{\lambda}^{(\alpha)} \; g_{\alpha\beta} \; n_{\mu}^{(\beta)} = g_{\lambda\mu} \tag{8}$$

Let us introduce the operators.

$$S^{(\alpha)} = -\frac{1}{M} W.n^{(\alpha)} = -\frac{1}{M} W^{\lambda} n^{(\alpha)}{}_{\lambda} \tag{9}$$

where in future for two four vectors $a.b = a^{\mu} b_{\mu}$. Note from

(4') that $S^{(0)} = 0$ and $S^{(\alpha)} = (0, \mathbf{S})$, $\mathbf{S} = (S^1, S^2, S^3)$ which are the operators " components of W in the tetrad $n^{(\alpha)}$ (p)"

$$\sum_{i=1}^{3} (S^i)^2 = -\frac{W^2}{M^2} \tag{10}$$

and

$$W_p^\mu = M \sum_{i=1}^{3} S^{(i)} n^{(i)}(p) \doteq M \mathbf{S} \cdot n \tag{10'}$$

From the commutation relation of the W^λ we can compute these of the $S^{(i)}$ and we find

$$[S^{(i)}, S^{(j)}] = i S^{(k)} \tag{11}$$

where i, j, k is a circular permutation of 1, 2, 3. Indeed the S^i are the generators of the little group of p; this little group is isomorphic to SO_3.

As is well known to physicists, from (11), we prove that for an irreducible representation

$$\sum_i {S^{(i)}}^2 = \mathbf{S}^2 = j(j+1), \quad 2j \text{ integer } \geq 0 \tag{12}$$

and from (10)

$$W^2 = -m^2 j(j+1)$$

This also proves that $\mathcal{H}_p$ has indeed dimension 2 j + 1.

To verify that **S** has the correct properties, consider the rest frame $P^0 = M$, $\mathbf{P} = 0$. Denote

$$J^i = M^{jk} \qquad N^i = M^{0i} \qquad \text{(i j k) a cyclic permutation of (123)}$$

then

$$W^0 = \mathbf{p}.\mathbf{J}, \quad \mathbf{W} = p^0 \mathbf{J} - \mathbf{P} \times \mathbf{N}$$

and in the rest-frame $P^0 = M$ so

$$\mathbf{S} = \frac{\mathbf{W}}{M} = \mathbf{J}.$$

For the mass zero case see L. Michel N. Cim. (Supl.) **14**, 99 (1959).

EXAMPLES. 1. Denote the mass operator by $M = \sqrt{P^2}$, and form a function F(W) which is a rational function of the Poincaré operators, e.g.,

$$\Sigma = \frac{1}{M}\left[W - \frac{W^0 P}{(P^0 + M)} \right] \qquad (13)$$

Note that $[\mathbf{P}, \Sigma] = 0$, $\Sigma \times \Sigma = i\,\Sigma$, while in the rest frame $\Sigma = \mathbf{S}$, Σ is then also a good candidate for a spin operator, (see for instance several papers by Chakrabarty in JMP 1964-65 and also H. Bacry's thesis).

Of course, since neither **S** nor Σ are polynomials in P, W (but more general function of them) they do not belong to $\mathscr{E}$, the enveloping algebra but to a much larger algebra. The trouble with this algebra is that it has lost its origin from the Lie algebra of the Poincaré group. It would just as well have been generated by the Lie algebra of SO(4, 1), the de Sitter group, whose infinitesimal generators $M^{\alpha\beta}$ (α, β = 0, 1, 2, 3, 4) can be identified with

$$\alpha \neq 4 \neq \beta \qquad M^{\alpha\beta} = M^{\mu\nu} \qquad \alpha = \mu \quad \beta = \nu$$

$$\alpha = 4 \quad \beta = \mu \qquad M^{4\mu} = [C, P^\mu] \qquad (P^\mu P_\mu)^{1/2} = - M^{\mu 4}$$

where $C = \frac{1}{2} M_{\mu\nu} M^{\mu\nu}$ and $(P^\mu P_\mu)^{1/2}$ is M, the positive square root of the operator P^2 for "physical" representations of the Poincaré group with non zero mass (i.e., $P > 0$).

2. Remarks on SU_6. For one year many physicists tried to define a relativistic SU_6 theory by trying to embed the Poincaré group P and SU_6 into a larger group G. The infinitesimal operators of the subgroup $SU_2 \times 1 < SU_2 \times SU_3 < SU_6$ had to be identified with the spin operators Σ where $\Sigma \times \Sigma = i\,\Sigma$, $[\mathbf{P}, \Sigma] = 0$. Hence Σ might be a way to describe polarization in SU_6. Let us try to fit in the rest of P in some way.

Let

$$Z = \frac{P \times W}{M(P^0 + M)} \qquad J' = J - \Sigma \qquad N' = N - Z \qquad \text{(orbital angular momentum)}$$

One may check that P^μ, J', and N' form a Lie algebra isomorphic to the Poincaré algebra, and the whole algebra commutes with Σ. We call P' the corresponding Lie group whose Lie algebra is that of P^μ, J', N'. Its corresponding mass operator $M'^2 = P^\mu P_\mu = M^2$ but for P', the corresponding $W'^2 = 0$ so P' is without spin.

So the operators P^μ, $J - \Sigma$, $N - Z$, Σ generate the Lie algebra of the direct product $P' \times SU_2$. But the physical Poincaré group (with generators P^μ, J, N) is not a subgroup of it. Indeed P , J, Σ, N, Z do not generate a finite Lie algebra.

On the other hand for the Galilei group we do have for the corresponding operators $[N, \Sigma] = 0$ and the physical Galilei group $\mathcal{G}$ is contained in $\mathcal{G}' \times SU_2$. On this basis we conclude that a Wigner supermultiplet theory with internal symmetry groups SU_2, SU_4, SU_6, ... can exist under Galilean but not Poincaré invariance.

