AMPLITUDE RECONSTRUCTION FOR USUAL QUASI TWO BODY

REACTIONS WITH UNPOLARIZED OR POLARIZED TARGET

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ABSTRACT

Tools for measuring joint polarization and polarization transfer are gathered. They allow the direct reconstruction of amplitudes in numerous quasi two body reactions with spinless beam and unpolarized or polarized target. Eight simple types of such reactions are worked out one by one; the practical results are summarized in Tables.

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1.   INTRODUCTION

Several experimental papers have been published [1,2,3,4]
in which quasi two body reaction amplitudes are reconstructed and
tabulated. Old [5,6,7] and recent [1,8] theoretical papers have
described the method for reconstructing the amplitudes from the data
in some simple cases. But each paper, limited to a particular
reaction, introduces its own arbitrary conventions and this may
obscure the future comparison of the experimental results for
similar reactions. This paper aims to a systematics of amplitude
reconstruction. It introduces a general terminology, it uses the
most commonly accepted quantization conventions and it presents in
details the practical method of amplitude reconstruction in the most
usual reactions with unpolarized or polarized target.

We focus our attention on quasi two body reactions. We suppose
therefore that resonances can be detected (we shall not enter into
the problems of background separation) and that they have well
known spins and parities. Furthermore, we restrict ourselves to
a direct, model independent amplitude reconstruction, based only
on angular momentum and parity conservation. Hence we shall not
make use of other first principles as unitarity and analyticity, nor
shall we relate amplitudes at different s and t values. Finally,
this paper is limited to the simple initial state with a spinless
meson beam and a spin 1/2 (polarized or unpolarized) baryon target.

In Table 1 we list the most common reaction types and tabulate
their number of amplitudes and observables for different target
polarizations. The reaction type refers uniquely to the spins of
the final particles and to the possibility of analyzing their
polarization from their decay. We assume that for the spin ℓ meson
resonance only the even polarization can be measured (e.g. $\rho \to \pi\pi$,
$K^* \to K\pi$), i.e. we do not consider the case of, e.g. $A_1 \to \rho\pi$.
On the contrary, for the baryonic spin j resonance we consider

both cases: i) measurement of even polarization only (j^e), this means no polarization measurement in the case $1/2^e$ (e.g. nucleons), and analysis of the simple, parity conserving decay in the case $3/2^e$, (e.g. $\Delta \rightarrow N\pi$). ii) Measurement of the whole polarization (j) by analysis of the parity violating decay of spin 1/2 baryons (e.g. $\Lambda \rightarrow p\pi$) or by analysis of the cascade decay of spin 3/2 resonances (e.g. $\Sigma^* \rightarrow \Lambda\pi$, $\Lambda \rightarrow p\pi$).

The tabulated number of amplitudes is the number of <u>real</u> independent amplitudes disregarding an overall phase.

For the observables the number tabulated in columns U and T is the number of real and imaginary components of a priori non vanishing multipole parameters and polarization transfer multipole parameters (cf. sect. 2 and 3). As we shall see, at fixed energy and momentum transfer they are not independent, they satisfy some linear and non linear (rank) constraints[*]. For this reason the number of observables is presented as the sum of two numbers: the number of independent observables + the number of constraints (when the second number is zero it has been omitted). Column U corresponds to an unpolarized target experiment, column T corresponds to a transversally polarized target experiment; the observables of column T includes of course the observables of column U. In column L is shown the number of independent observables or constraints that must be added to column U for an experiment with pure longitudinal polarization or to column T for an experiment with transverse and longitudinal polarization. Of course the number of independent observables can never be bigger than the number of amplitudes, but often it is one unit smaller; this is the case when the polarization of one spinning initial or final particle is not at all considered. This is an application of a general theorem derived by Simonius [9], cf. Appendix 2.

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(*) Since we observe only quadratic expressions of the amplitudes, there are often discrete ambiguities in their reconstruction; the non linear constraints can be used to remove some of these ambiguities.

Practically the ghost amplitude appears as a relative phase between two sets of transversity amplitudes.

An analysis of the numbers in Table 1 suggests the following comments:

i) For higher spins the number of amplitudes increases linearly while the number of observables increases quadratically. When the number of observables becomes much bigger than the number of amplitudes it seems reasonable to communicate the amplitudes themselves, in so far as the statistics of the experiment allows their reconstruction.

ii) For some types of reactions all amplitudes but one can be reconstructed with unpolarized target, while for others amplitude reconstruction requires polarized target. For both reaction types, the most simplest cases are those for which the number of observables in columns U and T are set in boxes. It would be reasonable to give priority for the reconstruction of amplitudes to these types of reactions.

iii) As mentioned above, in all reaction types with unpolarized target and in those with a spin $1/2^e$ particle, it remains one ghost amplitude, the relative phase between two sets of transversity amplitudes. Since helicity amplitudes are linear combinations of amplitudes in both sets, they are ghosts too. Furthermore, parity conservation in the reaction has a simpler form in transversity quantization, and the observables are closer to the transversity amplitudes. All these arguments favor the reconstruction of transversity amplitudes. However to facilitate the comparison with models which use helicity amplitudes, for each reaction type we give the relations between the two kinds of amplitudes.

Table 1 lists only general reaction types. In order to appreciate how many usual reactions correspond to each type, we present in Tables 2 and 3 several lists of such reactions. We have

considered only reactions with practicable beam and targets and with
well classified final particles (nonets or decuplets). We obtain
in this way a list of 252 reactions belonging to the 8 types set in
boxes in table 1 and whose amplitude reconstruction will be treated
in detail below.

Furthermore in table 2 and 3 the isospin relations between
these reactions are explicitly given. For reactions related via
one isospin channel, the ratios of their amplitudes to the isospin
amplitudes are fixed coefficients which are given as the coefficients
of (f) in the brackets. Then the measurement of the amplitudes for
only one reaction yields the amplitudes of all reactions in the same
column. For reactions related via two isospin channels the amplitudes
satisfy the relations indicated in part c) of each table. In these
cases the reconstruction of amplitudes for two independent reactions,
because of the overall phase and eventually of the ghost phase, does
not fix the amplitudes of a third reaction. But this reconstruction
for three (two by two linearly independent) reactions fixes the
amplitudes of all the reactions in the same column[*] up to an
overall and eventually one ghost phases. The ghost phase of the
whole set of reactions can be fixed by an experiment with polarized
target for only one reaction of the set.

Sect. 2 and 3 expose the general tools for the measurement of
observables. In sect. 4 the concrete recipes for the amplitude
reconstruction in each type of reaction are given. The hurried reader
can directly skip to the reaction type in sect. 4 he is interested in.
If its experiment uses polarized target we advise him to read
sect. 4.1 in order to see the connection of the terminology with
the standard Wolfenstein parameters. For all other necessary tools
he will find references to sect. 2 and 3.

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(*) The amplitude reconstruction for all other reactions in the
column supplies checks of isospin invariance, and even for only three
reactions, when several amplitudes with their relative phases are
reconstructed.

2.   SOME BASIC TOOLS OF QUASI TWO BODY REACTION ANALYSIS

In this section we present some well known features of the formalism used in the study of quasi two body reactions. For more details one refers to [10,11,12,13].

Let us first precise our notations. A quasi two body reaction will be denoted by

$$1 + 2 \rightarrow 3 + 4.$$

1 denotes the beam, 2 the target, and we call 3 and 4 respectively the particles which share some physical properties with the beam and the target (for instance 1 and 3 are mesons, 2 and 4 are baryons) so that the t- and u-channels are well defined. The 4-momenta of the particles are denoted by p_i (i = 1,2,3,4) with $p_1 + p_2 = p_3 + p_4$, their spins by j_i and their masses by m_i $(m_i \neq 0)$.

2.1 Covariant quantization systems

To describe the polarization of the initial and final states one must fix a quantization frame for each spinning particle. Several different choices are possible, the most popular are the helicity and transversity frames in the s-, t- and u-channels[*]. Unfortunately there is not as yet a universal agreement on the definition of these quantization frames; the most usual conventions are the following.

i)   For each particle and for each channel, the transversity quantization axis $T_n(3)$ and the helicity second axis $H_n(2)$ are along the "Basel normal" n to the reaction plane, defined by

$$n \cdot p_i = 0 \quad (i = 1,2,3), \quad n^2 = -1, \quad \det(n,p_1,p_2,p_3) > 0 \qquad (2.1)$$

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(*) Cf. refs. [10,11].

where the last condition is equivalent to $\vec{n} \cdot \vec{p}_1 \times \vec{p}_3 > 0$ in the laboratory system, or in the center of mass system.

ii) For each particle and for each channel the helicity and the transversity frames have the same first axis, $^T n^{(1)} = {}^H n^{(1)}$.

