CERN

POLARIZATION ANALYSIS IN HIGH ENERGY EXPERIMENTS

L. MICHEL

Institut des Hautes Etudes Scientifiques
Bures-sur-Yvette

P A R T I

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Introduction.

The interest for studying polarization is obvious. When a cross section has been measured in a high energy physics reaction, the measurement of polarization in the same kinematical conditions yields, for each independent polarization parameter, as much information, i.e. the value of a Hermitian form of the amplitudes.

The dependence of cross sections on energy, momentum transfer etc..., has already given several beautiful and simple physical laws or mechanisms (e.g. rising of cross section as s log s , pomeron or Regge poles exchange); there is still a lot to discover on the dependence of polarization on this same variables; and generally the study of polarization is the best tool for studying the importance of angular momentum transfers.

It is true that hadrons are composite systems so the trend of elementary particle physics is to go away from pure hadronic physics as it has historically gone away from molecular, atomic and nuclear physics. Of course these branches of physics are still very lively (and their polarization effects are well studied!). One can doubt that CERN will use only neutrino beams and forget its other potentialities! I would guess even the contrary; as elsewhere CERN will also do more and more sophisticated hadronic physics (not to forget that the ISR are unique up to now); this includes the use of polarized targets (already well developped here) and the use of polarized beams. In some respect these experiments are fascinating: indeed it is more exciting to find unexpected results; and for the last ten years most of the measured polarization effects have flatly contradicted the current (and too much simple) theoretical models. Presently there are unexpectially large polarization effects in some inclusive reaction; in p-p reactions different measured polarization parameters, such as C_{nn} have an extraordinary dependence on momentum transfer. It is an subjective attitude, but physicists are usually fascinated by what they do not yet understand.

For the unified weak and electromagnetic interactions, polarization is as much important. All neutrinos and antineutrinos are polarized (respectively left and right circular polarization) and electrons and positions are also very easily polarized in storage rings. Strongly polarized high energy photons beams have been obtained in Stanford from lasers and the electron beam.

Finally, one can even say that some measurement of polarization cannot be avoided in high energy physics! Indeed, whatever the beam and the interaction, all produced resonnances are recoqnized only by their decay modes, and to observe them is to observe partially their polarization. It seems that such polarization measurements are not fully exploited. One of the reason might be that most physicists (especially outside CERN) do not know the polarization domain corresponding to their experiment when particles of spin $> \frac{1}{2}$ are involved.

Indeed for each experiment physicist determine the domain of observable values of the energy momenta of the involved particle imposed by energy and momentum conservation (this is the Dalitz plot or a generalization of it). Similarly for each experiment in which polarization is observed, there is a Polarization Domain imposed by angular momentum conservation. It is a great handicap for the experimentalists not to know the polarization domain of their experiment; mainly it is a loss for the physical interpretation of the experimental results. Indeed the distribution of the data inside the polarization domain gives the significance of the experiment. Different part of the polarization domain correspond to different mechanism for the reaction. Of course theoretical physicists cannot ignore the polarization domain, as they usually do. Not to speak of a healthy, and to me absolutely necessary scientific curiosity, every theorist interested by the success of a model should know how much stronger than those of the great conservation laws are the predictions of the model. The smaller is the domain of model predictions inside the polarization domain, the better is the corresponding experiment for testing the model. Fig. 1 illustrate some of those points.

These five lectures cannot deal with every aspect of polarization. We shall of course recall the essential of the covariant description of polarization. But the main aim of these lectures of the Academic Training program will be to help physicists to determine themselves the polarization domain of the experiment they consider or they perform. Paradoxically it is not easy to find this help in text-books or even in the printed literature. For the simplest strong interaction reactions, the polarization domain for particles of spin 1 and 3/2 observed by strong two body decays (ρ, K^{+}, φ, and Δ) was, to my knowledge, first published respectively by Pierre MINNAERT [66] and Manuel DONCEL [67] , in 1966 and 1967. Since, we joined our efforts for a systematic study of polarization domains implied by general conservation laws, including hadronic internal symmetry, and some general model predictions. We gave together a set of lectures at the "Ecole d'Eté de Physique des particules" (Gif-sur-Yvette, 1970) which edited our 300 page lecture notes. Of course, the overlap between these notes and the present ones will be relatively small. The present lectures, less technical, will mainly survey some of the published or unpublished work we have done together since 1970. I thank CERN for inviting me to give these lectures and present some work of Doncel, Minnaert and I.

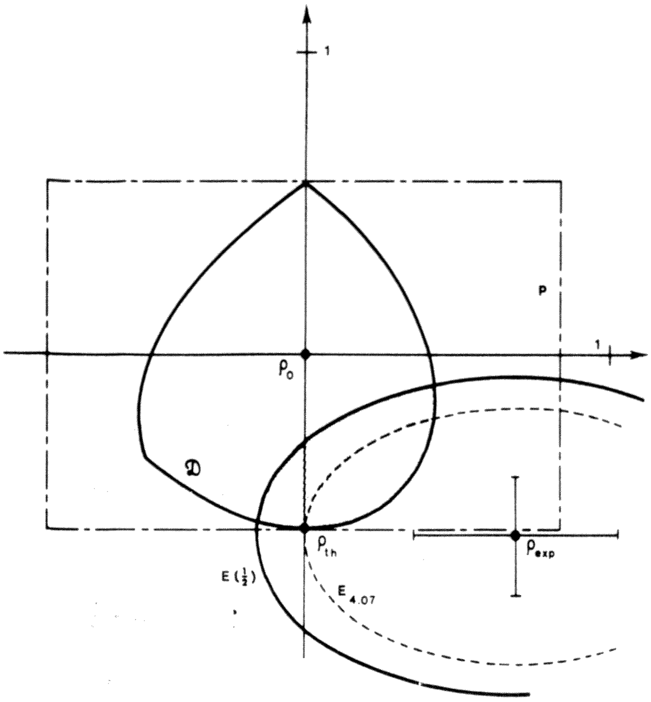
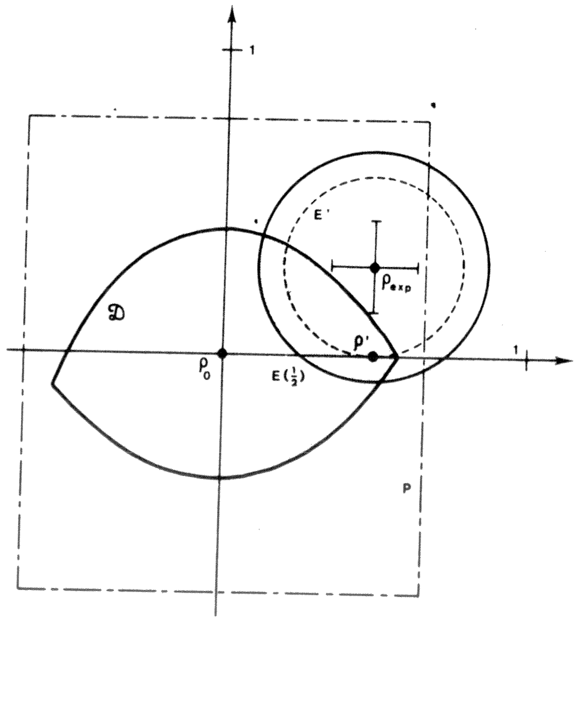


Fig. 1a

Fig. 1b



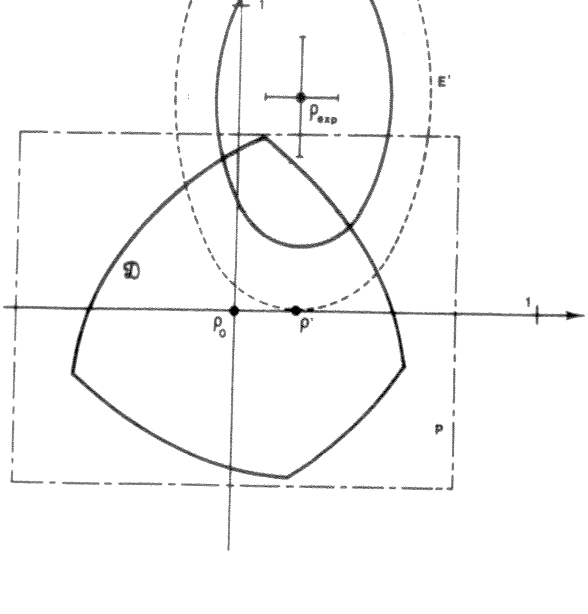


Fig. 1c

Caption of Fig. 1a, b, c.