IV. 2 THE DENSITY MATRIX FOR POLARIZATION.

Consider the Hilbert space $\mathcal{H}_p$ of particle states of given energy momentum p and a normalized state $|n, p\rangle \in \mathcal{H}_p$. In the following we drop the momentum dependence. The projection operator for $|n\rangle$ is $P_n = |n\rangle\langle n|$ where

$$\text{trace } P = \langle n | n\rangle = 1, \quad P_n = P_n^* = P_n^2.$$

Let A be an observable, then its expectation value for the state $|n\rangle$ is

$$\langle n | A | n\rangle = \text{Trace } (A\, P_n)$$

Note that P_n is invariant under a phase change in $|n\rangle$.

Given a set of states $|n\rangle$, n = 1, 2, ... orthonomal, with probabilities C_n for finding physical quantities in these states, $\Sigma\, C_n = 1$, the expectation value for A is

$$\text{trace } Ap = \sum_n C_n \langle n | A | n\rangle$$

where $p(p) = \sum_n C_n P_n$ is called the density matrix for the physical system. Note that we have assumed that the states $|n>$ mix incoherently.

Properties of the Density Matrix.

(1) $\rho^* = \rho$
(2) Trace $\rho = 1$,
(3) Trace $\rho^k \leq 1$, k positive integer,
(4) $< n \mid \rho \mid n> \geq 0, \quad \forall \mid n > \epsilon \mathcal{H}_p$.

If $\rho^k = \rho$ for some k we say that ρ describes a pure state corresponding to a single vector $\mid n >$ and $\rho \sim P_n$.

Case of $M > 0$, $j = 1/2$.

For spin 1/2, dim $\mathcal{H}_p = 2$ and the polarization operator W may be expressed in terms of the Pauli matrices

$$\tau = (\tau^1, \tau^2, \tau^3), \; \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The most general two dimensional operator satisfying the conditions for a density matrix is

$$\rho(p) = \frac{1}{2} [I + \zeta(p) . \tau]$$

$$= \frac{1}{2} + \zeta . \mathbf{S}$$

$$= \frac{1}{2} - \frac{1}{M} \zeta . (W_p . n)$$

where $0 \leq \zeta^2 \leq 1$ and $\mathbf{S} = \frac{1}{2}\tau$

Let $s = \zeta . n$ then

$$\rho(p) = \frac{1}{2} - \frac{W^\mu}{M} s_\mu \qquad 0 \leq \zeta^2 = -s^2 \leq 1, \; s.p = 0 \qquad (13)$$

which is a fully covariant expression. The vector ζ is called the stokes vector and $|\zeta|$ specifies the degree of polarization. Observe that

$$\rho^2 = \frac{1}{4}[1 + \zeta^2 + 2\zeta \cdot \tau] = \rho + (\zeta^2 - 1)/4$$

Represent ζ by the unit sphere and its interior (Poincaré sphere), then

(a) origin represents unpolarized state

(b) the surface, completely polarized states

(c) Interior represents partially polarized states with degree of polarization $\zeta^2 = -s^2$.

Longitudinal and transverse polarization. These concepts have a meaning only when a time axis is chosen. Let $t = (1, 0)$ be the unit vector along the time axis. Then, if the particle is not at rest, t and p^μ determine a plane in four space, let l be a unit vector in this plane orthogonal to p^μ and $l^2 = -1$, i.e.

$$
\begin{aligned}
l^2 &= -1 \\
l.p &= 0 \\
l^\mu &= \frac{E}{m\,|p|}\,p^\mu - \frac{m}{|p|}\,t^\mu \\
E &= \sqrt{p^2 + m^2}
\end{aligned}
\qquad (14)
$$

Consider the tetrad $\frac{p}{M}$, $n^{(1)}$, $n^{(2)}$, l where

we write

$$S^\mu = \zeta^{(1)} n^{(1)\mu} + \zeta^{(2)\mu} n^{(2)\mu} + \zeta^{(3)} l^\mu$$

transverse part   logitudinal part

This decomposition is certainly not covariant, and the chosen time axis must be specified.

Higher Spin Generalization.

For the case of a massive particle of given energy-momentum the little group is just SO_3 which acts in $\mathscr{H}_p$ by means of its irreducible unitary representations D^j. Under such a transformat on one readily shows that the density matrix transforms as

$$\rho \rightarrow \rho' = D^j \ \rho \ D^{j\,*}$$

where $D^{j\,*}$ is the hermitian conjugate to D^j. In terms of components

$$\rho' \alpha\beta = D^j \quad \overline{D}^j \quad \rho \sigma \tau$$
$$= Q\, \alpha\beta, \ \sigma \tau \rho$$

and ρ behaves like the components of a tensor whose irreducible components are

$$\sum_{i=0}^{2j} D^{(i)}$$

How we have seen, the polarization density matrix is a polynom in W. To reduce it into irreducible tensors for the little group it is easier just to pass through the operators $S^{(i)}$. For instance

j = 2 (quadrupole) $S^{(i)} S^{(j)} + S^{(j)} S^{(i)} - \frac{2}{3} S^2 \delta^{ij}$

Generally to construct a higher spin multipole consider sums

$$\xi \ ijk \ldots\ldots S^{(i)} S^{(j)} S^{(k)} \ldots\ldots$$

where ξ ijk .... is completely symmetric in (ijk...) and $\Sigma_i \xi_{iijk..} = 0$. To obtain a covariant expression replace $S^{(i)}$ by its covariant expression in terms of W.

The result for spin j has the form

$$\rho(p) = \frac{1}{2j+1} - s_\alpha \frac{W^\alpha}{m} + s_{\alpha\beta} \frac{W^\alpha W^\beta}{m^2} - s_{\alpha\beta\gamma} \frac{W^\alpha \mu 4^\beta W^\gamma}{m^3}$$

$$+ (-1)^{2j} \; s_{\alpha_1\alpha_2\cdots\alpha_{2j}} \frac{W^{\alpha_1} W^{\alpha_2} \ldots W^{\alpha_{2j}}}{m^{2j}}$$

where $s_{\alpha_1\alpha_2\cdots\alpha_{2j}}$ is a 2j rank tensor completely symmetric in

$(\alpha_1 \ldots\ldots \alpha_{2j})$ and $p^\alpha s_{\alpha\alpha_2\cdots\alpha_{2j}} = 0$, $s_{\alpha_1\alpha_2\alpha_3\cdots\alpha_{2j}} = 0.$

For a more detailed and complete account see C. Henry and E. de Rafael, Ann. Inst. Henri Poincaré, Vol II A, 87, 1965.