Note that with these two conventions, the transversity second axis $^T n^{(2)}$ and the helicity quantization axis $^H n^{(3)}$ have opposite directions[(*)]. Furthermore the transversity frame is transformed into the helicity frame by a rotation[(**)] $\tilde{R} = (-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ of $+\frac{\pi}{2}$ around the common axis $n^{(1)}$. The unitary representations $D^j(\tilde{R})$ of this rotation have several useful properties (cf. Ref. [13], [14]).

iii) For each particle i (i = 1,2,3,4) and for each channel a (a = s,t,u), the helicity quantization axis $^H_a n_i^{(3)}$ and the transversity second axis $^T_a n_i^{(2)}$ are defined by

$$^H_a n_i^{(3)} = - {}^T_a n_i^{(2)} = \varepsilon \, q_i(a), \qquad \varepsilon = \pm 1, \tag{2.2}$$

with

$$q_i(a) = [\sinh \phi_i(a)]^{-1} (\hat{p}_i \cosh \phi_i(a) - \hat{p}_{ai})$$

where $\hat{p}_i = p_i/m_i$, $\cosh \phi_i(a) = \hat{p}_i \cdot \hat{p}_{ai}$, $\sinh \phi_i(a) > 0$, and where ai is the particle associated to particle i in the channel a, i.e., for i = (1,2,3,4), si = (2,1,4,3), ti = (3,4,1,2),

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(*) This means that in any channel the x,y,z axes are related by $(^T x, {}^T y, {}^T z) = (^H x, - {}^H z, {}^H y)$. Ref. [14] and the Cracow group use the same convention although a misprint in ref. [14] says the contrary. Many other conventions are occasionally used. Ref. [2] and [3] use $(-{}^T x, {}^T y, {}^T z) = (^H x, {}^H z, {}^H y)$. Ref. [8] uses $(^T x, {}^T y, {}^T z) = (^H z, {}^H x, {}^H y)$. Ref. [15] compares the conventions of ref. [8] and that of the text and it recommends the latter.

(**) We use the Euler angles and the rotation matrices of Rose [16].

$ui = (4,3,2,1)$ [(*)].

iv) The sign ε in eq. (2.2) is a last convention to be chosen.
Jacob and Wick [17] and Cohen-Tannoudji, Morel and Navelet [18]
define $\varepsilon = +1$ so that the s-helicity quantization axis of particle i,
$_s \overset{H\rightarrow(3)}{n}_i$, be along the 3-momentum $\vec{p}_i$ of this particle, in the center of
mass system $(\vec{p}_1 + \vec{p}_2 = 0 = \vec{p}_2 + \vec{p}_4)$. On the contrary, Gottfried and
Jackson [19] define $\varepsilon = -1$ so that their t-helicity quantization
axis $_t \overset{H\rightarrow(3)}{n}_i$, in the rest system of particle i, is along the 3-momentum
$\vec{p}_{ti}$ of the particle ti associated to i in the t-channel.

For convenience in this paper we quantize the spin of the target
in the s-channel transversity frame with $\varepsilon = +1$, i.e., in the
laboratory system the second axis $\vec{n}^{(2)}$ is in the direction of the
beam 3-momentum $\vec{p}_1$. Then the longitudinal polarization of the target
(with respect to the beam) is in the y-direction, and the transverse
polarization is in the (x,z) plane with the z-direction along the
normal to the reaction plane (cf. Fig. 1).

For the final particles we quantize the spins in a transversity
frame. We do not precise the channel since all subsequent equations
are independent of this choice of channel.

2.2 Amplitudes

At fixed energy and momentum transfer, the Hilbert space
$\mathcal{H}_e = \mathcal{H}_1 \otimes \mathcal{H}_2$ of initial particles has $(2j_1 + 1)(2j_2 + 1)$ dimensions.
The transition operator for the reaction is a linear map T between

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(*) We call these frames the s-, t-, or u-helicity frames and the
s-, t-, or u-transversity frames. Some people use the following
vocabulary for helicity frames: t-helicity frame = Jackson-frame,
s-helicity frame = helicity frame; and they extend it to transversity
frames: t-transversity frame = Jackson transversity frame,
s-transversity frame = helicity transversity frame! We find that this
last expression is an awful barbarism since helicity and transversity
are two mutually exclusive notions.

the two spaces. The transversity and helicity amplitudes are the $\mathcal{N} = \prod_{i=1}^{4} (2j_i + 1)$ matrix elements of the transition operator in the transversity and helicity bases. For channel a they are respectively denoted by ${}_a T^{\lambda_3 \lambda_4}{}_{\lambda_1 \lambda_2}$ and ${}_a H^{\lambda_2 \lambda_4}{}_{\lambda_1 \lambda_2}$. They are related by

$$
{}_a T = [D^{j_3}(\tilde{R}) \otimes D^{j_4}(\tilde{R})] \, {}_a H \, [D^{j_1}(\tilde{R})^\dagger \otimes D^{j_2}(\tilde{R})^\dagger] \tag{2.4}
$$

The reflection through the reaction plane, called Bohr-symmetry or B-symmetry, leaves invariant the 4-momenta of the four particles. It acts on the polarization space of a spin-parity j^η particle by the operator $B(j) = \eta \, D^j(n,\pi)$, the product of the parity η of the particle times the unitary representation of the rotation by π around the normal to the reaction plane. If parity is conserved in the reaction, the transition operator is invariant by B-symmetry i.e.

$$
[B(j_3) \otimes B(j_4)] \ T \ [B(j_1)^\dagger \otimes B(j_2)^\dagger] = T, \tag{2.5}
$$

and the matrix elements of T satisfy the relations

$$
\eta \, (-1)^{\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2} \ T^{\lambda_3 \lambda_4}{}_{\lambda_1 \lambda_2} = T^{\lambda_3 \lambda_4}{}_{\lambda_1 \lambda_2} \tag{2.6a}
$$

$$
\eta \, (-1)^{j_3 - \lambda_3 + j_4 - \lambda_4 + j_1 - \lambda_1 + j_2 - \lambda_2} \ H^{-\lambda_3 - \lambda_4}{}_{-\lambda_1 - \lambda_2} = H^{\lambda_3 \lambda_4}{}_{\lambda_1 \lambda_2} \tag{2.6b}
$$

where η is the relative parity of the particles, i.e., $\eta = \eta_1 \eta_2 \eta_3 \eta_4$ with η_i = parity of particle i.

2.3 Density matrices

For a single particle of spin j and fixed 4-momentum, the polarization state is described by a Hermitian, positive, trace one operator acting on the Hilbert space $\mathcal{H}(j)$ and represented by a matrix $\rho(j)$. For a system of two particles (spins j and j') of fixed

4-momenta, the polarization operator acts on the space $\mathcal{H}(j) \otimes \mathcal{H}(j')$ and is represented by the joint density matrix $\rho(j,j')$. When the particles are uncorrelated, this matrix can be written in the form of a tensor product $\rho(j) \otimes \rho(j')$.

For later use, especially for the study of decay angular distributions, it is useful to introduce the polarization multipole parameters of these density matrices.

i) The single particle density matrix $\rho(j)$ is expanded on a set of basis matrices $T(j)^L_M$ (L = 0,..., 2j; M = -L,...,+L), the matrix elements of which are Clebsch-Gordan coefficients

$$(T(j)^L_M)^\lambda{}_{\lambda'} = \langle jL\ \lambda'M | j\lambda \rangle \tag{2.7}$$

The multipole expansion of $\rho(j)$ reads

$$\rho(j) = \frac{1}{2j+1} \left[\mathbb{1} + \sum_{L=1}^{2j} (2L+1) \sum_{M=-L}^{+L} \overline{t^L_M}\ T(j)^L_M \right], \tag{2.8}$$

the expansion coefficients t^L_M are the multipole parameters. We have exhibited the trace of the matrix. We could have written an expansion from L = 0 to 2j, with $T(j)^0_0 = 1$ and $t^0_0 = 1$.

ii) A similar multipole expansion can be written for the joint density matrix $\rho(j,j')$. The set of basis matrices is the tensor products $T(j)^L_M \otimes T(j')^{L'}_{M'}$ and the corresponding multipole parameters are denoted by $t^{LL'}_{MM'}$. The expansion is

$$\rho(j,j') = \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} \frac{(2L+1)(2L'+1)}{(2j+1)(2j'+1)} \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \overline{t^{LL'}_{MM'}}\ T(j)^L_M \otimes T(j')^{L'}_{M'} \tag{2.9}$$

Note that the multipole parameters $t(j)^L_M$ and $t(j')^{L'}_{M'}$ of the single particle density matrices[(*)] $\rho(j) = \mathrm{tr}_{j'}\ \rho(j,j')$ and $\rho(j') = \mathrm{tr}_j\ \rho(j,j')$

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(*) tr_j (and $\mathrm{tr}_{j'}$) represent the partial trace in the space $\mathcal{H}(j)$ (and $\mathcal{H}(j')$), e.g., $\rho(j)^\mu{}_\nu = (\mathrm{tr}_{j'}\ \rho(j,j'))^\mu{}_\nu = \sum_\lambda \rho(j,j')^{\mu\lambda}{}_{\nu\lambda}$.

are $t(j)^{L}_{M} = t^{Lo}_{Mo}$ and $t(j')^{L'}_{M'} = t^{o\,L'}_{o\,M'}$.