These three Figures are taken from: M. Daumens, C. Massas, L. Michel, P. Minnaert, Nucl. Phys. B 53 (1973) 303 . They are based on an experimental paper, published in CERN (collaboration of seven groups) where it is claimed that the polarization measurements of Y^{*+} (1385) in:

$\pi^+ p \to K^+ Y^{*+}$ or $K^- p^+ \to \pi^- Y^{*+}$ and $\pi^+ p^+ \to Y^{*+} + \ldots$ or $K^- p^+ \to Y^{*+} + \ldots\ldots$

(the last two are inclusive reactions) are in favour of the quark model. Seven polarization parameters can be measured, so $\dim \mathcal{D} = 7$. These figures show a two dimensional cut of $\mathcal{D}$ and of $E_{\frac{1}{2}}$ (the confidence $\frac{1}{2}$ ellipsoid in a χ^2-test) by the 2-plane determined by the 3 points: ρ_{exp} , ρ_{th}, ρ_o (= unpolarized state).

Fig. 1a is for both exclusive reactions together; the statistics is very poor the errors are as large as the size of the polarization domain, volume $E_{\frac{1}{2}} \sim 7 \times$ volume $\mathcal{D}$; nothing is measured, so the χ^2-test is good with any model. For the exclusive reaction, the theoretical domain $\mathcal{D}$ is the intersection of $\mathcal{D}$ by a 4-dimensional subspace $\mathcal{E}_{th}$; and ρ_{th} is taken as the nearest point from ρ_{exp} in $\mathcal{E}_{th}$ (i.e. the orthogonal projection of ρ_{exp} on $\mathcal{E}_{th}$) ; volume $E_{\frac{1}{2}}$ / volume $\mathcal{D}$ is respectively .1 and .02 for 1b and 1c; the data is much more significant. However the better the data, the poorer the agreement with the assumed predictions of the quark model: in 1c the disagreement cannot be worse. This imposes two remarks:

i) it is unlikely that the simplest predictions of the quark model used by the author apply to their inclusive reaction and the truth is that this paper is not a test of the model.

ii) In general the data of inclusive reaction is well inside $\mathcal{D}$ (see part II). The fact that it favours the boundary of the domain is very striking and begs explanation.

I. What is polarization? How to measure it.

I.1 The polarization of a particle with a known energy-momentum $\underline{p}$ (with $\underline{p}^2 = m^2$ in units $c=\hbar=1$) is what has to be still observed in order to know completly the state of the particle.

This can be extended to a set of several particles (e.g. those produced in a collision) of known energy-momenta $\underline{p}_i$; we then speak of the joint polarization of the particles; this is a much reacher concept that the set of the polarizations of each particle separately because it includes the polarization correlations.

We are interested only by a relativistic theory of polarization, but to begin with the simplest notion, we first consider the polarization of a single particle of spin j (2j integer ≥ 0) and mass $m > 0$ and known energy momentum $\underline{p}$. We can therefore choose a frame where this particle is at rest. Then its Hilbert space of state vectors $\mathcal{H}_j$ is finite dimensional: $\dim \mathcal{H}_j = 2j+1$ and every Hermitian operator on $\mathcal{H}_j$ is a polarization observable!

Invariance under the connected# Poincaré (= Lorentz inhomogeneous) group $\mathcal{P}_+^\uparrow$ requires for a particle of energy-momentum $\underline{p}$ the global invariance of its set of states by the "little group" $\mathcal{L}_p$ of Lorentz transformations※ leaving $\underline{p}$ invariant.

$$\mathcal{L}_p = \{\Lambda \in \mathcal{L}_+^\uparrow \, , \, \Lambda\underline{p} = \underline{p} \} \, . \tag{1}$$

We will deal later with space and time reflections

※ Here Lorentz transformations means simply elements of the Lorentz group. To speak of a pure Lorentz transformation of velocity $\vec{v}$, we will say a Lorentz boost or simply a boost. Note that this is not a covariant concept: it depends on the chosen frame.

For a particle at rest, the little group $\mathcal{L}_p$ is the rotation group SO(3) and Poincaré covariance is reflected by the action of SO(3) on $\mathcal{H}_j$: the space of state vectors of a particle of spin j at rest carries the 2j+1 dimensional irreducible linear representation, up to a phase, of the rotation group. Wigner has shown us that it is equivalent to consider the 2j+1 dimensional irrep (= irreducible linear representations) of SU(2), the covering group of SO(3) .

All useful formulae concerning these representations will be given in the appendix AI. of this chapter. We just recall here that the rotation of angle θ around the oriented axis of unit vector $\vec{n}$ is "covered" in SU(2) by the unitary matrices $\pm u(\vec{n}, \theta)$ with

$$u(\vec{n}, \theta) = e^{-\frac{i}{2}\theta\, \vec{n}.\vec{\sigma}} = (u(\vec{n}, \theta)^{-1})^{*} = u(\vec{n}, -\theta)^{*} \qquad (2)$$

where $\vec{\sigma} = \{\sigma_i \, , \, i=1,2,3\}$ are the three Pauli matrices.

Note that

$$u(\vec{n}, 2\pi) = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \qquad (3)$$

Of course SU(2) and SO(3) have same Lie algebra whose generators J_i , satisfy:

$$[J_i, J_j] = i\, \epsilon_{ijk}\, J_k \qquad (4)$$

are the observable of angular momentum.

I.2 Spin ½ particles.

The space $\mathcal{H}_{\frac{1}{2}}$ is the space of spinors $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$. The rotation groups acts by (2) on this space. The four matrices I , σ_1, σ_2, σ_3 form a complete basis of the space $\mathcal{L}(\mathcal{H}_{\frac{1}{2}})$ of linear operators. A pure polarization state is represented by a normalized state vector $\in \mathcal{H}_{\frac{1}{2}}$, defined up to a phase.

$$< \xi, \xi > = \bar{\xi}_\alpha \xi_\alpha = 1 \tag{5}$$

The angular momentum, which is pure spin angular momentum since the particle is at rest, is $\vec{j} = \frac{1}{2}\vec{\sigma}$. However $\vec{\sigma}$ is also called the polarization vector and its expectation value

$$\vec{s} = < \xi, \vec{\sigma}\, \xi > = \mathrm{tr}\, \vec{\sigma}\, \xi > < \xi = \bar{\xi}_\alpha \vec{\sigma}_{\alpha\beta} \xi_\beta \tag{6}$$

satisfies, from (5)

$$\vec{s}^{\,2} = 1 \tag{7}$$

To summarize, we see that totally polarized states (= pure states) of a spin $\frac{1}{2}$ particle at rest form a 2 dimensional sphere S_2 of radius 1 in the 3-dimensional space. There is also a one to one correspondance between the pure states, the point of S_2 and the rank one Hermitean projectors on $\mathcal{H}_{\frac{1}{2}}$

$$\rho = \rho^* = \rho^2 \qquad \mathrm{tr}\, \rho = 1 \tag{8}$$

Indeed $\rho = \xi > < \xi = \frac{1}{2}(I + \vec{s}.\vec{\sigma})$ (9)

Often the polarization state is not completly known; for instance in an experiment we may know, thanks to a homogeneous vertical magnetic field, that the polarization vector $\vec{s}$ is vertical, but with a probability c_+ to be up and c_- to be down ($c_{\pm} > 0$ $c_+ + c_- = 1$). Then we will say that the particle is partially polarized with the polarization

$$\vec{s} = c_+ \vec{s}_+ + c_- \vec{s}_- \quad ; \tag{10}$$

this state is also called technically a "mixture" . It cannot be represented by a state vector, but it is represented by the density operator

$$\rho = c_+ \xi_+ > < \xi_+ + c_- \xi_- > < \xi_- = \frac{c_+}{2}(I + \vec{\sigma}.\vec{s}_+) + \frac{c_-}{2}(I + \vec{\sigma}.\vec{s}_-) = \frac{1}{2}(I + \vec{s}.\vec{\sigma}) \tag{11}$$

where $\vec{s}$ satisfy (10) (indeed $\vec{s}_{\pm}$ satisfy (7), and $c_+ + c_- = 1$, then $\vec{s}^2 \leq 1$).