IV.3 N - PARTICLE STATES.

We turn briefly to setting up the formalism for a system of N particles, whose single particle representations $[M_i, j_i]$ fix the single particle Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$. For a given configuration of momenta $(p_1, p_2, \ldots, p_N)$ if there is no correlation between the particles the N-particle configuration may be represented by

$$|p_1, j_1> \otimes |p_2, j_2> \otimes \ldots\ldots \otimes |p_N, j_N> \text{ in } \bigotimes_{\alpha=1}^{N} H_\alpha$$

After correlation, for instance interaction, the final state vector cannot generally be decomposed as a tensor product. This applies even to independent but identical particles, eg. bosons or fermions require

$$\sum_{\alpha=1}^{N} \otimes \; |p, j>$$

symmetrized
anti-symmetrized

For a system without correlation the N-particle density matrix may be written

$$\rho_N = \bigotimes_{\alpha=1}^{N} \rho_\alpha(p_\alpha)$$

acting in the space

$$\bigotimes_{\alpha=1}^{N} \mathscr{H}_p$$

corresponding to the subspace for a momentum configuration $(p_1, p_2, \ldots, p_N)$.

Let ρ_N^i be the density matrix for an initially uncorrelated N-particle state, and S the scattering operator in

$$\bigotimes_{\alpha=1}^{\infty} \mathscr{H}_\alpha$$

Then the final configuration density matrix is

$$\rho_{N'}^f = S\rho_N^i \quad S^*$$

Let $\rho_{N'}$ be the density matrix describing an arbitrary configuration of the N' final particles. The transition probability for finding a state initially described by ρ_N^i in the configuration ρ_N, after interaction is

$$\lambda = \text{trace}\ (\rho_{N'}^f \quad \rho_{N'}) = \text{trace}\ (\rho_{N'}\ S\,\rho_N^i\ \ S^*)$$

From the general form of the given in 4.2 we see that λ is linear in the polarization tensors $s_{\alpha_1, \alpha_2, \ldots}$ of the different particles, when ρ_N^i and $\rho_{N'}$ are independent particle states.

Perturbation Theory.

Suppose to a given lowest non-vanishing order of perturbation theory $S = I + i\,H + O(H)$, where H is hermitian, and consider a decay of one particle into N.

$$\begin{aligned}\lambda &= \text{trace}\ (\ \{I + i\,H\}\ \rho_1^i\ \ \{\ I - i\,H\ \}\ \rho_N) \\ &= \text{trace}\ [\rho_1^i\ \ \rho_N] + i\ \text{trace}\ H(\rho_1^i\rho_N - \rho_N\rho_1^i) \\ &\qquad + \text{trace}\ (H\rho_1^i\ \ H\rho_N) \\ &= \text{trace}\ (H\rho_1^i\ \ H\rho_N)\end{aligned}$$

as ρ_N and ρ_1^i lie in orthogonal subspaces, i.e., $\rho_1^i \rho_N = 0 = \rho_N \rho_1^i$.

IV.4 PARITY CONSERVATION.

Let the initial state be denoted by Σ_1 and the final state by Σ_2, the transition probability for the reaction $\Sigma_1 \rightarrow \Sigma_2$ is λ_{12}. Under a space reflection with respect to some plane let $\Sigma_1 \rightarrow \Sigma_1'$, $\Sigma_2 \rightarrow \Sigma_2'$. Then if space inversion or parity is a symmetry of our theory

$$\lambda_{12} = \lambda_{1'2'},$$

and inequality indicates a violation of the symmetry.

Define

$$\lambda_{12} = a + b, \quad \lambda_{1'2'} = a - b$$

then a is a scalar and b a pseudo-scalar under space-reflection, since λ_{12} and $\lambda_{1'2'}$, are exchanged by space inversion.

Note that positive probabilities impose $a \geq |b|$. To detect violation of parity we attempt experimentally to measure a non-zero b term. For this let us build Table 1.

Recall that $P(p^0, p) = (p_1^0 - p)$, $P(s^0, s) = (-s^0, s)$; $T(p^0, p) = (p^0, -p)$, $T(s^0, s) = (s^0, -s)$.

EXAMPLE. A decay of unpolarized particles into two or more particles, but only the energy momentum p and polarization s of a $m > 0$, spin 1/2 final particle are observed. Let P the energy momentum of the initial particle $P = p + p' + p''$.

Table 1 shows that we may form the scalar $(P.p)$ and the pseudo-scalar $(P.S)$. This identifies a and b above, and

$$\lambda_{12} = f(P.p) + g(P.p)\ (P.s) \tag{15}$$

since λ_{12} must be linear in S (see 4.3).

TABLE I

Transformation Properties of the Scalars made from p's and s's

Lorentz scalar	P	T	PT	
$s_i . s_j$, $p_i . p_j$, $s_i . p_j$	+	+	+	(i, j, l denote any four of s and p for all particles)
$\det(p_i, p_j, s_k, s_l)$ } $\det(p_i, p_j, p_k, p_l)$	-	-	+	
$\det(p_i, p_j, p_k, s_l)$ } $\det(p_i, s_j, s_k, s_l)$	+	-	-	
Mass zero λ	-	+	-	

Recall that $P(p^0, p) = (p_1^0, -p)$; $P(s^0, s) = (-s^0, s)$;

$T(p^0, p) = (p^0, -p)$, $T(s^0, s) = (s^0, -s)$

We can use for this 1/2 spin particle the general form given in 4.2. We denote by $\rho(p, s)$ the density matrix of the state for which we compute λ

$$\lambda_{12} = \text{Trace } S\rho' S\rho(p,s) = \text{trace } \rho^f \rho(p,s) = \text{trace } \left(f(I - 2\frac{W^\mu}{m_\mu} a) \frac{1}{2} (I - 2 \frac{W^\nu}{m_\nu} s)\right) \tag{16}$$

Where $a.p = s.p = 0$.