A density matrix ρ can be split into a B-symmetric part ρ^{B} and a B-antisymmetric part ρ^{A} satisfying the conditions

$$B\,\rho^{B}\,B^{\dagger} = \rho^{B}, \qquad B\,\rho^{A}\,B^{\dagger} = -\rho^{A}$$

i)   For a single particle, in transversity quantization, the matrix elements of ρ^{B} and ρ^{A} satisfy the relations

$$(\rho^{B})^{\lambda}_{\lambda'} = (-1)^{\lambda-\lambda'}\,(\rho^{B})^{\lambda}_{\lambda'}\ ,\quad (\rho^{A})^{\lambda}_{\lambda'} = -(-1)^{\lambda-\lambda'}\,(\rho^{A})^{\lambda}_{\lambda'} \tag{2.10}$$

and the multipole parameters satisfy the conditions

$$\boxed{\ t^{L\,B}_{M} = (-1)^{M}\,t^{L\,B}_{M}\ ,\qquad t^{L\,A}_{M} = -(-1)^{M}\,t^{L\,A}_{M}\ } \tag{2.11}$$

ii)   For a joint density matrix, in transversity quantization, the matrix elements satisfy

$$(\rho^{B,A})^{\lambda\mu}_{\lambda'\mu'} = \pm(-1)^{\lambda+\mu-\lambda'-\mu'}\,(\rho^{B,A})^{\lambda\mu}_{\lambda'\mu'}\ ,\quad \begin{cases} + \text{ for } B \\ - \text{ for } A \end{cases} \tag{2.12}$$

and the joint multipole parameters satisfy

$$\boxed{\ (t^{LL'}_{MM'})^{B,A} = \pm\,(-1)^{M+M'}\,(t^{LL'}_{MM'})^{B,A}\ \begin{cases} + \text{ for } B \\ - \text{ for } A \end{cases}\ } \tag{2.13}$$

In a quasi two body reaction with spin zero beam and spin $\frac{1}{2}$ target, the transition matrix has 2 columns $(T^{\lambda_3\lambda_4}_{o\pm\frac{1}{2}})$. If the target is unpolarized, the density matrix of the final particles, at fixed energy and momentum transfer, is obtained by[*]

$$\sigma\,\rho_{f} = \frac{1}{2}\,T\,T^{\dagger} \tag{2.14}$$

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[*]   This relation fixes the normalization of T.

(where σ is the differential cross section) and ρ_f has rank 2. If furthermore T is B-symmetric, ρ_f is B-symmetric too and in transversity quantization it can be written in the form of a direct sum $\rho_f = \rho_1 \oplus \rho_2$, each of which has rank one. This property enforces very strong relations between the matrix elements or the multipole parameters of ρ_f.

2.4 Decay angular distributions

If the final particles of the quasi two body reaction are unstable, the angular distribution of their decay products (and occasionally the cascade angular distribution) provides some information on their polarization state. In this paper we limit ourselves to two body decays, however we give some results for 3 body decays in sect. 2.4.4.

2.4.1 Single decay angular distribution

Assume first that only one of the final particles is unstable and let j be the spin of this particle ($j = j_3$ or $j = j_4$). In the rest system of this particle the kinematics of the decay is determined by the polar angle Θ and azimuthal angle ϕ of one of the decay products with respect to the quantization frame of the decaying particle. If M is the decay operator, the normalized angular distribution is defined by

$$I(\Theta,\phi) = \mathrm{tr}\, M\, \rho(j)\, M^\dagger \left/ \int \mathrm{tr}\, M\, \rho(j)\, M^\dagger d(\cos\Theta)\, d\phi \right. \qquad (2.15)$$

This angular distribution is linear in the multipole parameters, it can be written

$$I(\Theta,\phi) = \frac{1}{4\pi} + \sum_{L=1}^{2j} C(L) \sum_{M=-L}^{+L} \overline{t_M^L}\, Y_M^L(\Theta,\phi) \qquad (2.16)$$

where the coefficients $C(L)$ depend on the spins of the decay products and on the dynamics of the decay. If parity is conserved in the decay (e.g. $\rho \to \pi\pi$ $\Delta \to N\pi$) the coefficients $C(L)$ vanish for L = odd;

the angular distribution is an analyser of the even polarization. If parity is violated in the decay (e.g. $\Lambda \rightarrow p\pi$, $\Omega \rightarrow \Xi\pi$) no $C(L)$ coefficient vanishes; the angular distribution is an analyser of the complete polarization.

In many cases angular momentum conservation implies that only one amplitude contributes to the decay. Then the coefficients $C(L)$ are pure numbers. If two amplitudes contribute they depend on one dynamical parameter. Here are the values of the coefficients $C(L)$ for some usual decays

$$1^- \rightarrow 0^- 0^- \quad : \quad \sqrt{4\pi}\, C(2) = -\sqrt{2},$$

$$2^+ \rightarrow 0^- 0^- \quad : \quad \sqrt{4\pi}\, C(2) = -\sqrt{10/7}, \quad \sqrt{4\pi}\, C(4) = \sqrt{18/7},$$

$$2^+ \rightarrow 1^- 0^- \quad : \quad \sqrt{4\pi}\, C(2) = -\sqrt{5/14}, \quad \sqrt{4\pi}\, C(4) = -\sqrt{8/7}, \qquad (2.17)$$

$$\frac{1}{2} \rightarrow \frac{1}{2}\, 0 \quad : \quad \sqrt{4\pi}\, C(1) = \alpha,$$

$$\frac{3}{2}^+ \rightarrow \frac{1}{2}^+ 0^- \quad : \quad \sqrt{4\pi}\, C(2) = -1,$$

where α is the asymmetry parameter of the parity violating decay $\frac{1}{2} \rightarrow \frac{1}{2}\, 0$. With these known values of the $C(L)$ coefficients, the t_M^L are obtained by a maximum likelihood analysis of the angular distribution, or by a moment analysis which yields ($\Omega = (\Theta, \phi)$, $d\Omega = d(\cos\Theta)\, d\phi$)

$$\boxed{C(L)\, t_M^L = \langle Y_M^L(\Omega) \rangle \equiv \int I(\Omega)\, Y_M^L(\Omega)\, d\Omega} \qquad (2.18)$$

Experimentally the moments $\langle Y_M^L(\Omega) \rangle$ of the angular distribution are the mean values of the spherical harmonics $Y_M^L(\Theta, \phi)$ for all events in an ensemble of fixed energy and momentum transfer

$$\langle Y_M^L(\Omega) \rangle = \frac{1}{N} \sum_{i=1}^{N} Y_M^L(\Theta_i, \phi_i)$$

where the index i specifies the event which is considered, and N is the total number of events in the ensemble.