To summarize, the polarization states of a particle of spin $\frac{1}{2}$ at rest are in one to one correspondance with the points of the unit ball. Indeed each density operator ρ defines a vector:

$$\vec{s} = \mathrm{tr}\, \rho \vec{\sigma} \quad \text{with} \quad \vec{s}^2 \leq 1 \tag{12}$$

The points of the surface $\vec{s}^2 = 1$ represent the totally polarized (= pure) states. The points of the interior represent the partially polarized states (mixtures). The center $\vec{s} = 0$ represents the unpolarized state. The length $|\vec{s}| = \sqrt{\vec{s}^2}$ is the degree of polarization, it is an invariant for rotations. There is also a one-to-one correspondance with the "density" operators, that is the positive operators

$$\rho = \rho^* \geq 0 \tag{13}$$

of trace one.

Let us recall that an operator on a Hilbert space $\mathcal{H}$ is positive iff (if and only if) its expectation value for any vector $x \in \mathcal{H}$ is ≥ 0 ; $\langle x | \rho x \rangle \geq 0$. It is equivalent to say that ρ is Hermitian# and has all its eigenvalues ≥ 0 .

The ball $\vec{s} \leq 1$ is the polarization domain. As (10) shows, it is the convex hull of the set of pure polarization states. This is a general remark valid for any spin. We also note that every pure polarization state $\vec{s}$, with $\vec{s}^2 = 1$ can be transformed into any other pure polarization state by a rotation. However this is only true for spin $j = \frac{1}{2}$ (although the contrary is sometimes claimed in books on quantum mechanics).

Indeed the antihermitian part would have a pure imaginary expectation value.

I. 3 Spin $j > \frac{1}{2}$ particles.

The dimension of $\mathcal{H}_j$ is $2j + 1$ and $2(2j+1)$ real parameters are necessary to fix a vector of $\mathcal{H}_j$. If it is normalized to one and since an overall phase is arbitrary, 4j real parameters are necessary to fix a pure state or a rank one hermitian projector (they form the points of $P(2j, \mathbb{C})$ the complex projective space of dimension 2j.

Since the rotation group SO(3) has only three parameters, it cannot transform every pure state into every pure state for $j \geq 1$. For instance if $|z> \in \mathcal{H}_1$, it has 3 complex coordinates $z_i = x_i + i\, y_i$ and $|z_i z_i| = |\vec{x}^2 - \vec{y}^2 + 2i\, \vec{x}.\vec{y}|$ is rotationally invariant; it varies between 0 and 1 ; it is zero for circularly polarized state $\vec{x}.\vec{y} = 0$, $\vec{x}^2 = \vec{y}^2$ and one for longitudinal states $\vec{x} \times \vec{y} = 0$.
A useful geometrical representation for the pure states of polarization of a spin j particle has been given by H. Bacry [64]. These states are in one to one correspondance with the constellations# of 2j points on the unit sphere S_2. It generalizes directly the case $2j = 1$. To choose a quantization axis corresponds to the choice of a diameter on S_2. The basis states $|j; m>$ is represented by $j+m$ points at the North pole and $j-m$ at the South pole of this diameter. For instance for $j=1$, pure circularly polarized states are represented by 2 points together and longitudinal states by two points at the two extremities of a diameter. The action of the rotational group on S_2 is the usual action: for instance for spin $j=1$ all pure circularly polarized states can be transformed into each other; this is also true of longitudinal states. But other states cannot be transform into a basis states $|j;m>$ by a choice of coordinates! More generally pure polarization states form by two isometrical patterns of 2j points on S_2 are on the same orbit of the rotation group.

i.e. any set of 2j points not necessarily distinct.

The polarization domain of the polarization states of a spin-j particle at rest has dimension

$$\dim \mathcal{D}_j = (2j+1)^2 - 1 = 4j(j+1) \tag{14}$$

There is no simple geometrical visualization of this domain, but we will later study all its relevant properties. Now we just give a parametrization of this domain, by expanding density matrices in a convenient basis. It must take into account the action of the rotation group. Let $\mathbf{R} \to D_j(R)$ the $2j+1$ irrep (= irreducible unitary representation) of SU(2) on $\mathcal{H}_j$. The corresponding action on $\mathcal{E}(\mathcal{H}_j)$, the space of Hermitian operators on $\mathcal{H}_j$ is given by:

$$R.\rho \to D_j(R)\,\rho D_j(R)^{-1} = D_j(R)\,\rho D_j(R)^* \tag{15}$$

$$\text{i.e.} \quad (R.\rho)_{k\ell} = D_j(R)_{kk'}\,\overline{D_j(R)_{\ell\ell'}}\,\rho_{k'\ell'} \tag{16}$$

It is well known to physicists that $\overline{D_j(R)}$, the complex conjugate of the irrep $D_j(R)$, is equivalent to $D_j(R)$:

$$\overline{D_j(R)} = \Gamma\, D_j(R)\, \Gamma^{-1} \tag{17}$$

and that therefore the tensoriel product decomposition into irrep for SO(2) is

$$D_j \otimes D_j = \bigoplus_{L=0}^{2j} D_L \tag{18}$$

This means that the action of the rotation group on the space $\mathcal{E}(\mathcal{H}_j)$ of observables on $\mathcal{H}_j$ decompose into a direct sum of $2j+1$ inequivalent irreps of respective dimension $2L+1$ with $0 \leq L \leq 2j$. This corresponds to the decomposition of each density matrix into multipoles:

$$\rho = \frac{I}{2j+1} + \sum_{L=1}^{2j} \rho^{(L)} \tag{19}$$

the scalar is fixed by the normalization $\operatorname{tr} \rho = 1$, $L = 1$ is the dipole, $L = 2$ the quadrupole, etc.

i) Expansion in Cartesian tensors. Since the three generators J_i of the rotation group generated also the algebra of operators on $\mathcal{H}_j$, it seems very intuitive to use cartesian coordinates. In that case

$$\rho = \frac{1}{2j+1} I + s_i^{(1)} J_i + s_{ij}^{(2)} J_i J_j + \ldots = \frac{1}{2j+1} I + \Sigma_{L=1}^{2j} s^{(L)}_{i_1 .. i_L} J_{i_1} J_{i_2} .. J_{i_L} \quad ; \tag{20}$$

the irreducible tensors of rank L are completely symmetrical in their indices and traceless

$$s^{(L)}_{p(i_1,..i_2)} = s^{(L)}_{i_1 ... i_2} \quad , \quad s_{ii\, i_3 . \, . i_2} = 0 \tag{20'}$$

The equations (20) are simply a generalization of equation (9) establieshed for spin ½ (with the notation $\frac{1}{2}\vec{\sigma} = \vec{J}$) . However for larger spin, they become more and more cumbersome.

ii) Expansion into Irreducible spherical tensors. It is the most frequently used by physicists; indeed the matrix elements of these irreducible spherical tensors are easily obtained from tabulated Clebsch Gordan coefficients or 3j Wigner-coefficients.