We can also use the "Stokes vector" $\alpha^{(i)} = a.n^{(i)}$, $\zeta^{(i)} = s.n^{(i)}$. Then

$$\lambda_{12} = f(1 + \alpha . \zeta) = f(1 - a_\mu s^\mu) \tag{17}$$

By comparison with (15) we then find that

$$a_\mu s^\mu = - g/f \quad s_\mu P^\mu \text{ for all } s .$$

This implies that a is proportional to the projection of P in the 3-plane $\perp$ p and spanned by $n^{(1)}$, $n^{(2)}$, $n^{(3)}$. So a is in the 2-plane P, p and $\perp$ p. Calculation yield $a = \alpha \hat{e}$ with $|\alpha| = |\alpha|$ and

$$\hat{e} = \frac{m^2 + M^2 - (P-p)^2}{\sqrt{-\Delta(M, m, x)}} \frac{p}{m} - \frac{2m}{\sqrt{-\Delta}} P_1 \qquad (18)$$

where $\Delta(a, b, c) = (a+b+c)(a+b-c)(b+c-a)(c+a-b)$, $x = (P-p)^2$.

If a non-zero value of α is found experimentally then parity is violated.

For the reaction $\pi \to \mu + \nu$, s being the μ-polarization, $x = 0$ so

$$\hat{e} = \frac{m_\pi^2 + m_\mu^2}{m_\pi^2 - m_\mu^2} \frac{p_\mu}{m_\mu} - \frac{2\, m_\mu}{m_\pi^2 - m_\mu^2} p_\pi \qquad (19)$$

In the rest system of the π, $\hat{e}$ is the longitudinal polarization of the μ.

For the $\pi^\pm$ decay one finds for the μ polarization

$$s_\mu = \mp \hat{e}$$

which means that it is totally polarized.

Exercise. Apply this to the μ decay when one observes the electron polarization.

IV.5 TIME REVERSAL.

Let T denote time reversal, then

$$T(p^0, p) = (p^0, -p)$$

$$T(s^0, s) = (s^0, -s)$$

$$T\lambda = \lambda$$

Under application of T let the motion reversed systems be Σ_1, Σ_2. Then T invariance implies

$$\lambda_{12} = \lambda_{\hat{2}\,\hat{1}}$$

A better name for this symmetry should be "reversal of motion".

EXAMPLE. Consider a scattering process between massive particles

$$A + B \rightarrow C + D$$

$$p_1 + p_2 \rightarrow p_3 + p_4$$

and the polarization s of particle C is measured. Again looking at Table 1 reveals that $\det(p, p_1, p_3, s)$; $p = p_1 + p_2$; is the only pseudo-invariant. This has its maximum value where $s \perp$ the reaction plane. Let $\hat{p}$, $\hat{s}$ denote the corresponding kinematic quantities for the motion reversed process $\hat{C} + \hat{D} \rightarrow \hat{B} + \hat{A}$. T invariance implies a relation between transition probabilities

$$\lambda_{12}(p_1, p_2 ; p_3 \; p_4) = \lambda_{\hat{2}\hat{1}} (\hat{p}_3, \hat{p}_4 ; \hat{p}_1 \; \hat{p}_2)$$

For a theory in which P is also conserved PT conservation gives $P \lambda_{\hat{2}\,\hat{1}} = \lambda_{2\,1}$ so

$$\lambda_{12} \; (P_1, p_2 ; p_3, p_4) = \lambda_{21} (p_3, p_4 \; ; p_1, p_2)$$

which is the principle of detailed balance when all the polarizations are unmeasured.

Under PT, $P\,\hat{s} \rightarrow -s$ so that PT invariance implies

$$\det(p, p_1, p_3, s_3) = \det(p, p_3, p_1, -s_1) = \det(p, p_1, p_3, s_1)$$

To say that in sentence, PT invariance implies for a scattering process that the polarization producing power of the process is equal to its polarization analyzing power.

Time Reversal in Perturbation Theory.

From § 4.3 we find to lowest order

$$\lambda_{12} = \text{trace}\,(H\rho_1\ H\rho_2) = \lambda_{21}$$

Similarly

$$\lambda_{\hat{2}\,\hat{1}} = \lambda_{\hat{1}\,\hat{2}}$$

and T invariance to any order implies.

$$\lambda_{12} = \lambda_{\hat{2}\,\hat{1}}$$

so to <u>lowest order</u> it yields $\lambda_{12} = \lambda_{\hat{1}\hat{2}}$ <u>i.e.</u>, we compare the kinematic quantities det (p, p_1, p_3, s) and $\det(\hat{p}, \hat{p}_1, \hat{p}_3, \hat{s}) = -\det(p, p_1, p_3, s)$.

The presence of such a term will rule out T invariance in perturbation theory. This seems to be true in $n \to p + e + \nu$, but it required careful measurement. In the light of CP violation found in K_1^0, K_2^0 decay, this is worth studying again to find sensitive tests for T violation if T C P is good. Note however that for the exceptional case of K^0 decay the existence of K^0 - K^0 mass difference does show that second order weak coupling term cannot be neglected. So the term det $(p_K, p_\mu, p_\mu, S_\mu)$ in K $K^0 \to \mu^\pm + \pi^\mp + \nu$ decay can appear (transverse μ-polarization to decay plane see 4.8) without T violation. Then its time average vanishes. See B. G. Kenny and R. G. Sachs, Phys. Rev. 138 B, 943, 1965.

Charge Conjugation (C).

Since C has no effect upon p and s, in the decay of unpolarized particles we have studied the polarization should be the same for particle and anti-particle reactions.

As we have said, they are opposite for μ^+ and μ^- from π decay, so C is violated and CP violation does not show up there.

IV.6 POLARIZATION AND ISOTOPIC OR UNITARY SPIN CONSERVATION.