2.4.2 Joint decay angular distribution

If both final particles of the quasi two body reaction are
unstable, one may study the correlations between the directions
of their decay products. Let j and j' be the spins of the
final particles. The joint decay angular distribution reads

$$
I(\Theta,\phi;\;\Theta',\phi') = \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} C(L)\;C'(L') \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \overline{t_{MM'}^{LL'}}\; Y_M^L(\Theta,\phi)\,Y_{M'}^{L'}(\Theta',\phi')
$$

$$(2.19)$$

where $C(L)$ and $C'(L')$ are the coefficients of the single decays, with
$C(0) = C'(0) = 1/\sqrt{4\pi}$. These coefficients being known, the parameters
$t_{MM'}^{LL'}$ are obtained by a best fit analysis of the angular distribution
or by a moment analysis ($\Omega = (\Theta,\phi)$, $\Omega' = (\Theta',\phi')$, $d\Omega = d(\cos\Theta)d\phi$,
$d\Omega' = d(\cos\Theta')d\phi'$)

$$
C(L)\;C'(L')\;t_{MM'}^{LL'} = \langle Y_M^L(\Omega)\;Y_{M'}^{L'}(\Omega)\rangle \equiv \int I(\Omega,\Omega')Y_M^L(\Omega)Y_{M'}^{L'}(\Omega')d\Omega\;d\Omega'
$$

$$(2.20)$$

2.4.3 Cascade decay angular distribution

Consider the cascade decay $C \to A + B$, $A \to A_1 + B_1$ (with spins
$j(C) = j$, $j(A) = j(A_1) = \frac{1}{2}$, $j(B) = j(B_1) = 0$); the first decay
is parity conserving and the second decay is parity violating.
(e.g. $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$). We denote by Θ and ϕ the angles of A
with respect to the quantization frame of C, and by Θ_1 and ϕ_1
the angles of A_1 with respect to the quantization frame for A,
deduced from the quantization frame of C by a pure Lorentz
transformation (boost). Then the cascade angular distribution is

$$
I(\Theta,\phi;\;\Theta_1,\phi_1) = \sum_{L=0}^{2j} \sum_{L_1=0}^{1} \sum_{J\;even} C(L,J,L_1) \sum_{M,N,M_1} \langle J\,L_1\,NM_1|LM\rangle
$$
$$
\times\; \overline{t_M^L}\; Y_N^J(\Theta,\phi)\; Y_{M_1}^{L_1}(\Theta_1,\phi_1)
$$

$$(2.21)$$

where the coefficients $C(L,J,L_1)$ depend on the spins and parities of the particles and on the dynamics of the decays, if they involve more than one amplitude. The most usual decay of this type is

$$\frac{3}{2}^+ \rightarrow \frac{1}{2}^+ + 0^-, \quad \frac{1}{2}^+ \rightarrow \frac{1}{2}^+ + 0^-.$$ For this cascade decay the non vanishing coefficients $C(L,J,L_1)$ are

$$4\pi \, C(0,0,0) = 1, \qquad 4\pi \, C(2,2,0) = -1,$$
$$4\pi \, C(1,0,1) = -\alpha\sqrt{5/9}, \qquad 4\pi \, C(1,2,1) = -\alpha\sqrt{2/45} \qquad (2.22)$$
$$4\pi \, C(3,2,1) = \alpha\sqrt{7/5}$$

where α is the asymmetry parameter of the second decay. With these known values of the coefficients, the t_M^L parameters are deduced by a best fit adjustment of the decay angular distribution, or by a moment analysis $(\Omega = (\Theta,\phi), \; \Omega_1 = (\Theta_1, \phi_1))$

$$C(L,J,L_1) \, t_M^L = \sum_{N,M_1} \langle JL_1 \, NM_1 | LM \rangle \langle Y_N^J(\Omega) \, Y_{M_1}^{L_1}(\Omega_1) \rangle \qquad (2.23)$$

Note that in the above example, since for $L = 1$ one has two non vanishing coefficients, $C(1,0,1)$ and $C(1,2,1)$, the parameters t_M^1 can be measured by two different combinations of the moments.

2.4.4 Three body decays

Some well known unstable resonances undergo 3 body decays, (e.g. $\eta \rightarrow \pi^0 \pi^0 \pi^0$ or $\pi^+ \pi^- \pi^0$ $\omega \rightarrow \pi^+ \pi^- \pi^0$ $\phi \rightarrow \pi^+ \pi^- \pi^0$). The final state is determined by 5 quantities, often split into the 2 Dalitz plot variables and the 3 angles which fix the orientation of the decay plane. In the rest system of the decaying particle, let us denote by Θ and ϕ the angles of the normal to the decay plane with respect to the quantization frame of the decaying particle. Then the angular distribution $I(\Theta,\phi)$ can be written in the same form as eq. (2.16) with coefficients $C(L)$ depending on the spins and parities of the particles and on the dynamics of the decay. For the

most usual decay, i.e., $1^- \to 0^- 0^- 0^-$ (e.g. ω and ϕ decays), the non vanishing $C(L)$ coefficient is a pure number

$$1^- \to 0^- 0^- 0^- \quad : \quad \sqrt{4\pi} \; C(2) = -\sqrt{2} \; , \tag{2.24}$$

which happens to be equal to the coefficient of the two body decay $1^- \to 0^- 0^-$, (c.f. eq. (2.17)).

3. QUASI TWO BODY REACTIONS WITH POLARIZED TARGET

In this section we study the observables of a quasi two body reaction with polarized target. In the particular case of a spin zero beam and a spin 1/2 polarized target we give explicitly the structure of the final state density matrix and of the differential cross section in terms of the initial polarization. We also show how to measure the multipole parameters which describe the polarization transfer between the initial and final states.

3.1 Observables

Consider a quasi two body reaction $1 + 2 \to 3 + 4$. The beam and the target are prepared independently hence the initial state is described by a (Hermitian, positive, trace one) density matrix which is a tensor product $\rho_e = \rho(j_1) \otimes \rho(j_2)$. On the contrary, the polarizations of the final particles are generally correlated and the final state is described by a joint density matrix $\rho_f = \rho(j_3, j_4)$ which cannot generally be written in a tensor product form. These matrices are related through the transition matrix T by

$$\sigma \, \rho_f = T \, \rho_e \, T^\dagger. \tag{3.1}$$

We have denoted by σ the double differential cross section

$$\sigma \equiv \frac{d\sigma}{dt \, d\psi} = \mathrm{tr} \; T \, \rho_e \, T^\dagger \tag{3.2}$$

where t is the momentum transfer and ψ is, in the laboratory system, the angle between the Basel normal $\vec{n}$ and a direction $\vec{\ell}$, perpendicular to the beam direction, fixed by the initial polarization. If the initial state is unpolarized, $\rho_e = \mathbb{1}/n_1 \otimes \mathbb{1}/n_2$ with $n_i = (2j_i + 1)$,

the double differential cross section is denoted by σ_o

$$\sigma_o \equiv \frac{d\sigma}{dt\,d\psi}\bigg|_{\rho_e = \mathbb{1}/n_1 n_2} = \frac{1}{n_1 n_2}\, \text{tr}\, TT^\dagger \qquad (3.3)$$

In this case, the initial state has no preferential direction $\vec{\ell}$ in the laboratory and the double differential cross section σ_o is isotropic in ψ. Then one may consider the simple differential cross section

$$\frac{d\sigma}{dt} = \int_o^{2\pi} \sigma_o\, d\psi = 2\pi\, \sigma_o\ . \qquad (3.4)$$

At fixed energy and momentum transfer, a complete measurement of the reaction includes the measurement of the double differential cross section σ and of the joint final density matrix ρ_f, as functions of the initial polarization ρ_e and of the angle ψ. We call <u>observables</u> of the reaction the set of quantities which parametrize these functions and can be effectively measured. The measure is obtained by an analysis of the differential cross section and of the combined angular distribution of the normal $\vec{n}$ and of the decay products of the final particles (cf. sect. 3.3 below) for different initial polarizations.

3.2 <u>Description of the final state when the target is polarized</u>

From now on we assume that the initial state consists simply of a beam of spin zero particles and a target of spin $\frac{1}{2}$ particles.