Let us first recall that the (complex) vector space $\mathcal{L}(\mathcal{H})$ of linear operators on the n dimensional Hilbert space $\mathcal{H}$, is itself a n^2 -dimensional Hilbert space with the Hermitian scalar product:

$$< A , B > = \mathrm{tr}\, A^* B \quad . \tag{21}$$

In a spherical basis define the matrices T^L_M whose elements are

$$(T^L_M)^m{}_n = < jm | jn, LM > \qquad \text{(Clebsch-Gordan coefficients)} \tag{22}$$

These are not Hermitian:

$$T^{L\ *}_M = (-1)^M \, T^L_{-M} \tag{23}$$

They satisfy

$$< T^L_M , T^{L'}_{M'} > = \mathrm{tr}\ T^{L\,*}_M\ T^{L'}_{M'} = \frac{2j+1}{2L+1}\ \delta_{LL'}\ \delta_{MM'} \qquad (24)$$

$$T^o_o = I \qquad (24')$$

Define

$$t^L_M = < \rho,\ T^L_M > = \mathrm{tr}\ \rho\ T^L_M \qquad (25)$$

Then

$$\rho = \Sigma^{2j}_{i=0}\ \frac{2L+1}{2j+1}\ \sum^{L}_{m=-L}\ \overline{t^L_M}\ T^L_M \qquad (26)$$

with $t^o_o = 1$ (27)

Since we expand a Hermitian matrix ρ into a non Hermitian basis, the coordinates t^L_M must satisfy a relation:

$$\overline{t^L_M} = (-1)^M\ t^L_{-M} \qquad (28)$$

Let ρ' be the transformed of ρ by the rotation R

$$\rho' = D_j(R)\ \rho\ D_j(R)^* \qquad (29)$$

its coordinates t'^L_M are related to the t^L_M by

$$t'^L_{M'} = t^L_M\ D_L(R)^M{}_{M'} \qquad (30)$$

iii) <u>The real multipole parameters</u>

It is paradoxical and a little akward to choose a basis of non Hermitian Operators T^L_M for studying the density operators, elements of $\mathcal{E}(\mathcal{H})$ the <u>real</u> vector space of Hermitian operators on $\mathcal{H}_j$. The space $\mathcal{E}(\mathcal{H})$ is an Euclidean space with the real scalar product (identical to (21) for Hermitian matrices)

$$(\rho_1,\ \rho_2) = \mathrm{tr}\ \rho_1\ \rho_2 \qquad (31)$$

To choose a basis of Hermitian matrices, we can take the real part and imaginary part of the T^L_M . Moreover D.M.M. (Doncel-Minnaert-Michel) have proposed a more convenient normalization:

$$Q^L_0 = \sqrt{2L+1}\; T^L_0 \tag{32}$$

$$M > 0 \qquad Q^L_M = (-1)^M \sqrt{2L+1}\; \frac{1}{2} (T^L_M + T^{L\,*}_M) \tag{32'}$$

$$M < 0 \qquad Q^L_{-M} = (-1)^M \sqrt{2L+1}\; \frac{-i}{2} (T^L_M - T^{L\,*}_M) \tag{32''}$$

so

$$(Q^L_M , Q^{L'}_{M'}) = \mathrm{tr}\, Q^L_M \; Q^{L'}_{M'} = (2j+1)\; \delta_{LL'}\, \delta_{MM'} \tag{33}$$

then

$$\rho = \frac{1}{2j+1} \left(I + \sum_{L=1}^{2j} \sum_{M=-L}^{L} q^L_M\, Q^L_M \right) \tag{34}$$

where

$$q^L_M = (\rho, Q^L_M) = \mathrm{tr}\, \rho\, Q^L_M \tag{35}$$

This formalism generalizes directement what we wrote for spin $\frac{1}{2}$

$$j = \tfrac{1}{2} , \; Q^1_0 = \sigma_3 , \; Q^1_1 = \sigma_1 , \; Q^1_{-1} = \sigma_2 \tag{36}$$

For instance we can define the degree of polarization d_ρ by

$$d^2_\rho = \frac{1}{2j} \sum_{LM} (q^L_M)^2 = \frac{2j+1}{2j}\; \mathrm{tr}(\rho-\rho_0)^2 \tag{37}$$

where ρ_0 is the density matrix of the unpolarized state.

$$\rho_0 = \frac{1}{2j+1}\; I \tag{38}$$

That is d is proportional to the distance of ρ from ρ_0 and reach its maximum for the pure states

$$\rho = \rho^* = \rho^2 , \quad \mathrm{tr}\, \rho = 1 \tag{39}$$

(These are the conditions for Hermitian projections of rank one).

iv) The matrix elements. In many papers ρ is simply given by its matrix elements. This obscure the rotational covariance by it has some advantages for the geometrical point of view we will develop. We just remark that

$$(\rho', \rho'') = \sum_{ij} \overline{\rho'_{ij}}\, \rho''_{ij} \tag{40}$$

Finally, we end this long section by some remarks. Not only these different kind of basis are choosen in the physics literature, but to give numerically the expansion coefficients or the matrix elements one has also to choose a coordinate frame in space-time, and even much more, as we will see, for the covariant description of polarization. The great variety of choice which appears in the literature makes the life much more difficult for the physicists. Some have tried to impose a universal convention (e.g. Madison convention in nuclear physics). This dictatorship is ridiculous: every choice of convention has advantages and nuisances depending on the type of experiment. But one must always remember that physical laws are independent from the choice of coordinate systems! Naturally we will essentially use here the intrinsic characteristic of matrices e.g. their spectrum = set of eigenvalues, their rank = nb of non zero eigenvalue, their kernel, their support, their image etc. Similarly we will present the polarization domain geometrically with minimum of reference to coordinate systems.

I. 4 Polarization of light.

Light polarization is the oldest known; indeed it was discovered by Malus [09] in 1808. However we did not start from photon polarization because their are essential differences between the $m=0$ and $m\neq0$ cases. Let us stress first the similarity. A formalism equivalent to 2×2 density matrix was made by S okes [52] in 1852 for light polarization (and this much before quantum mechanics!) while the description by the Poincaré sphere [92] was introduced in 1892. The surface of the sphere represent totally polarized states. Choose a pole axis; the North and South pole represents repsectively right and left circular polarization while the equator represent elliptical polarization (the shape and orientation of the ellipse depend respectively on the latitude and the longitude). The inside of the sphere represents partially polarized states, the center representing the unpolarized state.

The same sphere, the same formalism of 2 by 2 matrices deal with both cases: massive spin $\frac{1}{2}$ particles and massless spin 1 as the photon. Why? To understand plainly, we have to consider special relativity. While Lorentz transformations can induce on the spin $\frac{1}{2}$, $m \neq 0$ polarization sphere any rotation, they induce on the Poincaré sphere for light polarization only rotations around the pole axis.

I.5 Covariant description of Polarization: The polarization operator.

The invariance group of special relativity, the Poincaré group has ten linearly independent Hermitian generators: four for the translations: P^λ, observables of momentum (P^i) and energy (P^o) and six for the Lorentz transformations $M^{\mu\nu} = -M^{\nu\mu}$, observables of the relativistic antisymmetric angular momentum tensor: $M^{ij} = \epsilon_{ijk} J^k$, $M^{oi} = N^i$ where J^k is the non relativistic angular momentum.

The Lie algebra of the Poincaré group $\mathcal{P}$ is represented by the commutation relations

$$[P^\lambda, P^\mu] = 0 \ , \ [P^\lambda, M^{\mu\nu}] = ig^{\lambda\mu} P^\nu - ig^{\lambda\nu} P^\mu \qquad (42)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\, g^{\mu\rho} M^{\nu\sigma} - i\, g^{\mu\sigma} M^{\nu\rho} + i\, g^{\nu\sigma} M^{\mu\rho} - i\, g^{\nu\rho} M^{\mu\sigma} \qquad (43)$$

(This can be computed directly from the 5 x 5 matrices $\left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array}\right)$ which represents the Poincaré operation: translation by $\underline{a}$ followed by a Lorentz transformation Λ). Since the polarization of a particle is what one must observe in order to have a complete knowledge of the particle state when one already knows the energy-momentum, the observable of polarization must commute with the P^{λ}'s. We see that the angular momentum component are excluded. I do not know who found first the solution and when. It seems to be Pauli - unpublished#.

Consider the polar tensor $\tilde{\underline{\underline{M}}}$ of $\underline{\underline{M}}$, of components

$$\tilde{M}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma} \qquad (41)$$

One finds that $P^{\mu} \tilde{M}_{\mu\nu} = -\tilde{M}_{\nu\mu} P^{\mu}$ (not obvious since P and $\tilde{M}$ do not commute! is has to be check) and Pauli denoted it by

$$\underline{W} = \underline{P}.\tilde{\underline{\underline{M}}} = -\tilde{\underline{\underline{M}}}.\underline{P} \qquad (42)$$

Then, from (42) and (43) one finds that the components of the P's and that of the W's commute

$$[P^{\lambda} , W^{\mu}] = 0 \qquad (43)$$

So $\underline{P}.\underline{W} = \underline{W}.\underline{P}$; but from (42) $\underline{W}.\underline{P} = -\underline{P}\underline{W}$, hence

$$\underline{P}.\underline{W} = 0 = \underline{W}.\underline{P} \qquad (44)$$

$\underline{W}$ is therefore the good condidate we are looking for and we call it the Polarization operator. It is an axial vector operator. Finally one finds that $\underline{P}^2$ and $\underline{W}^2$ commute with all P^{λ} and $M^{\mu\nu}$ and therefore will all observables one can built from $\underline{P}$ and $\underline{\underline{M}}$ (i.e. the kinematical observables).