We want to study the relations between the polarizations in different processes related to each other by an internal symmetry group such as isospin or unitary spin invariance. The simplest non trivial case will be to study the polarization of a spin 1/2 particle in three reactions with two isospin (or unitary spin) channels. For example, study the Σ polarization in

$$\text{a)}\quad \pi^+ + p^+ \to \Sigma^+ + K^+$$

$$\text{b)}\quad \pi^- + p^+ \to \Sigma^0 + K^0$$

$$\text{c)}\quad \pi^- + p^+ \to \Sigma^- + K^+$$

when the possible value of isospin T are 1/2 and 3/2. (Process a is pure $T = 3/2$.) So for given energies momenta and polarizations of the π, p, Σ, K, the corresponding amplitudes f_a, f_b, f_c depends linearly on f 3/2, f 1/2 through coefficients which are Clebsch Gordan coefficients. Hence these 3 amplitudes satisfy a linear relation

$$\lambda_a f_a + \lambda_b f_b + \lambda_c f_c = 0 \tag{20}$$

In that particular case

$$-\lambda_a = \frac{\sqrt{2}}{2} \quad \lambda_b = \lambda_c$$

The corresponding cross-sections are $\sigma_i = |f_i|^2$ (with $i = a, b, c$). They depend only in the modulus of the f_i and relation (20) for the modulus means, if

$$a = |\lambda_a f_a|, \; b = |\lambda_b f_b|, \; c = |\lambda_c f_c|,$$

the three positive numbers a, b, c must be the lengths of the sides of a triangle. Equivalent mathematical formulation of this fact are

$$a \leq b + c, \; b \leq c + a, \; c \leq b + a \tag{21}$$

or

$$|a - b| \leq c, \quad |b - c| \leq a, \quad |c - a| \leq b. \tag{21'}$$

or

$$-2ab \leq a^2 + b^2 - c^2 \leq 2ab, \tag{21''}$$

or

$$\Delta(a, b, c) = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) = -(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2) \geq 0 \tag{21'''}$$

Of course one does not measure pure kinematical states but unpolarized or partially polarized states, and some energy band, or some geometric average on momentum direction, for instance measure of a <u>total</u> cross section. Then the measured quantity is

$$\sigma = \int |f|^2 \, d\mu = \sum_i \sum_s \int |f(E_i, \Omega_i, \delta_i)|^2 \rho(E, \Omega, \delta_i) \, dE_i \, d\Omega_1 \, dE_2 \, d\Omega_2 \ldots$$

where $d\mu$ is a positive measure on the energies and directions of momenta E_i, Ω_i, discrete values of the spin. The averaged cross sections to be compared must be the same average on spin, the same integration on the other kinematical variable; they should differ only by the internal symmetry variable. Then they satisfy the same kind of triangular relation:

$$\Delta(|\lambda_a| \sqrt{\sigma_a}, |\lambda_b| \sqrt{\sigma_b}, |\lambda_c| \sqrt{\sigma_c}) \geq 0 \tag{22}$$

as is shown by the following lemma.

LEMMA. Let $a(x)$, $b(x)$, $c(x)$ positive valued functions defined on a domain D and such that for every $x \in D$, $\Delta(a, b, c) \geq 0$. Let

$$A^2 = \int_D a^2(x) \, d\mu(x), \quad B^2 = \int_D b^2 \, d\mu, \quad C^2 = \int_D c^2 \, d\mu$$

where $d\mu$ is a positive measure. Then $\Delta(A, B, C) \geq 0$.

PROOF. Relation 21'' is also true when integrated by $\int d\mu$ $-2\int a\, b\, d\mu \leq A^2 + B^2 - C^2 \leq 2\int a\, b\, d\mu$. Schwartz's inequality which expresses that $\int (a\lambda + b)^2 d\mu \geq 0$ for any real λ yields ($A, B, C > 0$)

$$(\int a\, b\, d\mu)^2 \leq \int a^2\, d\mu \, . \int b^2\, d\mu = A^2 B^2 .$$

Hence

$$-2AB \leq -2\int a b d\mu \leq A^2 + B^2 - C^2 \leq 2\int a b d\mu \leq 2AB$$

so $\Delta(A, B, C) \geq 0$.

If one observes only the Σ polarization, it is orthogonal to the scattering plane and can be measured by its component ξ for Σ^+, η for Σ^0, ζ for Σ^- which satisfies

$$-1 \leq \xi \leq 1 , \quad -1 \leq \eta \leq 1 , \quad -1 \leq \zeta \leq 1 \tag{23}$$

Let $x = \lambda_a^2 \sigma_a$, $y = \lambda_b^2 \sigma_b$, $z = \lambda_c^2 \sigma_c$ where the σ's are the cross section for unpolarized Σ production and energy and momentum (fixed or average) given once for all. The corresponding cross section with total polarization up (+) or down (-) are $\sigma_i^{(\pm)} = \sigma_i^{(\epsilon)}$ (where $\epsilon = \pm$).

Then the observed polarizations are

$$\xi = \frac{\sigma_a^+ - \sigma_a^-}{\sigma_a + \sigma_a} = \frac{\sigma_a^+ - \sigma_a^-}{2\sigma_a} , \quad \eta = \ldots, \quad \xi = \ldots .$$

so $\sigma_a^{(\pm)} = \frac{1}{2} \sigma_a (1 + \epsilon\xi)$, etc... for η, ζ .

We can write again

$$\lambda_a^2 \sigma_a^{(\epsilon)} = x_\epsilon = \frac{1}{2} x (1 + \epsilon\xi) , \quad y_\epsilon = \frac{1}{2} y (1 + \epsilon\eta),$$

$$z_\epsilon = \frac{1}{2} z (1 + \epsilon\zeta) .$$

We know that we have not only (22) but also

$$\Delta(\sqrt{x_\epsilon}, \sqrt{y_\epsilon}, \sqrt{z_\epsilon}) \geq 0. \tag{24}$$

Let us denote

$$4\,\Delta\,(\sqrt{x_\epsilon}\,,\ \sqrt{y_\epsilon}\,,\ \sqrt{z_\epsilon}\,) = -C_\epsilon\,(\xi\,,\eta\,,\zeta\,) \tag{25}$$

with

$$\begin{aligned} C_\epsilon\,(\xi,\eta,\zeta) = {} & \xi^2\ x^2 + \eta^2 y^2 + \zeta^2 z^2 - 2\,\eta\zeta\ yz - 2\,\xi\zeta\ xz - 2\,\xi\eta\ xy \\ & + 2\,\epsilon\,[\,\xi\,x(x-y-z) + \eta\,y(y-z-x) + \zeta z(z-x-y)\,] \\ & -\ \Delta\,(\sqrt{x},\ \sqrt{y},\ \sqrt{z}\,) \end{aligned} \tag{26}$$

Condition (24) reads

$$C_+ \le 0, \quad C_- \le 0 \tag{26'}$$

In the cartesian coordinates $C_+ = 0$, $C_- = 0$ are the survaces of two cones. Condition (27) gives the intersection of their interior which is inside the cube (23).