3.2.1 <u>The initial state</u>

The initial density matrix ρ_e is a 2 x 2 matrix, which in the laboratory system is described by the polarization pseudo vector $\vec{\zeta}$ ($\vec{\zeta}^2 \leq 1$), also called the Stokes vector. The projection of this vector on a plane perpendicular to the beam fixes the direction $\vec{\ell}$ alluded to previously. As we defined it in sect. 2.1, for s-transversity quantization, in the laboratory system the $\vec{n}^{(2)}$ axis

of the target is in the direction of the beam momentum $\vec{p}_1$, while $\vec{n}^{(3)}$ is along the Basel normal $\vec{n}$ to the reaction plane and $\vec{n}^{(1)}$ is perpendicular to the beam and to this normal. Then the density matrix ρ_e, in s-transversity quantization, is

$$\rho_e = \frac{1}{2}\, (\mathbb{1} + x\,\tau_x + y\,\tau_y + z\,\tau_z) \qquad\qquad (3.5)$$

where τ_x, τ_y, τ_z are the Pauli matrices, and x,y,z are the projections of the vector $\vec{\zeta}$ on the s-transversity axes $\vec{n}^{(1)}$, $\vec{n}^{(2)}$, $\vec{n}^{(3)}$ respectively. These components can be written

$$\vec{\zeta}\ :\quad x = P_T \sin\psi,\quad y = P_L,\quad z = P_T \cos\psi \qquad (3.6)$$

where ψ is the angle between $\vec{n}$ and $\vec{\ell}$ with the sign of $\vec{n} \times \vec{\ell} \cdot \vec{p}_1$, see fig. 1. By definition P_T is the length of the projection of $\vec{\zeta}$ on the (x,z) plane; it is the degree of transverse polarization, $0 \leqslant P_T \leqslant 1$. P_L is the projection of $\vec{\zeta}$ on the beam; it may be positive or negative and its modulus $|P_L|$ is the degree of longitudinal polarization, $0 \leqslant |P_L| \leqslant 1$. Note that $P_T^2 + P_L^2 = \vec{\zeta}^2$ is the degree of polarization of the target. It is important to remark that in general the initial state is not B-symmetric. Indeed, the matrices $\mathbb{1}$ and τ_z are B-symmetric in transversity quantization, but τ_x and τ_y are not. Then, except in the case of normal polarization, $\vec{\zeta} \simeq \vec{n}$, the initial state is not invariant by reflection through the reaction plane.

3.2.2 The density matrix of the final state

The density matrix ρ_f computed from eq. (3.1) with the initial state (3.5) is linear in the components of $\vec{\zeta}$. It can be written in the form

$$\sigma\,\rho_f = \sigma_0(\rho_0 + z\,\rho_z + x\,\rho_x + y\,\rho_y), \qquad\qquad (3.7)$$

where ρ_0 is the density matrix of the final state when the target is unpolarized ($\vec{\zeta} = 0$). If parity is conserved in the reaction,

the transition matrix T is B-symmetric, and since the matrices $\mathbb{1}$ and τ_z are B-symmetric, the matrices ρ_o and ρ_z are B-symmetric too and have non vanishing trace, while the matrices ρ_x and ρ_y are B-antisymmetric and hence traceless. The density matrix ρ_o has trace 1, whereas the trace of ρ_z depends on the dynamics of the reaction; this trace, P_R, is sometimes called the "reaction polarization"

$$\text{tr } \rho_o = 1, \quad \text{tr } \rho_z = P_R, \quad \text{tr } \rho_x = \text{tr } \rho_y = 0. \tag{3.8}$$

By definition the 4 matrices $\sigma_o \rho_\alpha = \tfrac{1}{2} T \tau_\alpha T^\dagger$ ($\alpha = 0,x,y,z$, $\tau_o = \mathbb{1}$) are not independent of each other. It is easy to show that the so-called "polarization transfer matrix" (cf. Appendix 1 and ref. [20])

$$W = \sum_\alpha \tilde{\tau}_\alpha \otimes \sigma_o \rho_\alpha = \sigma_o \quad \begin{array}{|c|c|} \hline \rho_o + \rho_z & \rho_x + i\rho_y \\ \hline \rho_x - i\rho_y & \rho_o - \rho_z \\ \hline \end{array} \tag{3.9}$$

($\sim$ = transposition in the initial space) must be positive and have rank 1, since σW can be written

$$W = \frac{1}{2} \tilde{T} \; \tilde{T}^\dagger \tag{3.9'}$$

where $\tilde{T}$ is the column matrix obtained from the transition matrix T by transposition in the initial space, i.e., its elements are

$$\tilde{T}^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = T^{\lambda_3 \lambda_4}_{\lambda_1 \lambda_2} .$$

The matrix W is B-symmetric; in transversity it can be written in the form of a direct sum $W = W_1 \oplus W_2$. The line and column indices of W_1 satisfy $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = $ even, those of W_2 satisfy $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = $ odd. The rank 1 condition on W implies that either $W_2 = 0$ and rank $W_1 = 1$ or $W_1 = 0$ and rank $W_2 = 1$. Which submatrix is nul depends on the relative parity of the particles. From eq. (2.6a), if $\eta = +1$, $W_2 = 0$, if $\eta = -1$, $W_1 = 0$. The nullity of W_1 (or W_2) yields linear constraints between the elements of ρ_o and ρ_z and of ρ_x and ρ_y, while the rank 1 condition for W_2 (or W_1) gives quadratic constraints between the

elements of all matrices. Often the decay of the final particles
does not allow a complete measurement of the matrices W_1 and W_2.
Then one may obtain constraints on the observable parameters by
elimination of the unobserved quantities from the previous equations.
This elimination keeps the degree of the linear constraints, but it
generally raises the degree of the quadratic constraints.

3.2.3 Multipole expansions

i) If only the final particle 4 has spin ($j_3 = 0$, $j_4 = j$), the
density matrix ρ_f is a single particle density matrix. Its multipole
expansion is (cf. eq. (2.8))

$$\frac{\sigma}{\sigma_0} \, \rho_f = \frac{1}{2j+1} \left[(1 + P_R z) \, \mathbf{1} + \sum_{L=1}^{2j} \sum_{M=-L}^{+L} (2L+1) \{ \overline{t_M^L} + z \, \overline{{}^z t_M^L} + x \, \overline{{}^x t_M^L} + y \, \overline{{}^y t_M^L} \} \, T(j)_M^L \right],$$

(3.10)

where t_M^L and ${}^z t_M^L$ are the multipole parameters of the B-symmetric
matrices ρ_0 and ρ_z, and ${}^x t_M^L$ and ${}^y t_M^L$ are the parameters of the
B-antisymmetric matrices ρ_x and ρ_y. We have exhibited the trace
$(1 + P_R z)$ of the matrix $\frac{\sigma}{\sigma_0} \, \rho_f$; we could keep this term inside the
summation $(L = 0, 2j)$, with the conventions $t_0^0 = 1$, ${}^z t_0^0 = P_R$.

ii) If both final particles 3 and 4 have spin ($j_3 = j$, $j_4 = j'$),
the density matrix ρ_f is a joint density matrix $\rho_f(j,j')$. Its
multipole expansion is (cf. eq. (2.9)).

$$\frac{\sigma}{\sigma_0} \, \rho_f = \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} \frac{(2L+1)(2L'+1)}{(2j+1)(2j'+1)} \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \{ \overline{t_{MM'}^{LL'}} + z \, \overline{{}^z t_{MM'}^{LL'}} + x \, \overline{{}^x t_{MM'}^{LL'}} + y \, \overline{{}^y t_{MM'}^{LL'}} \} \, T(j)_M^L \otimes T(j')_{M'}^{L'}$$

(3.11)

where $t_{MM'}^{LL'}$, ${}^z t_{MM'}^{LL'}$, ${}^x t_{MM'}^{LL'}$, ${}^y t_{MM'}^{LL'}$ are the multipole parameters
of the B-symmetric matrices ρ_0 and ρ_z and B-antisymmetric matrices

ρ_x and ρ_y respectively, with the conventions $t^{oo}_{oo} = 1$ and $^z t^{oo}_{oo} = P_R$.

The multipole parameters $^\alpha t^L_M$ and $^\alpha t^{LL'}_{MM'}$ ($\alpha = x,y,z$) are called "polarization transfer" multipole parameters.

3.3 Measurement of the observables of a reaction

If the final particles are unstable and undergo two body decays, the final state is characterized by the production angle ψ and by the decay angles Θ,ϕ and Θ',ϕ'.