One can prove even that P^{λ} and W^{μ} generate the center of the algebra of polynomial in P^{λ}, $M^{\mu\nu}$ i.e. the universal enveloping algebra of the Poincaré Lie Algebra. The carrier space of irreducible representations of this algebra are the spaces of state-vectors of one particle $\mathcal{H}$. For such irreducible representation P^2 and W^2 are multiple of the unit operator i.e. $P^2 = m^2 I$, $m^2 \geq 0$.

#The first printed quotation of it, quoting Pauli, is Lubanski [42].

Let us calculate W^2. We follow here Bargmann and Wigner [48]. The strategy is that the $\underline{W}^{\lambda}$ must generate the little group $\mathcal{L}_p$ for the states of the particle with energy momentum $\mathbf{p}$.

The W^{λ}'s do not commute among each other:

$$[W_{\lambda}, W_{\mu}] = i\, \epsilon_{\lambda\mu\nu\rho}\, P^{\nu} W^{\rho} \tag{45}$$

Consider a particle state with a fixed energy momentum and define a associated tetrad $n_{\mu}^{(\alpha)}(p)$ such that

$$\underline{n}^{(\alpha)} . \underline{n}^{(\beta)} = g^{\alpha\beta} \tag{46}$$

with right handedness

$$\epsilon^{\lambda\mu\nu\rho}\, n_{\lambda}^{(\alpha)}\, n_{\mu}^{(\beta)}\, n_{\nu}^{(\gamma)}\, n_{\rho}^{(\delta)} = -\, \epsilon^{\alpha\beta\gamma\delta} \tag{46'}$$

By completion one has

$$g_{\alpha\beta}\, n_{\mu}^{(\alpha)}\, n_{\nu}^{(\beta)} = g_{\mu\nu} \tag{47}$$

We define operators $S^{(\alpha)}(p)$ by

$$S^{(\alpha)} = -\, \underline{n}^{(\alpha)} . \underline{W} \tag{48}$$

so from (47)

$$\underline{W} = \Sigma_{\alpha}\, S^{(\alpha)}(p)\, \underline{n}^{(\alpha)}(p) \tag{49}$$

i.e. the $S^{(\alpha)}$ are the components of W in the tetrad and

$$\underline{W}^2 = \Sigma_{\alpha} S^{(\alpha)}(p)\, g_{\alpha\beta}\, S^{(\beta)}(p) \tag{50}$$

We must now consider two cases.

<u>1^{rst} case</u> $M \neq 0$. We choose of course

$$\underline{n}^{(o)} = \mathbf{p}/m \tag{51}$$

so from (44) and (46), (49), (50)

$$S^{(o)} = 0 \quad \text{and} \quad \underline{W}^2 = \Sigma_{i=1}^{3}\, S^{(i)^2} \tag{52}$$

We can now compute, with the help of (45), (49),

$$[S^{(i)}, S^{(j)}] = n_\lambda^{(i)} n_\mu^{(j)} [W^\lambda, W^\mu] = i\, \varepsilon^{\lambda\mu\nu\rho} P_\nu n_\lambda^{(i)} n_\mu^{(j)} \sum_k n_\rho^{(k)} S^{(k)} \;.$$

For the particle state we consider $n^{(o)}.P = m$ (from 51) using now (46') we obtain

$$[\frac{S^{(i)}}{m}, \frac{S^{(j)}}{m}] = i\, \varepsilon_{ijk} \frac{S^{(k)}}{m} \qquad (53)$$

This is the commutation relations of the generators of the little group $\mathcal{L}_p$. All physicist know how to deduce from it the value of the Casimir operator $\sum_i \frac{S^{(i)^2}}{m^2}$

Hence we deduce from (51)

$$\underline{W}^2 = -m^2\, j(j+1)I \qquad (53')$$

$\underline{2^{nd}\ \text{case}}$ $\underline{P}^2 = 0$

In that case $\underline{p}$ is light like, so

$$\underline{p}.n^{(o)} = E \;, \quad \underline{p} = E(n^{(o)} + n^{(3)}) \qquad (54)$$

Hence from (48) and (44)

$$S^{(o)} + S^{(3)} = 0 \qquad (55)$$

and, from (50)

$$\underline{W}^2 = -(S^{(1)^2} + S^{(2)})^2 \qquad (56)$$

Taking on account (55) one can again compute the commutation relations of the $S^{(i)}$

$$[\tfrac{1}{E} S^{(3)}, S^{(1)}] = i\, S^{(2)} \quad [\tfrac{1}{E} S^{(3)}, S^{(2)}] = -i\, S^{(1)} \quad [S^{(1)}, S^{(2)}] = 0 \qquad (57)$$

The commutation relations of the generators of the little group of $\mathcal{L}_p$ define the Lie algebra of $E(2)$, the two-dimensional Euclidean group where $S^{(1)}$, $S^{(2)}$ generate the translations and $\frac{1}{E} S^{(3)}$ the rotations. We have two cases to consider.

α) $W^2 < 0$

This case corresponds to a mass zero particle with an infinite number of degrees of freedom for polarization when $\underline{p}$ is fixed. It was study in detail by Wigner [39],[48]; such type of particle is unknown in Nature

β) $\underline{W}^2 = 0$

This case is in some respect the limiting case of $m \to 0$ (see e.g. (53)). Then $\underline{P}^2 = \underline{P}.\underline{W} = \underline{W}^2 = 0$ implies.

$$\underline{W} = \lambda \, \underline{P} \tag{58}$$

Since $\underline{P}$ is a vector and $\underline{W}$ an axial vector, λ is a pseudo-scalar. It is the helicity. It is the eigenvalue of $\frac{1}{E} S^{(3)}$. Since the Poincaré group has at most a double covering representating the eigenvalue of the rotation generator in E(2) must satisfy

$$2\lambda \text{ is integer} \tag{59}$$

When the space of one-particle states is not invariant by parity, we have only one possible sign for λ e.g.

$\lambda = -\frac{1}{2}$ for neutrinos, $\lambda = \frac{1}{2}$ for antineutrinos.

But for the photon $\lambda = \pm 1$, for the gravitation $\lambda = \pm 2$.

I.6 Relativistic invariance and Polarization

The fondamental paper on this subject is Wigner [39]. But this deep and long paper is not easy to read. We just gives here some glimpses in order to show the relations between the different preceeding sections. For the particle at rest, of spin j we needed a 2j+1 dimensional space $\mathcal{H}_j$ to describe its polarization. For a covariant theory we will need a copy of this space for each energy momentum $\underset{\sim}{p}$. Here is the rigourous way to do it. Let $\mathcal{H}_j$ be a finite dimensional Hilbert space of dimension $2j+1$ if $m \neq 0$, 2 or 1 if $m = 0$; let Ω_m^+ be the mass shell $\underset{\sim}{p}^2 = m^2 \geq 0$, $p^o > 0$. The one particle state is the Hilbert space $\mathcal{H}$ of functions χ from Ω_m to $\mathcal{H}_j$;

$$\mathcal{H} = \{\chi, \Omega_m^+ \xrightarrow{\chi} \mathcal{H}_j\} \tag{60}$$

For a fixed $\underline{p}$, $\chi(p) \in \mathcal{H}_j$ where the scalar product is

$$< \chi'(\underline{p}) \; , \chi''(\underset{\sim}{p}) > \; = \; \Sigma_\alpha \; \overline{\chi'_\alpha}(p) \; \chi''_\alpha(p) \tag{61}$$