Let us study these two cones.

$$C_+(\xi,\ \eta\,,\zeta\,) = C_-\ (-\xi\ ,\ -\eta,\ -\zeta)$$

so they are symmetrical by inversion through the origin.

$$C_-(\xi + 2,\ \eta + 2,\ \zeta + 2) = C_+\,(\xi,\ \eta,\ \zeta\,)$$

so the translation (2, 2, 2) carries C_- on C_+.

Hence their summits are (- 1, - 1, - 1) and (1, 1, 1). Indeed

$$C_\epsilon\,(-\epsilon\ ,\ -\epsilon\ ,\ -\epsilon\,) = 0.$$

Their intersection is a fourth degree curve which decomposes into a conic at infinity and a conic in the plane

$$\xi\,x(x - y - z) + \eta\,y(y - z - x) + \zeta z(z - x - y) = 0.$$

It is an ellipse. Also the cones are inscribed inside the cube (23). For instance the intersection with the plane $\zeta = -1$ is

$$C_+(\xi, \eta, -1) = ((\xi + 2)\ x - (\eta + 2)\ y)^2 = 0$$

i.e., the generators (+ 1) x = (η + 1) y counted twice. See Fig. 1.

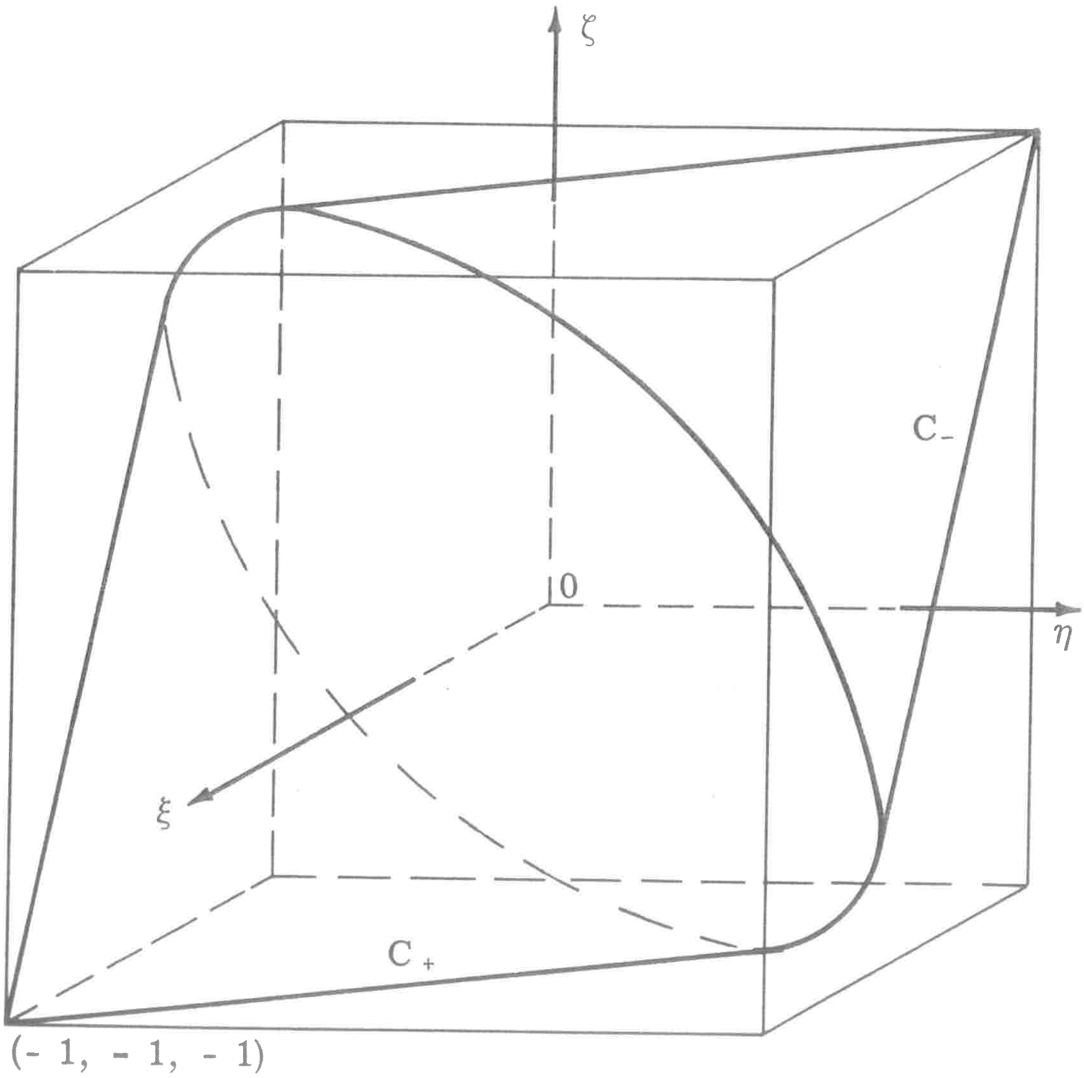

Fig. 1. Three corresponding polarizations in a given direction ξ, η, ζ, must be inside the double cone inscribed in the cube $-1 \leq \xi, \eta, \zeta \leq 1$.

If one polarization (let us say ζ) is not measured, the two measured polarizations ξ, η must be inside the apparent contour of the double cone by vertical projection on the horizontal plane ξ, η.