3.3.1 The double differential cross section

From the general form (3.7) of $\sigma\rho_f$ and from the trace conditions (3.8), one gets

$$\sigma = \text{tr } \sigma\rho = \sigma_o(1 + P_R z) \tag{3.12}$$

Then from the value (3.6) of the z-component of the polarization vector $\vec{\zeta}$, the ψ dependence of the double differential cross section is

$$\boxed{\sigma(\psi) = \sigma_o(1 + P_R P_T \cos \psi)} \tag{3.13}$$

i)   The unpolarized double differential cross section σ_o may be obtained by several different ways

$$\boxed{\begin{aligned} \sigma_o &= \sigma(\psi)\big|_{P_T=0} & \text{(3.14a)} \\[2mm] \sigma_o &= \sigma\left(\frac{\pi}{2}\right) & \text{(3.14b)} \\[2mm] \sigma_o &= \frac{1}{2}\left(\sigma(0) + \sigma(\pi)\right) & \\[2mm] \sigma_o &= \frac{1}{2\pi}<\sigma(\psi)> \equiv \frac{1}{2\pi} \int_o^{2\pi} \sigma(\psi)\, d\psi & \text{(3.14d)} \end{aligned}}$$

One may verify that all these ways lead to the same result.

ii) Similarly the asymmetry $P_R P_T$ of the differential cross section can be obtained by several ways. We denote by $\sigma\uparrow$ (resp. $\sigma\downarrow$) the cross sections of the events with the polarization vector $\vec{\zeta}$ above (resp. under) the reaction plane

$$\sigma\uparrow = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma(\psi) \, d\psi \; , \qquad \sigma\downarrow = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sigma(\psi) \, d\psi \; , \qquad (3.15a)$$

and we use the notation

$$\langle f(\psi) \rangle = \int_{0}^{2\pi} f(\psi) \, d\psi \qquad (3.15b)$$

Then the asymmetry can be obtained by

$$P_R P_T = \frac{\sigma(o) - \sigma(\pi)}{\sigma(o) + \sigma(\pi)} \qquad (3.16a)$$

$$P_R P_T = \frac{\pi}{2} \, \frac{\sigma\uparrow - \sigma\downarrow}{\sigma\uparrow + \sigma\downarrow} \qquad (3.16b)$$

$$P_R P_T = \frac{\langle \sigma(\psi) \; 2 \cos\psi \rangle}{\langle \sigma(\psi) \rangle} \qquad (3.16c)$$

3.3.2 Production and single decay angular distribution

Assume first that only one final particle undergo a two body decay (e.g. $\pi p \to \pi\Delta$, $\Delta \to \pi N$; or $\pi p \to \rho N$, $\rho \to \pi\pi$). Then the final state is characterized by 3 angles ψ, Θ, ϕ. Let us call T the reaction transition matrix and M the decay transition matrix. The normalized combined angular distribution is defined by

$$I(\psi; \Theta, \phi) = \frac{\text{tr } MT \rho_e T^\dagger M^\dagger}{\int \text{tr } MT \rho_e T^\dagger M^\dagger d(\cos\Theta) d\phi \, d\psi} \cdot \qquad (3.17)$$

From eq. (3.1) this can be written

$$I(\psi; \Theta, \phi) = \frac{\text{tr } M \frac{\sigma}{\sigma_o} \rho_f M^\dagger}{\int \text{tr } M \frac{\sigma}{\sigma_o} \rho_f M^\dagger d(\cos\Theta) \, d\phi \, d\psi} \cdot \qquad (3.18)$$

Then, by comparison of the multipole expansion of $\frac{\sigma}{\sigma_0}\rho_f$ (cf. eq. (3.10)) with the expansion (2.8) and from the usual decay angular distribution (2.16) one gets the combined normalized angular distribution

$$I(\psi;\Theta,\phi) = \frac{1}{2\pi}\left[\frac{1 + P_R P_T \cos\psi}{4\pi} + \sum_{L=1}^{2j} C(L) \sum_{M=-L}^{+L} \{\overline{t_M^L} + P_T \cos\psi\ \overline{{}^z t_M^L} + P_T \sin\psi\ \overline{{}^x t_M^L} + P_L\ \overline{{}^y t_M^L}\}\ Y_M^L(\Theta,\phi)\right]$$

(3.19)

where the coefficients $C(L)$ are defined in sect. 2.4 and are, for the most usual decays, well known numerical coefficients (cf. eq. (2.17)). By inspection of this expression one sees that it allows the measure of the quantities $P_R P_T$, $t_M^L + P_L\ {}^y t_M^L$, $P_T\ {}^z t_M^L$, $P_T\ {}^x t_M^L$, either by a best fit adjustment or by moment analysis which yields

$$P_R P_T = \langle 2\cos\psi\rangle \equiv \int I(\psi;\Omega)2\cos\psi\ d\Omega\ d\psi \qquad (3.20a)$$

$$C(L)\ (t_M^L + P_L\ {}^y t_M^L) = \langle Y_M^L(\Omega)\rangle \equiv \int I(\psi;\Omega)\ Y_M^L(\Omega)\ d\Omega\ d\psi \qquad (3.20b)$$

$$C(L)\ P_T\ {}^z t_M^L = \langle 2\cos\psi\ Y_M^L(\Omega)\rangle \equiv \int I(\psi;\Omega)\ Y_M^L(\Omega)2\cos\psi\ d\Omega\ d\psi \qquad (3.20c)$$

$$C(L)\ P_T\ {}^x t_M^L = \langle 2\sin\psi\ Y_M^L(\Omega)\rangle \equiv \int I(\psi;\Omega)\ Y_M^L(\Omega)2\sin\psi\ d\Omega\ d\psi \qquad (3.20d)$$

with $\Omega = (\Theta,\phi)$ and $d\Omega = d(\cos\Theta)\ d\phi$. Note that eq. (3.20a) is a particular case of eq. (3.20c) for $L = 0$; with the conventions ${}^z t_0^0 = P_R$, $C(0) = 1/\sqrt{4\pi}$) and is equivalent to eq. (3.16c). By different choices of the initial polarization $\vec{\zeta}$, i.e. of P_T and P_L, one easily deduces from these equations the value of the observables:

i) The target is unpolarized, i.e., $P_L = P_T = 0$. One must first verify that the angular distribution is isotropic around the direction of the beam. Then eq. (3.20b) gives the B-symmetric parameters t_M^L, and one must verify that the B-antisymmetric moments vanish.

(We recall that in transversity quantization B-symmetric parameters have M = even and B-antisymmetric parameters have M = odd).

ii) The target is transversally polarized, i.e., $P_T \neq 0$, $P_L = 0$. Eq. (3.20c) gives the B-symmetric polarization transfer parameters $^z t^L_M$, and one must verify that the B antisymmetric moments vanish. Similarly, eq. (3.20d) gives the B-antisymmetric polarization transfer parameters $^x t^L_M$ and one must verify that the corresponding B-symmetric moments vanish.

Furthermore one may verify that eq. (3.20b) yields the same results as in case i).

iii) The target is longitudinally polarized, i.e., $P_T = 0$, $P_L \neq 0$. One must verify that the angular distribution is isotropic around the common direction of the beam and of the polarization vector $\vec{\zeta}$. Then eq. (3.20b) gives the B-symmetric parameters t^L_M (which should be equal to the parameters obtained in i)) and the B-antisymmetric polarization transfer parameters $^y t^L_M$.

iv) The target is arbitrarily polarized, i.e., $P_T \neq 0$, $P_L \neq 0$. One obtains all the parameters. Eq. (3.20b) gives the B-symmetric parameters t^L_M and the B-antisymmetric parameters $^y t^L_M$. Eq. (3.20c,d) give the B-antisymmetric parameters $^z t^L_M$ and $^x t^L_M$, and one may verify that their B-symmetric moments vanish.

Of course, if the decay is parity conserving, this analysis yields only the L = even parameters (cf. sect. 2.4.1). The moments with L = odd must be found compatible with zero.

3.3.3 Production and joint decay angular distribution

If both final particles (3 and 4) are unstable and undergo two body decay (e.g. $\pi p \to \rho \Delta$ $\rho \to \pi \pi$, $\Delta \to N \pi$) the combined production and joint decay angular distribution is (cf. eq. (2.9, 2.19, 3.11))

$$I(\psi;\Theta,\phi,\Theta',\phi') = \frac{1}{2\pi} \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} C(L) \, C'(L') \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \{ \overline{t_{MM'}^{LL'}} +$$

$$+ P_T \cos\psi \, \overline{{}^z t_{MM'}^{LL'}} + P_T \sin\psi \, \overline{{}^x t_{MM'}^{LL'}} + P_L \, \overline{{}^y t_{MM'}^{LL'}} \}$$

$$Y_M^L(\Theta,\phi) \, Y_{M'}^{L'}(\Theta',\phi') \qquad (3.21)$$

where $t_{oo}^{oo} = 1$, ${}^z t_{oo}^{oo} = P_R$, $C(0) = C'(0) = 1/\sqrt{4\pi}$, $Y_o^o = 1/\sqrt{4\pi}$. The moment analysis of this distribution gives ($\Omega = (\Theta,\phi)$, $\Omega' = (\Theta',\phi')$).

$$C(L) \, C'(L') \, (t_{MM'}^{LL'} + P_L \, {}^y t_{MM'}^{LL'}) = \langle Y_M^L(\Omega) \, Y_{M'}^{L'}(\Omega') \rangle$$

$$C(L) \, C'(L') \, P_T \, {}^z t_{MM'}^{LL'} = \langle 2\cos\psi \, Y_M^L(\Omega) \, Y_{M'}^{L'}(\Omega') \rangle \qquad (3.22)$$

$$C(L) \, C'(L') \, P_T \, {}^x t_{MM'}^{LL'} = \langle 2\sin\psi \, Y_M^L(\Omega) \, Y_{M'}^{L'}(\Omega') \rangle$$

A similar discussion to that of the preceding section 3.3.2 can be made. We shall not repeat it. We only recall that in the present case, the B-symmetric parameters (in transversity) have $M + M'$ = even and the B-antisymmetric ones have $M + M'$ = odd. Furthermore, if both decays are parity conserving, one gets only the L = even and L' = even parameters; all the moments with L or L' odd must vanish. If the decay in (Θ',ϕ') is parity violating one gets the multipole parameters with L = even and L' = even or odd. All other moments vanish.