The scalar product in $\mathcal{H}$ is

$$\ll \chi', \chi'' \gg \; = \int_{\Omega_m^+} < \chi'(p), \chi''(p) > d\mu(\underline{p}) \tag{62}$$

where $d\mu(\underline{p})$ is the Lorentz invariant measure on Ω_m^+ i.e.

$$d\mu(\underline{p}) = \delta(\underline{p}^2 - m^2) \; \theta(p^o) \; d^4\underline{p} = \frac{1}{2} \frac{d^3\vec{p}}{\sqrt{\vec{p}^2 + m^2}} \tag{63}$$

So the function χ is the wave packet. Of course plane waves are not normalizable; it would correspond to $\chi(p) = \delta(p - p_{fixed})$. However there exist a notion of direct integral of Hilbert space and $\mathcal{H}_j(p) \; d\mu(p)$ is the integrand

$$\mathcal{H} = \int_{\Omega_m^+}^{\oplus} \mathcal{H}_j(p) \; d\mu(p) \tag{64}$$

where $\mathcal{H}_j(p)$ is the set of value of the χ's at energy momentum $\underset{\sim}{p}$. Every operator on $\mathcal{H}$ can also be written as a direct integral. e.g.

$$\underline{P} = \int^{\oplus} \underset{\sim}{p} \; I(p) \; d\mu(p) \tag{65}$$

where $I(p)$ is the identity operator on $\mathcal{H}_j(p)$.

Similarly when a tetrad $n^{(\alpha)}(p)$ is choosen for each $\underset{\sim}{p}$

$$W = \sum_\alpha \int^{\oplus} S^\alpha(p) \; n^{(\alpha)}(p) \; d\mu(p) \tag{66}$$

This should make very clear the previous section. Indeed $\mathcal{H}_j(p)$, technically speaking is not a subspace of $\mathcal{H}$; but we will always consider relations among the integrands and in general, for particles with fixed $\underset{\sim}{p}$, we have to deal only with finite dimensional spaces and matrices.

Translations by $\underline{a}$ just multiply the function χ by a phase $e^{-i\underline{p}.\underline{a}}$. Lorentz transformations Λ transforment $\mathcal{H}_j(p)$ into $H_j(\Lambda p)$ and acts on this space by a unitary matrix depending both on Λ and $\underset{\sim}{p}$. Some convention has to be made; it can be reduced to the choice of tetrad. For $m \neq 0$, some physicists like to choose a frame at rest (i.e. in their lab, or the center of mass) and boost this frame for each particle to obtain the corresponding $\underset{\sim}{p}$; this is for instance very usual in nucleon-nucleon scattering literature. Another general method $(m \geq 0)$ it to determine the tetrad of each particle by using the energy momenta of the other involved particles. This determination is then independent of the frame (center of mass, lab, Breit,etc...); the formalism is more elegant; we will use it, but the least possible.

When tetrads are fixed, the representation of $\bar{\mathcal{L}}_p$, the covering of the litt group, is completely fixed (up to an overall phase inherent to quantum mechanics).

If we denote by $R(n, \varphi)$ the "rotations" around $\underline{n}$ of angle φ and by Σ_n the reflections through the hyperplane orthogonal to $\underline{n}$, we have for instance

$$D_j(R(\underline{n}^{(3)},\varphi))^m{}_{m'}=e^{-im\varphi}\ \delta_{mm'} \tag{67}$$

It will be convenient to introduce the matrices (DMM)

$$(B_j)^m{}_{m'} = (-1)^{j-m}\ \delta_{mm'}\ ,\ (\Delta_j)^m{}_{m'} = \delta_{m,-m'}\ \ (\Gamma_j)^m{}_{m'} = (-1)^{j-m}\ \delta_{m,-m'} \tag{68}$$

so $$B_j^{\ 2} = I = \Delta_j^{\ 2} \quad \Gamma_j = B_j\ \Delta_j = (-1)^{2j}\ \Delta_j B_j \tag{69}$$

The Γ_j matrix , introduced by Wigner transforms $D_j(R)$, in spherical coordinates, into its complexe conjugate: (equat.(17)):

$$\overline{D_j(R)} = \Gamma D_j(R) \Gamma^{-1} \tag{70}$$

and satisfy

$$\Gamma^T = (-1)^{2j} \Gamma \quad , \quad \Gamma^2 = (-1)^{2j} I \tag{71}$$

Then the "rotations" of π around the tetrad axis are represented by

$$D_j(R(\underline{n}^{(1)},\pi)) = e^{-i\pi j} \Delta_j \ , \ D_j(R(\underline{n}^{(2)},\pi) = \Gamma_j \ , \ D_j(R(\underline{n}^{(3)},\pi)) = e^{-i\pi j} B_j \tag{72}$$

They verify the general relation:

$$(D_j(R(\underline{n}, \pi))^2 = D_j(R(2\pi)) = (-1)^{2j} I \tag{73}$$

Finally we use the general relation:

$$\Sigma_{\underline{n}} = P_{\circ}R(\underline{n}, \pi) \tag{74}$$

where P is the parity operator (reflection through the origin).

So, for instance,

$$D_j(\Sigma_{\underline{n}^{(3)}}) = \eta\, e^{-i\pi j} B_j \ , \ D_j(\Sigma_{\underline{n}^{(2)}}) = \eta\, \Gamma_j \tag{75}$$

where η is the intrinsic parity of the particle.

To summarize, we need only to use the finite dimensional space $\mathcal{H}_j$ and matrices for most of our problems. The usual non covariant polarization formalism for $m \neq 0$ can be read in a covariant fashion by replacing rotations and reflections by the corresponding operations of the little group $\mathcal{L}_p$. In other words the generator of the rotations J_i or the non relativistic spin operator are simply replaced, up to a normalization factor, by $\frac{S^{(i)}}{m} = -\frac{W}{m} . \underline{n}^{(i)}$ (equation 48). For instance, the covariant expression for polarization density matrix, corresponding to (20) is (Michel [59])

$$\rho(p) = \frac{I}{2j+1} - S_\lambda \frac{W^\lambda}{m}(p) + S_{\lambda\mu} \frac{W^\lambda}{m} \frac{W^\mu}{m} + \ldots$$

$$= \frac{I}{2j+1} + \Sigma_{i=1}^{2j} (-1)^L S^{(2)}_{\lambda_1\lambda_2\ldots\lambda_L} \frac{W^{\lambda_1}}{m}(p) \frac{W^{\lambda_2}}{m}(p) \ldots\ldots \frac{W^{\lambda_2}}{m}(p) \tag{76}$$

where $\underline{W}^{\lambda}(p)\, d\mu(p)$ is the integrand of the operator W^{λ}, and $s_{\lambda_1 \ldots \lambda_k}$ are Minkowski tensors of rang k, completely symmetrical in their k indices, with zero partial trace and orthogonal to p :

$$s_{\lambda_1 \lambda_2 \ldots \lambda_k} = s_{\lambda'_1 \ldots \lambda'_k} \quad \lambda'_i = \text{Permutation of } \lambda_i \; , \; s^{\lambda}{}_{\lambda\lambda_2 \ldots \lambda_k} = 0 \quad p^{\lambda} s_{\lambda\lambda_2 \ldots \lambda_k} = 0 \tag{77}$$

(For an explicit covariant multipole expansion of $\rho(p)$, see e.g. de Rafael [66].)

For instance, for a spin $\frac{1}{2}$ particle of energy momentum p, its polarization state is completely characterised by the axial vector $\underline{s}$, expectation value of $\underline{W}_{(p)}$ and satisfying

$$p^2 = m^2 \quad p.\underline{s} = 0 \quad \sqrt{-\underline{s}^2} = \text{degree of polarization.} \tag{78}$$

It is not the place to study here the covariant polarization operator in Dirac theory (Michel and Wightman [55]) or the covariant equation for the precession of the polarization of a particle of spin j moving in a slowly varying electromagnetic field (it depends only of the dipole polarization $\underline{s}$; Bargmann et al. [59]). The latter was the object of an Academic Training course of J.S. Bell.

I.7 Polarization correlations.