I computed directly this relation in N. Cim. 22, 203, 1961. Equation (17) of this paper reads in to-day notation.

$$2\cos^2\omega_z - 1 - \xi\eta \leq (1-\xi^2)^{1/2}(1-\eta^2)^{1/2} \qquad (27)$$

where ω is the angle between sides $\sqrt{x}$ and $\sqrt{y}$ in the triangle with sides $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$. I made a silly mistake passing from (17) (here 27) to equation (18) of the quoted letter. The equivalent of (24) is

$$D = (D_1 \cup D_2) \cap D_3$$

which happens to be

$$D = D_1 \cup (D_2 \cap D_3)$$

because

$$D_1 \subset D_3$$

where

$$D_1 : \xi^2 + \eta^2 - 2\xi\eta\cos 2\omega_z - \sin^2 2\omega_z \leq 0 \text{ inside of ellipse} \qquad (28)$$

$$D_2 : -\xi\eta + \cos 2w_z \leq 0 \qquad \text{hyperbola} \qquad (28')$$

$$D_3 : -1 \leq \xi \leq 1, \ -1 \leq \eta \leq 1 \qquad \text{inside of square.} \ (28'')$$

The domain of possible value of (ξ, η) is drawn in Fig. 2.

Of course if $\omega_z = 0$ or π the triangle $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ is flat (because for instance one of the two isospin channels amplitude is negligible) then $\xi = \eta$ the first diagonal in Fig. 2 and the main diagonal of the cube in Fig. 1.

The opposite case, $\omega_z = \pi/2$ gives no relation between ξ and η ($D = D_3$, the whole square). The intersection of the two cones C_ϵ is for instance in the plane $\xi + \eta = 0$ (for $z = x + y$) which of course still yield relations between ξ, η, ζ.

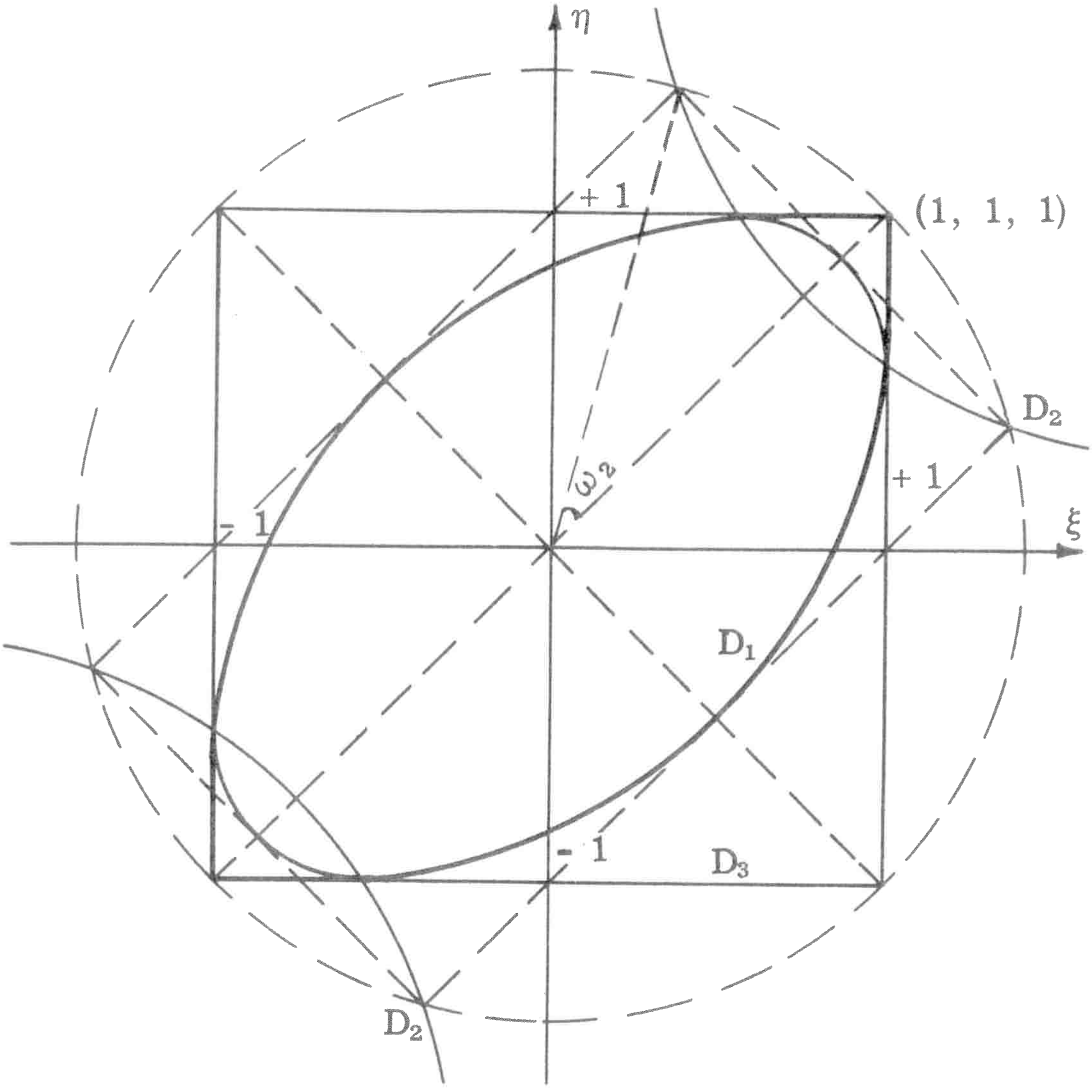

Fig. 2. If two of three corresponding polarizations are measured, they should be in the domain $D_1 \cup (D_2 \cap D_3)$ where D_1 is the interior of the ellipse, D_2 that of the hyperbole (i.e., contains the foci), D_3 that of the square. (See equation 28.)

IV.7 PRECESSION OF POLARIZATION IN AN ELECTROMAGNETIC FIELD.