3.3.4 Production and cascade decay angular distribution

Assume again that only one final particle decays, but that it undergoes a cascade decay of the type discussed in sect. 2.4.3 (e.g. $Kp \to \pi\Sigma^*$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$). Then one may study the angular distribution of the production and of the cascade decay. It reads (cf. eq. (2.21))

$$I(\psi;\Theta,\phi;\Theta_1,\phi_1) = \frac{1}{2\pi} \sum_{L=0}^{2j} \sum_{L_1=0}^{1} \sum_{J \text{ even}} C(L,J,L_1) \sum_{M,M_1,N} \langle JL_1NM_1|LM\rangle \times$$

$$\times \{\overline{t_M^L} + P_T\cos\psi\ \overline{{}^z t_M^L} + P_T\sin\psi\ \overline{{}^x t_M^L} + P_L\ \overline{{}^y t_M^L}\}Y_N^J(\Theta,\phi)Y_{M_1}^{L_1}(\Theta_1,\phi_1)$$

<div align="right">(3.23)</div>

The moment analysis of this angular distribution yields ($\Omega = (\Theta,\phi)$, $\Omega_1 = (\Theta_1,\phi_1)$).

$$C(L,J,L_1)(t_M^L + P_L\ {}^y t_M^L) = \sum_{N,M_1} \langle JL_1NM_1|LM\rangle \langle Y_N^J(\Omega)\ Y_{M_1}^{L_1}(\Omega_1)\rangle$$

$$C(L,J,L_1)\ P_T\ {}^z t_M^L = \sum_{N,M_1} \langle JL_1NM_1|LM\rangle \langle 2\cos\psi\ Y_N^J(\Omega)\ Y_{M_1}^{L_1}(\Omega_1)\rangle$$

$$C(L,J,L_1)\ P_T\ {}^x t_M^L = \sum_{N,M_1} \langle JL_1NM_1|LM\rangle \langle 2\sin\psi\ Y_N^J(\Omega)\ Y_{M_1}^{L_1}(\Omega_1)\rangle$$

<div align="right">(3.24)</div>

A discussion identical to that of sect. 3.3.2 can be made. Note however that in this case, with the $C(L,J,L_1)$ coefficients given in eq. (2.22) all parameters (L = even and L = odd) can be measured.

3.3.5 More complex combined production and decay angular distributions

One may consider more intricate situations. For example, if the two final particles are unstable and one of them undergo a cascade decay (e.g. $Kp \to \rho\Sigma^*$, $\rho \to \pi\pi$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$) the complete angular distribution involves 7 angles. Still more complex is the case where both final particles undergo cascade decays (e.g. $Kp \to A_1\Sigma^*$, $A_1 \to \rho\pi$, $\rho \to \pi\pi$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$). Then the complete angular distribution involves 9 angles.

The expressions of such angular distributions are easily written down, however the present day experimentalists are not yet interested in such complex reactions with polarized target.

4. AMPLITUDE RECONSTRUCTION IN USUAL REACTIONS

In the previous sections we have shown the way of measuring the observables of a reaction with unpolarized target (sect. 2) or with polarized target (sect. 3). They are embodied in the final polarization $\sigma\rho_f$ or in the transfer polarization matrix W, which are quadratic expressions of the transition matrix T or $\tilde{T}$, namely (c.f. eq. (2.14) and (3.9')).

$$\sigma\rho_f = \frac{1}{2}\, T\, T^\dagger, \qquad\qquad W = \frac{1}{2}\, \tilde{T}\, \tilde{T}^\dagger$$

We call amplitudes the elements of these transition matrices. Their reconstruction consists essentially in obtaining an explicit expression for T or $\tilde{T}$ by inverting the quadratic expressions $TT^\dagger$ or $\tilde{T}\tilde{T}^\dagger$. Theoretically this can easily be done, and one obtains T or $\tilde{T}$ up to some unknown phases (by the procedure of "conventional amplitude reconstruction" of Appendix 1,2). Practically each concrete case needs a separate study since generally the observable matrices $\sigma\rho_f$ or W are not completely measured.

In this section we present the practical method of reconstructing the amplitudes in usual reactions with unpolarized and/or polarized target. For pedagogical purposes we first recall the method of reconstructing the amplitudes in the simplest reaction type $\pi p \to K\Lambda$, by measurement of the classical Wolfenstein parameters P,R,A. In view of further generalizations to higher final spins, we introduce, already in this simple case, the multipole formalism and some complex spin rotation parameters. This is discussed in sect. 4.1, and summarized in table 4. In the following sections we present the details of the generalization to reactions of the types $\pi p \to K\Sigma^*$, $\pi\Delta$, $K^*\Lambda$, ρN as simple comments to the tables 5 to 10, in which all the recipes for measurements and amplitude reconstructions have been gathered.

4.1 Reactions of type $\pi p \to K\Lambda$

4.1.1 The Wolfenstein parameters

It is well known that reactions of the type $\pi p \to K\Lambda$ with polarized target and analysis of the final Λ polarization are completely described, at fixed energy and momentum transfer, by σ_o, the unpolarized differential cross section, and P,R,A, the 3 Wolfenstein parameters (cf. ref.[21]) which satisfy one quadratic constraint (cf. table 4)).   Indeed these 4 real numbers supply the whole phenomenological information, namely the differential cross section σ and the final polarization components (X,Y,Z) as functions of the initial ones (x,y,z). In the right part of table 4) we show these functions when the polarizations are quantized in s-transversity frames (cf. sect. 2.1). The simple inspection of these functions shows that each one of the 4 parameters can be measured twice. That both experimental procedures must supply the same result constitutes a Wolfenstein theorem which will be proven below.

4.1.2 The complex spin rotation parameters

A complete measurement of the reaction $\pi p \to K\Lambda$ involves a measure of the differential cross section and an analysis of the combined angular distribution of the normal to the reaction plane and of the Λ decay products, as it was discussed in sect. 3.3. This angular distribution is given in table 4a) and its moment analysis yields the polarization transfer parameters t_M^L, $^z t_M^L$, $^x t_M^L$, $^y t_M^L$ as shown in the same table. There are 8 real, a priori non vanishing, parameters, given in table 4b) (in transversity quantization). They are not independent; they satisfy some linear and quadratic constraints. The method to derive systematically these constraints was given in sect. 3.3.2. In the present case we first write the B-symmetric matrices ρ_o and ρ_z, and the B-antisymmetric ones ρ_x and ρ_y, which appear in the matrix W (eq. 3.9), in the form

$$\rho_0 = \begin{pmatrix} P_0 & \\ & P'_0 \end{pmatrix}, \quad \rho_z = \begin{pmatrix} {}^zP_0 & \\ & {}^zP'_0 \end{pmatrix}, \quad \rho_x = \begin{pmatrix} & {}^xR_0 \\ \overline{{}^xR_0} & \end{pmatrix}, \quad \rho_y = \begin{pmatrix} & -i\,{}^yR_0 \\ i\,\overline{{}^yR_0} & \end{pmatrix}$$

Then the polarization transfer matrix W reads

$W = \sigma_0$

$\lambda_p\ \lambda_\Lambda$	$+1/2$ $+1/2$	$+1/2$ $-1/2$	$-1/2$ $+1/2$	$-1/2$ $-1/2$
$+\tfrac12\ +\tfrac12$	$P_0 + {}^zP_0$			${}^xR_0 + {}^yR_0$
$+\tfrac12\ +\tfrac12$		$P'_0 - {}^zP'_0$	$\overline{{}^xR_0} - \overline{{}^yR_0}$	
$-\tfrac12\ +\tfrac12$		${}^xR_0 - {}^yR_0$	$P_0 - {}^zP_0$	
$-\tfrac12\ -\tfrac12$	$\overline{{}^xR_0} + \overline{{}^yR_0}$			$P'_0 + {}^zP'_0$