We consider only two particle-states and let the reader to generalize. Let p_1, p_2 the energy momenta of the particles; then the joint polarization density matrix $\rho(p_1, p_2)$ is a Hermitian positive operator on the space $\mathcal{H}_{j_1 j_2} = \mathcal{H}_{j_1}(p_1) \otimes \mathcal{H}_{j_2}(p_1)$ of dimension $(2j_1+1)(2j_2+2)$ (replace $2j+1$ by 2 or 1 when m=0). Given the two tetrads $\underline{n}^{(\alpha)}(p_1)$, $\underline{n}^{\beta}(p_2)$ we can expand ρ in a basis of $\mathcal{H}_{j_1 j_2}$. Since it is the representation $D_{j_1} \otimes D_{j_2}$ of SU(2) which acts on $\mathcal{H}_{j_1 j_2}$, through the little groups $\mathcal{L}_{p_1}$, $\mathcal{L}_{p_2}$, it can be very convenient for instant to use bimultipoles

$$\rho = \sum_{L_1=0}^{2j_1} \sum_{L_2=0}^{2j_2} \rho^{(L_1,L_2)} \tag{79}$$

with

$$\rho^{(L_1,L_2)} = \sum_{M_1 M_2} \frac{2L_1+1}{2j_1+1} \frac{2L_2+1}{2j_2+1} t^{L_1\ L_2}_{M_1\ M_2} T^{L_1}_{M_1} \otimes T^{L_2}_{M_2} \tag{79'}$$

$$= \frac{1}{(2j_1+1)(2j_2+1)} \sum_{M_1 M_2} q^{L_1 L_2}_{M_1 M_2} Q^{L_1}_{M_1} \otimes Q^{L_2}_{M_2} \tag{79''}$$

The space of linear operators $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ on a tensor product is itself a tensor product

$$\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) \tag{80}$$

So given two operators $A \in \mathcal{L}(\mathcal{H}_1)$; $B \in \mathcal{L}(\mathcal{H}_2)$ one can form the operator $A \otimes B$ on $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Of course this is not the most general case of operator on $\mathcal{H}_{12}$; we call these operator <u>decomposable</u>. General operators on $\mathcal{H}_{12}$ are linear combination of decomposable operators:

$$X = \Sigma_{i=1}^{n^2} A_i \otimes B_i \qquad n = \min(n_1, n_2) , n_i = \dim \mathcal{H}_i \tag{81}$$

We recall that for decomposable operators

$$A \otimes B_1 + A \otimes B_2 = A \otimes (B_1+B_2) , (A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2 \tag{82}$$

and this extends easily to general operators by linear combinations.
Also

$$X^* = \Sigma_i A_i^* \otimes B_i^* \tag{83}$$

$$\mathrm{tr}\, X = \Sigma_i \ \mathrm{tr}\, A_i \ \mathrm{tr}\, B_i \tag{84}$$

We will need furthermore a notion of partial trace:

$$\mathrm{ptr}_2\, X = \Sigma_i\, A_i \,\mathrm{tr}\, B_i \qquad \mathrm{ptr}_1\, X = \Sigma_i\, B_i \,\mathrm{tr}\, A_i \tag{85}$$

so

$$\mathrm{tr}(\mathrm{ptr}_1\, X) = \mathrm{tr}(\mathrm{ptr}_2\, X) = \mathrm{tr}\, X \tag{86}$$

For the polarization density matrix we will use the short hand.

$$\rho_1 = \mathrm{ptr}_2\, \rho \qquad\qquad \rho_2 = \mathrm{ptr}_1\, \rho \tag{87}$$

They have a clear meaning: ρ_1 (respectively ρ_2) describes the polarization of the particle 1 (resp. 2) when one does not abserve the polarization of the other particle. More generally, let A_1 and A_2 two polarization analysers for the particle 1 and 2 respectively. After the observation A_1 (resp. A_2) the polarization of the particle 2 (resp. 1) is desribed by $\mathrm{ptr}_1\, \rho(A_1 \otimes I)$ (resp. $\mathrm{ptr}_2\, \rho(I \otimes A_2)$); the special case $A=I$ corresponding to no observation at all of polarization.

Hence we can define the <u>polarization correlation:</u>

$$C_{12} = \rho - \rho_1 \otimes \rho_2 \tag{89}$$

Indeed if the particle 1 polarization is analyzed by A_1 the polarization of 2 is described by

$$\mathrm{ptr}_1\, \rho(A_1 \otimes I) = \rho_2\, \mathrm{tr}\, \rho_1 A + \mathrm{ptr}_1\, C_{12}(A_1 \otimes I)\ ; \tag{90}$$

if $C_{12} = 0$, this polarization is always ρ_2 , independently of A (the normalizing factor $\mathrm{tr}\rho_1 A$ corresponding to the change of flux intensity) and it is only if $\mathrm{ptr}_1\, C_{12}(A_1 \otimes I) = 0$ for all possible A , i.e. $C_{12} = 0$, that the polarization of 2 is independent from the observation A on the polarization of 1 i.e. there are no polarization correlations.

The normalization in expansions (79') are so chosen that the multipole coefficients $t^{L_1}_{M_1}$, $t^{L_2}_{M_2}$ or $q^{L_1}_{M_1}$, $q^{L_2}_{M_2}$ of ρ_1 and ρ_2 are given by

$$t^{L_1}_{M_2} = t^{L_1\,0}_{M_1\,0} \;,\; t^{L_2}_{M_2} = t^{0\,L_2}_{0\,M_2} \;,\; q^{L_1}_{M_1} = q^{L_1\,0}_{M_1\,0} \;,\; q^{L_2}_{M_2} = q^{0\,L_2}_{0\,M_2} \tag{91}$$

so that the bimultipole expansion of the correlation matrix is

$$(C_{12})^{L_1\,L_2}_{M_1\,M_2} = t^{L_1\,L_2}_{M_1\,M_2} - t^{L_1\,0}_{M_1\,0}\, t^{0\,L_2}_{0\,M_2} \tag{92}$$

or the corresponding expression for the q's.

If, except for $t^{00}_{00} = 1$, $t^{L_1 0}_{M_1 0} = 0 = t^{0\,L_2}_{0\,M_2}$ then each particle separately is unpolarized but there can be a non vanishing polarization correlation since it is given by $t^{L_1\,L_2}_{M_1\,M_2}$ when $L_1\, L_2 \neq 0$. For example in the π^o decay or the S_o state of positronium annihilation each one of the two γ is unpolarized when its polarization is observed separately;but if both polarization is observed in coincidence there is a maximal polarization correlation: the polarization of the second photon can be completly predicted from the nature of the analyser which has observed the first one and it is a pure state if the analyser observes only photons in pure state. (The 4 x 4 density matrix can be written $\rho = \frac{1}{4}(I_4 - \Sigma_i\, \sigma_i \otimes_{\sigma_i})$ see eg. Bernstein, Michel [60] for details). This was pointed out for positronium by Wheeler [46] who proposed an experimental test. In general experimental correlation of polarization must satisfy Bell's inequalities [64] [71]. They are well checked by the nucleon-nucleon data; that seems to rule out all proposed deterministic theories with hidden variables. Experimentally, there is of course no polarization correlations between systems prepared independently as e.g. a beam and a target. Theoretically an interesting extension is the theorem: if the partial polarization ρ_1 or ρ_2 is a pure state, there is no correlation (see DMM; it was proven independently by d'Espagnat [71]).

Another theorem which is important to know is:

If ρ is a pure state, spectrum ρ_1 = spectrum ρ_2 up to zeros.

I. 8. How is polarization measured?

One really understands what is polarization when he knows how it can be measured. Polarization measurement could be the subject of a thick treatise; new experimental techniques are found and more will be found. On the other hand, one gets too often the impression, from general lectures, that the prototype of polarization measurement is the Stern-Gerlach erperiment. This is far from the truth, even on the level of fundamental concepts. The principle of the Stern-Gerlach experiment is used in molecular beam techniques for analysing - or producing - polarization. When two Stern-Gerlach experiments are performed successively on the same beam, for measuring two different components of the spin operator $\vec{\sigma}$ (I prefer to think covariantly of the polarization operator $\underline{W}$), since the components σ_i (or W^λ) do not commute between each other, the results of the successive experiments depend on the order in which they are performed.