We turn to the question of a particle with spin interacting with a macroscopic electromagnetic field **E** and **B**. The field is assumed not to vary appreciably over a region the size of the particle. This allows us to consider only direct interactions, and no terms of the form (μ . grad) **B**. Mean values of the observables P^μ and W^λ satisfy classical equations. Let the spin vector be **s**, v the velocity, e the charge, m the mass and g, g' the magnetic and electric dipole factors. The non relativistic equations of motion for v and s are

$$\frac{dv}{dt} = \frac{e}{m}(v \times B + E) \qquad \mu = \frac{g_e}{2m} \quad s \tag{29}$$

$$\frac{ds}{dt} = \mu \times B + \delta \times E \qquad \delta = \frac{g'_e}{2m} \quad s \tag{29'}$$

The last equation is exact in the rest frame of the particle (here the units are so that c = 1). These equations were made covariant, for g = 2, g' = 0, by Thomas (Phil. Mag. 3. 1. 1927), by noting that the linearity of the equations must continue to be true in any frame. Let

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} = F$$

be the electromagnetic field tensor. Denote proper time derivative by dots, i.e.,

$$u = \dot{x} = \frac{dx}{d\tau}, \quad \dot{u} = \frac{du}{d\tau}, \quad \frac{d}{dt} = \frac{1}{\gamma}, \frac{d}{d\tau}, \quad \gamma = (1 - v^2)^{-1/2}$$

$$u = \dot{x} = \gamma(1, v)$$

(29') may then be written (the scalar products are Lorentz products) (most general linear equation in F and u, which specializes to (13) in the rest frame).

$$\dot{u} = \frac{e}{m} \; F \, . \, u \tag{30}$$

with u.s = 0

$$\dot{s} = \frac{e}{m} \; M \, . \, s \tag{30'}$$

where

$$M.s \;=\; \left[F + P \frac{(g-2)}{2} \; \mathbf{F} + \frac{g'}{2} \; \mathbf{F}' \;) \right] . \; s \tag{31}$$

in which $F'\mu\nu = \epsilon \;\; \mu\nu\rho\sigma F^{\rho\sigma}$, $P = 1 - u \otimes u$ a projection on to the plane perpendicular to u. When g = 2, g' = 0 the equations for u and s are identical (see Bargmann, Michel, Telegdi, Phys. Rev. Let. **2**. 435, 1959).

If we write

$$M = [\; F + \; P(a\,\mathbf{F} + a'\;\mathbf{F}')\;\; P] \tag{32}$$

then

$$\dot{u} \;=\; \frac{e}{m} \;\; M \, . \, u$$

$$\dot{s} \;=\; \frac{e}{m} \;\; M \, . \, s$$

$$\dot{n}^{(\alpha)} \;=\; \frac{e}{m} \;\; M \, . \, n^{(\alpha)} \tag{33}$$

where $n^{(\alpha)}$ is the tetrad associated with u. Then M, an antisymmetric tensor is an infinitesimal homogeneous Poincaré transformation. (Note that when there is no electric dipole, g' = 0, and **E** = 0, u . **F'** s is a constant of the motion.)

A useful case is to decompose the precession of polarization into the normal one

$$\dot{n}^{(\alpha)} = \frac{e}{m} \;\; \mathbf{F} \, . \, n^{(\alpha)} \tag{34}$$

and the anomalous one which is described by the motion of the Stokes vector in the frame (34). This is particularly useful when a and a' are small (e.g., G. Charpak, F. J. M. Farley, R. C. Garwin, T. Muller, S. C. Sens, A. Cichichi, N. Cim. 37. 1241. 1965) then the second motion is slow (is a short for Σ_i i = 1, 2, 3)

$$\zeta^i = s.n^{(i)} \dot{\zeta}^i = \dot{s} . n^{(i)} + s . \dot{n}^{(i)}$$

$$= \frac{e}{m} \ (- s . \mathbf{M} . n^{(i)} + s . \mathbf{F}. n^{(i)})$$

$$= \frac{e}{M} \ s . (aF + a' \ F') . n^{(i)}$$

$$= \frac{e}{m} \Sigma_j \ n^{(i)} . (a \ \mathbf{F} + a' \mathbf{F}) . n^{(j)} \ \zeta^{(j)} = \Omega^{ij} \ \zeta^j$$

Ω is the infinitesimal rotation with respect to the " normally precessing" frame.

As another example of application, choose a time axis t = (1, 0) and compute the rate of change of the longitudinal polarization.

Longitudinal Polarization.

Let $|\mathbf{v}| = v$, $\hat{v} = \mathbf{v}/v$, and $l = \gamma\,(v, \hat{v})$, the longitudinal vector ($u.\ l = 0$, $l^2 = -1$) $\quad l = (1/v)\ u + (1/\gamma v)\ t$

Write

$$s = l \cos\varphi + r \sin\varphi, \text{ with r transverse vector}$$

$$r^2 = -1, \ r . l = r.u = r.t = 0$$

then $\cos\varphi$ determines the degree of longitudinal polarization.

Exercise. Prove $\dot{\varphi} = \frac{e}{m} \frac{1}{\gamma v} \ t . \mathbf{M} . r$

IV.8 POLARIZATION AND THE DIRAC EQUATION.

As example of application to Dirac equation see Wightman, Les Houches, 1960 (Relations de dispersion et particules élémentaires, Hermann Paris 1961).

EXAMPLE. Study the decay $K^{\pm} \rightarrow \pi^{0} + \mu^{\pm} + v$

$$P = p' + p + q \qquad \begin{array}{ll} \text{momenta} & P,\ p',\ p,\ q \\ \text{masses} & M,\ m',\ m,\ 0. \end{array}$$

Introduce the real form factors f, g, f', g' such that the box amplitude is $z = (f + ig)\ P + (f' + i\,g')(p + q)$. Then the transition probability for decay into a μ of charge $\lambda = -1$ and polarization s is proportional to

$$\mathcal{M} = [p - \lambda\, m\, s,\ z,\ q,\ \bar{z}\ ] - i\, m\, (s,\ z,\ q,\ \bar{z})$$

where

$$[a,\ b,\ c,\ d] = (a.b)\ (c.d) - (a.c)\ (b.d) + (a.d)(b.c) =$$
$$= \text{Pfaffian of } (a,\ b,\ c,\ d)$$

$$(a,\ b,\ c,\ d) = \text{determinant } (a,\ b,\ c,\ d)$$

Polarization in the decay plane contributes only to the Pfaffian. If the determinant is $\neq 0$, there is a transverse polarization of the $\mu \perp$ to the decay plane (in the rest frame of the K) and it shows a violation of time-reversal.