(columns labelled $\lambda'_p\ \lambda'_\Lambda$)

It is the direct sum of the external matrix W_1 (with $\lambda_p - \lambda_\Lambda$ = even)
and the internal one W_2 (with $\lambda_p - \lambda_\Lambda$ = odd). The rank of W must
be 1 since the other two particles are spinless. This condition
imposes either

$$W_2 = 0, \text{ rank } W_1 = 1 \longleftrightarrow P_0 = {}^zP_0, \quad P'_0 = {}^zP'_0, \quad {}^xR_0 = {}^yR_0, \quad |R_0|^2 = P_0 P'_0,$$

or

$$W_1 = 0, \text{ rank } W_2 = 1 \longleftrightarrow P_0 = -{}^zP_0, \quad P'_0 = -{}^zP'_0, \quad {}^xR_0 = -{}^yR_0, \quad |R_0|^2 = P_0 P'_0.$$

In each case there are 4 linear constraints and one quadratic constraint.
For our reaction, with relative parity $\eta = \eta_\pi \eta_p \eta_K \eta_\Lambda = +1$, the first
alternative is realized, and the corresponding linear constraints are
equivalent to the Wolfenstein theorem. The parameters P_0 and P'_0 are
real, the parameter $R_0 = {}^xR_0 = {}^yR_0$ is complex. They are also called
"the spin rotation parameters" of the reaction. The definition of
complex spin rotation parameters will be useful in the generalizations
below. The relation of P_0, P'_0, R_0 with the Wolfenstein's A,P,R is
given in table 4c, where we also show their relations with the
polarization transfer multipole parameters and the linear constraints
between the latter.

Finally, table 4d gives the quadratic constraint in terms of A,P,R and P_o, P'_o, R_o and table 4e shows the polarization transfer in terms of the density matrix elements in transversity quantization which are closer to the amplitudes we intend to reconstruct.

4.1.3 Reconstruction of amplitudes

Table 4f introduces the terminology for the helicity and transversity amplitudes (c.f. sect. 2.2). Their relations with the spin rotation parameters are given in table 4g.

Note that with unpolarized target, i.e. by measuring only σ_o and P (or σ_o and P_o) the moduli a and a' of the transversity amplitudes is determinated and only their relative phase is ghost.

On the contrary, for the helicity amplitudes, the moduli are not determined with an unpolarized target.

4.2 Reactions of type $\pi p \rightarrow K\Sigma^*$ $(O^- \frac{1}{2}^+ \rightarrow O^- \frac{3}{2}^+)$

In table 2a, 30 examples of such reactions are listed. For any of them, with unpolarized target but with analysis of the cascade decay of the final baryon, the transversity amplitudes can be measured, up to one ghost phase, by the procedure indicated in table 5. The determination of the ghost phase needs a polarized target and can be performed following the procedure described in table 6.

4.2.1 Reactions with unpolarized target

This section is a comment of table 5. Part a) gives the method for measuring the even multipole parameters by a moment analysis of the two body decay of the Σ^* (cf. sect. 2.4.1). Part b) gives the method for measuring all the multipole parameters by a moment analysis of the cascade decay: $\Sigma^* \rightarrow \Lambda\pi$, $\Lambda \rightarrow p\pi$ (cf. sect. 2.4.3). In fact, for L = even one has $L_1 = O$ and the Θ',ϕ' disappears; then by integrating on Θ',ϕ' , the same distribution as in Part a) is recovered,

with $C(L, J=L, L_1=0) = C(L)/\sqrt{4\pi}$. The parameters t_M^1 can be measured twice, once for J=0 and once for J=2. The compatibility of these two groups of measurements is a check of the experimental and theoretical assumptions (no biases, spin $\frac{3}{2}$ for the Σ^* ...) and can be used to reduce the experimental errors.

The a priori non vanishing multipole parameters are listed in part c). Of course all other multipole moments of the cascade angular distribution can be measured too. Their vanishing is a check of parity conservation in the production and in the $\Sigma^* \to \Lambda\pi$ decay. From these values of the multipole parameters, the density matrix elements are easily obtained (cf. eq. (2.8)). In part d) we give explicitly the non-vanishing density matrix elements in transversity quantization. Remark that since the matrix elements are linear combinations of the multipole parameters, they could be obtained directly by the method of moments as mean values of complicated linear expressions of spherical harmonics. This method can reduce the errors on density matrix elements, and should be applied when amplitude reconstruction is intended. Nevertheless one should perform the checks mentioned above.

The positivity and rank 2 conditions of the 4x4 density matrix (cf. sect. 2.3) impose to its elements the constraints written in part e). Part f) introduces some simple terminology for the transversity and helicity amplitudes which satisfy the B-symmetry conditions (cf. eq. (2.6)). We give also the relation between these amplitudes when the conventions of sect. 2.1 are used. Finally part g) shows the very simple connection of the transversity amplitudes with the measurable transversity density matrix elements.

Remark that the argument of Q and Q' give the relative phases between the amplitudes a and b and between a' and b'. But the relative phase between these two groups of amplitudes is ghost. Therefore the moduli of the helicity amplitudes are also ghosts.

4.2.2 Reactions with polarized target

The determination of the ghost phase requires an experiment with
a polarized target. Table 6 shows the method for measuring all
the observables of the reaction and gives the corresponding generalized
spin rotation parameters (for comparison, see sect. 4.1). Part a)
shows the combined production and cascade decay angular distribution
for an arbitrary target polarization (cf. sect. 3.3.4). Its moment analysis
yields the polarization transfer multipole parameters. In part b)
we list those which are not a priori vanishing. Of course the other
moments of the combined distribution could be measured and should be
found compatible with zero.

Part c) shows the linear constraints on the observed transfer
multipole parameters (cf. sect. 3.2.2) and introduces the linearly
independent generalized spin rotation parameters. The real P's and
the complex Q's can be measured with an unpolarized target by the left
side equations. The complex R's can be measured with a longitudinally
polarized target by the left side equations. Besides, all parameters
can be measured with a transversally polarized target by the right
side equations. Therefore, the experiments with only transverse
or only longitudinal target polarization are equivalent: both supply
the whole physical information. But the first experiment allows
the check of 8 linear constraints; the other 8 constraints can be
checked only when both kinds of experiments are performed.

The generalized spin rotation parameters must still satisfy some
quadratic constraints and some positivity conditions (cf. sect. 3.2.2)
given explicitly in part d). These parameters are the coefficients
of the polarization transfer from the target polarization to the
final particle density matrix as shown in part e).

At last part f) shows the observables in terms of the transversity
amplitudes defined in table 5f). The ghost phase between the
amplitudes a, b and the amplitudes a', b' is contained in the arguments

of the parameters R, R', R_1 and R_2. As emphasized above, it can be measured either with transverse or longitudinal target polarization.

4.3 Reactions of type $\pi p \to \pi\Delta$ ($0^- \frac{1}{2}^+ \to 0^- \frac{3}{2}^{+e}$)

In table 3a, 34 examples of this type of reactions have been listed. Their amplitudes can be completely reconstructed with a transversally polarized target. Table 7 that we now comment gives the recipes for the reconstruction by measuring some generalized spin rotation parameters (cf. sect. 4.1).

Part a) shows the combined angular distribution of the normal to the reaction plane and of the Δ decay products (cf. sect. 3.3.2) It allows the measurement of the polarization transfer multipole parameters by a moment analysis as indicated in the same part a). The list of those parameters which are not a priori vanishing is given in part b). Remark that $^y t_1^2$ can be measured only with a longitudinally polarized target. Of course all other moments of the combined angular distribution could be measured and should be found compatible with zero, as a check of parity conservation in the reaction and in the Δ decay.

Part c) introduces the generalized spin rotation parameters which are linear combinations of the transfer multipole parameters and as them could be directly measured by the moment method. The last line of this part c) shows that a longitudinally polarized target provides no new information, only a complex linear constraint can be checked. This part c) uses the same terminology as part c) of table 6 (reaction type $\pi p \to K\Sigma^*$). But for the Δ we can only measure the even multipole parameters, i.e., expressions of the type $P_1 + P_1'$ given by the left side equations and $P_1 - P_1'$ given by the right side equations; R_- is simply R-R' and can be obtained from the left side and the right side equations, whence the linear constraint.

The spin rotation parameters must satisfy the non linear rank constraints and the positivity conditions written in part d).