The practice of polarization measurement, at least in high energy physics, is fondamentally different. Indeed one measures simultaneously several components of the polarization i.e. expectation values of the products $W^{\lambda_1} W^{\lambda_2} \dots W^{\lambda_i}$. In term of Stern-Gerlach this would correspond to splitting the beam into many identical parts, each secondary beam passing through one different Stern-Gerlach. (There is no question of non commutativity of observables.) While Stern-Gerlach apparatus introduce an asymmetry which prefer one polarization state, another pinciple of polarization observation is to transform polarization into transition probability by a collision (which introduce necessarily some asymmetry) or a decay. Transition probabilities are always of the form of expectation values: $\lambda = \mathrm{tr}\,\rho\, A$, i.e. they are linear in ρ. Since angular momentum is preserved in the decay, the linear correspondance $\rho \overset{\lambda}{\longmapsto} \lambda(\rho)$ is invariant by $\mathcal{L}_p$ i.e. by "rotations".

For instance consider a two body decay $A \rightarrow B + C$ of a polarized particle A of energy momentum p_A. Energy momentum conservation requires $p_A = p_B + p_C$ so there are only two linearly independent four vector in the problem that we choose to be p_A and q :

$$\underline{q} = \frac{m_A}{\sqrt{\Delta(m_A^2, m_B^2, m_C^2)}} \left(\underline{p}_B - \underline{p}_C - \frac{m_B^2 - m_C^2}{m_A^2} \underline{p}_A\right) \tag{93}$$

where $\Delta(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx$ (94)

so $\underline{p}_A \cdot \underline{q} = 0 \quad \underline{q}^2 = 1$ (95)

From (77) (orthogonality with $\underline{p}$ of the tensors $S^{(L)}$), and "rotational" invariance (i.e. invariance by L_p) , the transition rate $\lambda(\rho)$, linear in ρ is of the form

$$\lambda(\rho) = \lambda_o + \Sigma_{L=1}^{2j} \lambda_L S^{(L)}_{\lambda_1 \ldots \lambda_L} q_{\lambda_1} \ldots q_{\lambda_L} \tag{96}$$

By parity $\underline{q}$ is changed into $-\underline{q}$ while the tensors $S^{(L)}$ are all invariant. Hence, if parity is conserved in the decay, λ is a scalar and all λ_L with L odd must vanish. Since the transition rate is independent of the odd multipoles we have the :

Theorem From the angular distribution of the decay products of a two body parity conserving decay one cannot obtain the odd (multipole) polarization.

Moreover the angular distribution, will be at most of degree 2j in $\underline{q}$ in any case.

Generally the decay angular distribution $\mathcal{J}(\underline{q})$ (if you do not like covariance, take the decaying particle at rest and replace $\underline{q}$ by θ , φ) is expanded into spherical harmonics.

$$(\underline{q}) = \sum_{LM} \bar{y}^{L}_{M} \; Y^{L}_{M}(\underline{q}) \quad \text{with} \quad \bar{y}^{L}_{M} = (-1)^{M} \, y^{L}_{-M} \tag{97}$$

The transition rate $\rho \to \lambda(\rho)$ is given by a linear map λ

$$\mathcal{L}(\mathcal{H}_j) \xrightarrow{\;\lambda\;} \mathcal{Y} = \bigoplus_{L=0}^{\infty} \mathcal{Y}^{(L)} \tag{98}$$

from the space of operators on $\mathcal{H}_j$ into the space $\mathcal{Y}$ of spherical function; for relativistic invariance this map must commute with the actions of $\mathcal{L}_p$ (or the "rotation" group) on $\mathcal{L}(\mathcal{H}_j)$ and on $\mathcal{Y}$; so between each irreducible representations of dimension 2L+1 , it is a multiple of the identity (this version of Schur lemma is known as Wigner Eckart theorem by quantum physicists); so, between the expansion coefficients of ρ and of $\mathcal{J}(\bar{q})$ one has the relation

$$y^{L}_{M} = \lambda_o C_L \, t^{L}_{M} \quad , \quad C_o = 1 \; , \quad C_L = 0 \;\; \text{if} \;\; L > 2j \tag{99}$$

As we have seen, parity conservation moreover implies

$$C_L = 0 \quad \text{for odd } L \; . \tag{100}$$

These coefficients C_L are characteristics of the decay and they may depend on its dynamics. However they are purely kinematical and depends only on the spin and parity value when there is only one amplitude in the decay (i.e. $D^{\eta_A}_{j_A}$ is of multiplicity one in $D^{\eta_B}_{j_B} \otimes D^{\eta_C}_{j_C}$ when $m_B \, m_C \neq 0$) .

This is the case, for parity preserving decays, when one of the decay product has spin zero (π or K , or particle α , etc...). Numerical tables of the C_L for all the kinematical cases have been gathered in D.M.M.

For two body parity conserving decays into two spin zero particle e.g. $\rho \to 2\pi$, $\phi \to K\bar{K}$, $K^{*} = K\,\pi$ one can measure only the even polarization (= even multipoles) sometimes also called "alignment"; when one of the decay product has spin $\neq 0$, the odd multipoles can be measured through some observation on this secondary particle polarization; this can be performed by a successive decays (see e.g. Minnaert [75], where earlier papers are also quoted, for the complete measurement of polarization from sequential decays). For instance the Y^* polarization can be completely measured from the decay $Y^{*\pm} \to \pi^{\pm} \Lambda^{o} \hookrightarrow p^{-} \pi^{+}$. From parity conserving three body decays, if one observes only the angular distribution of the normal to the decay plane, one obtains only the even polarization; the magnitude of the odd polarization multipoles is given as soon as one observes some angular distribution of the decay product into the decay, plane except for vector mesons e.g. $\omega^{o} \to \pi^{+}\pi^{-}\pi^{o}$.

The polarization of short lived resonnances, can only be observed through their decays. For long lived particles, a partial polarization measurement can be obtained by a scattering reaction, or successive scatterings with, in between, eventually, an electromagnetic field (one "calibrated" scattering, is enough for spin $\frac{1}{2}$ particles when the polarization direction is already know). Indeed unpolarized particles can be polarized by a scattering (this will be one of the subjects of next part); from time reversal invariance the same scattering can be used as polarization analyser.

References

H. BACRY [74] J. Math. Phys. 15 (1974) 1686

V. BARGMANN, E.P. WIGNER [48] Proc. Nat. Acad.Sci USA 34 (1948) 211

V. BARGMANN, L. MICHEL, V.L. TELEGDI [59] Phys. Rev. Letters 2 (1959) 435

J.S. BELL [64] Physics 1 (1964) 195

J.S. BELL [71] Foundation of Quantum Mechanics, p. 171, 40th Varenna course, Academic Press,(1971) New York

J. BERNSTEIN, L. MICHEL [60], Phys. Rev. 118 (1960) 871

DMM = M.G. DONCEL, L. MICHEL, P. MINNAERT "Matrices densité de polarisation", Ecole d'été de Physique des Particules, Gif-sur-Yvette, 1970.

M.G. DONCEL [67] N.Cim. 52A (A967) 617

B. d'ESPAGNAT [71] Lett. N. Cim. 2 (1971) 823

J.K. LUBANSKI [42] Physica 9 (1942) 310

E.L. MALUS [09] Sur une propriété de la lumière reflechie, Mém. Phys. Chim. Soc. D'Arcueil 2 (1809) 143

L. MICHEL, A.S. WIGHTMAN [55] Phys. Rev. 98 (1955) 1190

L. MICHEL [59] Suppl. N. Cim. 14 (1959) 95

P. MINNAERT [66a] Phys. Rev. Letters 16 (1966) 672

[66b] Phys. Rev. 151 (1966) 1306

[75] 3rd international Winter Meeting in Fundamental Physics p. 415 Edit. Instituto de Estudios Nucleares, Madrid (1975)

H. POINCARE [92] Théorie Mathématique de la lumière p. 275-285, Georges Carré Edit. Paris (1892)

E. de RAFAEL [66] Ann. Inst. H. Poincaré 2 (1966) 83

G.G. STOKES [52] Trans. Cambridge Phil. Soc. 9 (1852) 399

J.A. WHEELER Ann. New York, Acad. Scie. 48 (1946) 219

E.P. WIGNER [39] Ann. Math. 40 (1939) 149

E.P. WIGNER [48] Z. Phys. 124 (1948) 665